Correction to last time:

Naturality for S-W classes - required only for bundle maps that map fibers of E to fibers of E' isomorphically.

$$\begin{tikzcd} E \arrow[r, "\Phi"] \arrow[d] & E' \arrow[d] \\ M \arrow[r, "j"] & M' \end{tikzcd}$$

- Then, one wants $w_i(E) = j^* w_i(E')$.

(in particular, $E = j^* E'$ is an example.)

• Stiefel-Whitney numbers:

For M a closed smooth n-mfd, we have a unique fundamental homology class (with $\mathbb{Z}_2$-coefficients) $\mu_M \in H_n(M, \mathbb{Z}_2)$. Any deg $= n$ cohomology class $v \in H^n(M, \mathbb{Z}_2)$ can be paired to it:

$\langle v, \mu_M \rangle \in \mathbb{Z}_2$.

If $\begin{smallmatrix} E \\ \downarrow \\ M \end{smallmatrix}$ a v. bun., and $r_1, \dots, r_n \geq 0$ s.t. $r_1 + 2r_2 + \dots + n r_n = n$, then

we can form $\langle w_1(E)^{r_1} \cdots w_n(E)^{r_n}, \mu_M \rangle \in \mathbb{Z}_2$.

For $E = TM$, these a "S-W numbers of M", $\langle w_1(TM)^{r_1} \cdots w_n(TM)^{r_n}, \mu_M \rangle$
$=: w_1^{r_1} \cdots w_n^{r_n} [M]$

- S-W number of M associated with the monomial $w_1^{r_1} \cdots w_n^{r_n}$

Ex:

· for n even, $w_n(\mathbb{RP}^n) = (n+1) a^n \neq 0 \Rightarrow w_n[\mathbb{RP}^n] = 1 \in \mathbb{Z}_2 = \{0, 1\}$

· —"—, $w_1(\mathbb{RP}^n) = (n+1) a = a \neq 0$
$\Rightarrow w_1^n \neq 0 \quad \Rightarrow \quad w_1^n [\mathbb{RP}^n] = 1$

• for $\underset{2k-1}{\overset{\shortparallel}{n}}$ odd, $w(\mathbb{RP}^n) = (1+a)^{2k} = (1+a^2)^k \Rightarrow w_j = 0$ for j odd

but a monomial $w_1^{r_1} \cdots w_n^{r_n}$ of (odd) degree n must contain some w_j for j odd

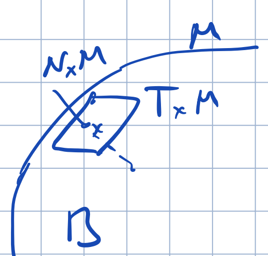
$\Rightarrow$ all S-W numbers for $\mathbb{RP}^n$ with n odd <u>vanish</u>.

<u>Thm</u> (Pontrjagin) If B is an $(n+1)$-dim. C^∞ mfd with boundary $\partial B = M$, then all S-W numbers of M are <u>zero</u>.

<u>Proof</u> Let $\mu_B \in H_{n+1}(B, M; \mathbb{Z}_2)$ - relative fund. class. Then $\partial \mu_B = \mu_M$
map $H_{n+1}(B, M) \xrightarrow{\partial} H_n(M)$

· for any $v \in H^n(M)$, we have $\langle v, \underbrace{\partial \mu_B}_{\mu_M} \rangle \overset{(*)}{=} \langle \delta v, \mu_B \rangle$
$H^n(M) \xrightarrow{\delta} H^{n+1}(B, M)$

$TB|_M = TM \oplus \underbrace{NM}$
$\shortparallel$ trivial line bundle
$i^* TB$, (outward normal - after choosing a metric on B
$i: M \hookrightarrow B$ = nonvanishing section)



So: $w_j(TM) = i^* w_j(TB)$ under $H^j(B) \xrightarrow{i^*} H^j(M)$

$\Rightarrow \delta(w_1^{r_1} \cdots w_n^{r_n}) = 0$ from $H^n(B) \xrightarrow{i^*} H^n(M) \xrightarrow{\delta} H^{n+1}(B, M)$

$\Rightarrow \langle w_1^{r_1} \cdots w_n^{r_n}, \mu_M \rangle \overset{=}{\underset{(*)}{}} 0$ $\square$

Converse: Thm (Thom)

If all S-W numbers of M are zero, then M can be realized as the boundary of some smooth compact manifold.

