INTERMEDIATE GEOMETRY AND TOPOLOGY EXERCISES 10, 11/12/2021. HAMILTONIAN VECTOR FIELDS.

1. (**Harmonic oscillator.**) Consider the plane $\mathbb{R}^2$ with coordinates q, p and standard symplectic form $\omega = dq \wedge dp$. Let $H = \frac{1}{2}(p^2 + q^2)$. Find the corresponding Hamiltonian vector field X_H and its flow in time $t \in \mathbb{R}$.

2. (**Pendulum.**) Consider the symplectic manifold $M = T^*S^1$ with coordinate $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ on the base and p the coordinate on the cotangent fiber, and with the canonical symplectic form of the cotangent bundle, $\omega = d\theta \wedge dp$. Let
$$H = \frac{1}{2}p^2 - \cos\theta$$
Find the corresponding Hamiltonian vector field X_H. Analyze qualitatively the flow ρ_t of X_H:
(a) Draw a sketch of the integral curves of X_H (orbits of ρ_t).
(b) Does ρ_t have constant orbits? Which of them are stable (perturbing the initial condition yields an orbit that stays near the constant one), which are unstable?
(c) Does ρ_t have non-closed orbits?
(d) Do closed orbits of ρ_t all have the same period? If not, how does the period behave depending on the orbit?
(e) Find a closed (integral) formula for the period of a periodic orbit.

3. (**Angular momentum.**) Consider $M = T^*\mathbb{R}^3$; denote $q \in \mathbb{R}^3$ a point in the base and $p \in (\mathbb{R}^3)^* \simeq \mathbb{R}^3$ a point in the cotangent fiber. For any vector $\nu \in \mathbb{R}^3$, define a Hamiltonian function $J_\nu = (\nu, q \times p)$ where $(,)$ is the interior product of vectors in $\mathbb{R}^3$ and $\times$ the exterior product.
(a) Describe the flow of the Hamiltonian vector field X_{J_ν}.
(b) Show that the Poisson brackets[1] are $\{J_\nu, J_\mu\} = J_{\nu\times\mu}$.
(c) Show that J defines a homomorphism of Lie algebras $\mathfrak{so}(3) \xrightarrow{J} C^\infty(T^*\mathbb{R}^3)$ which fits into a sequence of homomorphisms
$$\mathfrak{so}(3) \xrightarrow{J} C^\infty(T^*\mathbb{R}^3) \xrightarrow{X_{\dots}} \mathfrak{X}(\mathbb{R}^3)$$
Here the Lie algebra structure on C^∞ is given by the Poisson bracket and Lie algebra structure on vector fields is the usual Lie bracket of vector fields.

4. (**Integrals of motion.**)
(a) Let (M, ω) be a symplectic manifold, H a Hamiltonian function on M and I another function such that $\{H, I\} = 0$.[2] Show that if γ is an integral curve of the Hamiltonian vector field X_H, then I is constant along γ (i.e. $\frac{d}{dt}I(\gamma(t)) = 0$).

[1] For f, g two functions on a symplectic manifold (M, ω), the Poisson bracket is defined as $\{f, g\} := \mathcal{L}_{X_f} g$ – the Lie derivative of g along the Hamiltonian vector field corresponding to f.

[2] Such I is called an "integral of motion" – where the motion is understood as determined by X_H.

(b) Let $M = \prod_{i=1}^n T^*\mathbb{R}$. A point in M is given by $q_1, \ldots, q_n \in \mathbb{R}$ and $p_1, \ldots, p_n \in \mathbb{R}^*$; the symplectic form is $\sum_i dq_i \wedge dp_i$. Let $H = \sum_i \frac{1}{2}p_i^2 + \sum_{i \neq j} V(q_i - q_j)$ with $V \in C^\infty(\mathbb{R})$ some fixed function.[3] Show that the "total momentum" $P = \sum_i p_i$ is an integral of motion. Describe the flow of the Hamiltonian vector field X_P generated by P.

(c) Let $M = T^*\mathbb{R}^3$ with canonical symplectic structure, let $H = \frac{(p,p)}{2} + V(q)$ with $V(q) = f(||q||)$ a function on $\mathbb{R}^3$ depending only on the norm of q. Show that in this system, the angular momentum $J_\nu = (\nu, q \times p)$ for any vector $\nu \in \mathbb{R}^3$ is an integral of motion.

5. Let (M, ω) be a symplectic manifold of dimension $2n$ and let $H_1, \ldots, H_k$ be a collection of functions on M satisfying $\{H_i, H_j\} = 0$ for any i, j. Consider the map $M \to \mathbb{R}^m$ given by $\mu(x) = (H_1(x), \ldots, H_m(x))$. Assuming that $c \in \mathbb{R}^m$ is a regular value of μ, show that $\mu^{-1}(c)$ is a coisotropic submanifold of M. Show that if additionally $m = n$, then $\mu^{-1}(c)$ is a Lagrangian submanifold of M.[4]

[3] The physical interpretation of this system is: n particles on a real line (with positions q_i and momenta p_i), of mass 1, with a pairwise interaction via a force potential V depending only on the distance.

[4] Such a situation – a maximal collection of Poisson-commuting Hamiltonians on a symplectic manifold is called an *(Liouville-) integrable system.*