Ex: (1) for any M, S-W numbers of $M \amalg M$ are zero.

Indeed $M \amalg M = \partial(\underbrace{M \times [0,1]}_{B})$

(2) $M = \mathbb{R}P^{2k-1}$ has SW numbers $= 0 \Rightarrow \exists \underset{\dim = 2k}{B}$ s.t. $\partial B = \mathbb{R}P^{2k-1}$.

def two closed n-Manifolds M_1 and M_2 belong to the same unoriented cobordism class iff $M_1 \amalg M_2$ is the boundary of a compact $(n+1)$-mfd.

Cor (of Pontryagin-Thom theorems):
Two closed n-mfds M_1, M_2 belong to the same cobordism class iff all of their corresponding SW numbers are equal.

recall Künneth f-la: $H^j(M \times M', F) \cong \bigoplus_{r+s=j} H^r(M,F) \otimes H^s(M',F)$ field

• Lemma given v.bun. $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$, $\begin{matrix} E' \\ \downarrow \\ M' \end{matrix}$, $w\left(\begin{matrix} E \times E' \\ \downarrow \\ M \times M' \end{matrix}\right) = w(E) \otimes w(E')$

(in particular, $w_j(E \times E') = \sum_{i=0}^{j} w_i(E) \otimes w_{j-i}(E')$)

Proof: let $\tilde{E}, \tilde{E}'$ - pullbacks of E, E' to $M \times M'$ under $p_1 \swarrow$ $M \times M'$ $\searrow p_2$ to M, M'; then $E \times E' = p_1^* E \oplus p_2^* E'$
$\Rightarrow w(E \times E') = p_1^* w(E) \cup p_2^* w(E')$ ∎

Ex: $w(\mathbb{R}P^2 \times \mathbb{R}P^2) = (1 + a + a^2)(1 + b + b^2) = 1 + \underbrace{(a+b)}_{w_1} + \underbrace{(a^2 + ab + b^2)}_{w_2} + \underbrace{(a^2 b + a b^2)}_{w_3} + \underbrace{a^2 b^2}_{w_4}$

a, b — generators of $H^1(\mathbb{R}P^2, \mathbb{Z}_2)$ for the two copies of $\mathbb{R}P^2$

Compare with $w(\mathbb{R}P^4) = 1 + \underset{w_1}{c} + \underset{w_4}{c^4}$

SW numbers:

	$\mathbb{R}P^4$	$\mathbb{R}P^2 \times \mathbb{R}P^2$
w_1^4	1	0
$w_1^2 w_2$	0	0
$w_1 w_3$	0	0
w_2^2	0	1
w_4	1	1

$\underbrace{(a^4 + 4a^3 b + 6 a^2 b^2 + 4 a b^3 + b^4)}_{0}$ [fund class]

So: $\boxed{[\mathbb{R}P^4] \neq [\mathbb{R}P^2 \times \mathbb{R}P^2]}$ cobordism class

another notation: Ω^0_n, Ω^0_*

• $\mathcal{N}_n = \{\text{closed } n\text{-manifolds}\} / \text{unoriented cobordism}$ - additive group with $+ = \amalg$.
- it is a $\mathbb{Z}_2$-module.

• $\mathcal{N}_* = \bigoplus_{n \geq 0} \mathcal{N}_n$ - cobordism ring, $+ = \amalg$ (disj. union), $\cdot = \times$ (Cartesian product)

Ex: $\mathcal{N}_0 = \{\emptyset, pt\}$

$\mathcal{N}_1 = \{\emptyset\}$

$\mathcal{N}_2 = \{\emptyset, \mathbb{RP}^2\}$

$\mathcal{N}_3 = \{\emptyset\}$

$\mathcal{N}_4 = \{\emptyset, \mathbb{RP}^4, \mathbb{RP}^2\times\mathbb{RP}^2, \mathbb{RP}^4 \amalg \mathbb{RP}^2\times\mathbb{RP}^2\}$

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full answer for $\mathcal{N}_*$ - Dold-Thom theorem www.map.mpim-bonn.de/Unoriented_bordism

$\mathcal{N}_* = \mathbb{Z}_2[x_0, x_2, x_4, x_5, x_6, x_8, \dots]$

polynomial ring - one generator in each degree $i \neq 2^k - 1$

for i even, $x_i = [\mathbb{RP}^i]$

for i odd, $x_i = [\text{Dold manifold } P(2^r-1, s\cdot 2^r)]$, $\quad P(m,n) = \dfrac{S^m \times \mathbb{CP}^n}{(x,[y]) \sim (-x,[\bar{y}])}$

$\overset{\shortparallel}{2^r(2s+1)-1}$

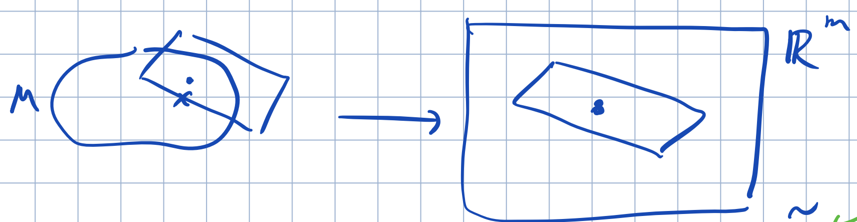
Classifying bundle & classifying maps (for real v.b.)

Recall: we have $\begin{array}{c}\tau^{(k)} \leftarrow \text{rank} \\ \downarrow \\ Gr_{\mathbb{R}}(k,m)\end{array}$ - tautological $rk = k$ bundle over the Grassmannian.

fact
- $Gr(k,m)$ is a compact C^∞ manifold

- for $i : M \hookrightarrow \mathbb{R}^m$ an immersed smooth n-mfd, one has the "Gauss map"

$$g : M \longrightarrow Gr(n,m)$$
$$x \longmapsto i_*(T_x M) \subset T_{i(x)}\mathbb{R}^m \cong \mathbb{R}^m$$



$(x, v \in T_x M) \longmapsto (\underset{\overset{\shortparallel}{(di)_x}}{i_*} T_x M, i_* v \in i_* T_x M)$

it extends to a bundle map

$$\begin{array}{ccc} TM & \xrightarrow{\tilde{g}} & \tau^n \\ \downarrow & & \downarrow \\ M & \xrightarrow[g]{} & Gr(n,m) \end{array}$$

So: $\boxed{TM = g^* \tau^n}$

Lemma For any rank $= k$ bundle $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$ over a compact base M,

there exists a map $f : M \to Gr(k, m)$ for m sufficiently large,
s.t. $\boxed{E \underset{\text{iso}}{\simeq} f^* \tau^k}$ $(*)$

Proof: Let $\{U_\alpha\}_{\alpha \in \{1 \ldots r\}}$ a (finite) trivializing cover, $\underbrace{\varphi_\alpha : U_\alpha \times \mathbb{R}^k \to \pi^{-1}(U_\alpha)}_{\text{trivializing maps}}$

Let $\{\psi_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$.

Let $h_\alpha = proj_2 \circ \varphi_\alpha^{-1} : \pi^{-1}(U_\alpha) \longrightarrow \mathbb{R}^k$ - a map linear on fibers

Let $h'_\alpha : E \longrightarrow \mathbb{R}^k$, $\underset{p \in E}{h'_\alpha(p)} = \begin{cases} \psi_\alpha(\pi(p)) \cdot h_\alpha(p) & \text{for } \pi(p) \in U_\alpha \\ 0 & \text{outside } U_\alpha \end{cases}$

Define $F : E \longrightarrow \mathbb{R}^{k \cdot r}$ - a continuous / smooth map
$p \longmapsto (h'_1(p), \ldots, h'_r(p))$ linear and injective on fibers of E.

Define $f : M \longrightarrow Gr(k, k \cdot r)$
$x \longmapsto F(E_x)$. By construction, $E \simeq f^* \tau^k$. $\square$

Terminology
- f satisfying $(*)$ is the "classifying map" for E.

$\begin{matrix} \tau^k \\ \downarrow \\ Gr(k, m) \end{matrix}$ is the "universal bundle".
(or "classifying bundle")