INTERMEDIATE GEOMETRY AND TOPOLOGY EXERCISES 11, 11/19/2021.

1. (**Coisotropic reduction.**) Let (M, ω) be a symplectic manifold and $C \subset M$ a coisotropic submanifold. Consider the subbundle χ of the tangent bundle of C given by $\chi_x = (T_x C)^\perp$ for $x \in C$ (where $(T_x C)^\perp$ is the symplectic orthogonal of $T_x C$ in $(T_x M, \omega_x)$). χ is called the "characteristic distribution" on C.[1]
   (a) Show that an equivalent definition of χ is:

$$\chi_x = \ker(\omega_x|_{T_x C})^\# \tag{1}$$

   where $(\omega_x)|_{T_x C} : T_x C \times T_x C \to \mathbb{R}$ is the restriction of the symplectic form evaluated at x to tangent vectors to C and $(\omega_x|_{T_x C})^\# : T_x C \to T_x^* C$ is the corresponding linear map. (One can also write (1) as $\chi = \ker(\omega|_C)$.)
   (b) Show that the distribution χ is involutive. (And thus, by Frobenius theorem, integrates into a foliation.)
   (c) Assume that χ integrates to a fibration, i.e., that there exists a (smooth) fiber bundle $\pi \colon C \to B$ such that $\chi = \ker d\pi$ is the corresponding vertical tangent bundle on C. Then the base B (the space parameterizing the leaves λ of the foliation integrating χ; one denotes it C/χ or $\underline{C}$) is called the coisotropic reduction of C. Show that $\underline{C}$ is a symplectic manifold, with a natural symplectic 2-form $\underline{\omega}$ inherited from ω on M.
   (d) (Example.) Let $M = T^*\mathbb{R}^2$ with base coordinates q_1, q_2, cotangent fiber coordinates p_1, p_2 and the canonical symplectic form $\omega = \sum_{i=1}^2 dq_i \wedge dp_i$. Let C be the submanifold cut out by the equation $p_2 = 0$. Describe the coisotropic reduction $\underline{C}$.
   (e) (Example.) Let again $M = T^*(\mathbb{R}^2 \backslash \{0\})$ and let C be defined by the equation $q_1 p_2 - q_2 p_1 = 0$. Describe the coisotropic reduction $\underline{C}$.
   (f) (Example.) Let $M = T^*\mathbb{R}$ with C given by $\frac{p^2+q^2}{2} = E$ for some fixed positive number E. Describe $\underline{C}$.

2. (**Poisson algebras, Casimir elements.**) Let $\mathfrak{g}$ be a Lie algebra.
   (a) Show that there exists a unique Poisson algebra structure on the symmetric algebra $S^\bullet \mathfrak{g}$ with standard commutative associative algebra structure of the symmetric algebra and with Poisson bracket satisfying $\{X, Y\} = [X, Y]$ for $X, Y \in \mathfrak{g} = S^1 \mathfrak{g}$.[2]

[1] Recall that a "distribution" D on a manifold N is a subbundle of the tangent bundle TN. A distribution is said to be "involutive" if for any $X, Y \in \Gamma(N, D)$ – two vector fields on N belonging (parallel) to D, their Lie bracket $[X, Y]$ is also in D. Frobenius theorem says that an involutive distibution is integrable – (locally) integrates to a foliation, $N \underset{\text{open}}{\supset} U = \cup_\alpha \lambda_\alpha$, with λ_α submanifolds ("leaves"), of dimension equal to the rank of D, so that $D_x = T_x \lambda$ where λ is the leaf of the foliation through x. In adapted local coordinates $x_1, \ldots, x_n$ on N, leaves of the foliation are given by equations $x_1 = x_1^0, \ldots, x_{n-k} = x_{n-k}^0$, where k is the rank of D and $x_{\cdots}^0$ are fixed numbers.

[2] This Poisson algebra is called the Kirillov-Kostant-Souriaux Poisson algebra.

(b) Assume that $\mathfrak{g}$ is equipped with a symmetric bilinear form $K : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ which is non-degenerate (i.e. $K^\# : \mathfrak{g} \to \mathfrak{g}^*$ is an isomorphism) and invariant (i.e. $K(\mathrm{ad}_X Y, Z) + K(Y, ad_X Z) = 0$ for $X, Y, Z \in \mathfrak{g}$). Let T^a be some basis in $\mathfrak{g}$ and T_a the dual basis in $\mathfrak{g}$ (with respect to K). Set $Q = \sum T^a T_a \in S^2\mathfrak{g}$ (the "quadratic Casimir element"). Prove that Q is a central element for the Poisson bracket of (2a), i.e., $\{Q, u\} = 0$ for any $u \in S^\bullet\mathfrak{g}$.[3]

(c) Let (M, ω) be a symplectic manifold and consider the Poisson algebra structure on $C^\infty(M)$. Describe all central elements of the Poisson bracket.

3. (**Coadjoint orbits.**) Let G be a Lie group, $\mathfrak{g}$ its Lie algebra and $\mathfrak{g}^*$ the linear dual space of $\mathfrak{g}$. $\mathfrak{g}^*$ carries the coadjoint action $\mathrm{Ad}^* : G \times \mathfrak{g}^* \to \mathfrak{g}^*$ of G defined by $\langle \mathrm{Ad}_g^* \xi, X\rangle = \langle \xi, \mathrm{Ad}_{g^{-1}} X\rangle$ with $\xi \in \mathfrak{g}^*$, $X \in \mathfrak{g}$, $g \in G$, $\langle , \rangle$ the canonical pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$, $\mathrm{Ad}_g X = \frac{d}{dt}\big|_{t=0}(ge^{tX}g^{-1})$ the adjoint action of G on $\mathfrak{g}$. Likewise, $\mathfrak{g}^*$ carries the cadjoint action of the Lie algebra $\mathfrak{g}$, $\mathrm{ad}^* \colon \mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g}^*$ defined by $\langle \mathrm{ad}_X^* \xi, Y\rangle = -\langle \xi, [X, Y]\rangle$. One has an equivalence relation on $\mathfrak{g}^*$ where two points ξ, ξ' are equivalent if $\xi' = \mathrm{Ad}_g^*(\xi)$ for some $g \in G$. Equivalence classes of this relation are called "coadjoint orbits."

(a) Let $O \subset \mathfrak{g}^*$ be a coadjoint orbit and $\xi \in O$ a point in it. Show that the tangent space $T_\xi O \subset T_\xi \mathfrak{g}^* \simeq \mathfrak{g}^*$ has the form $T_\xi O = \{\mathrm{ad}_X^*(\xi) \,|\, X \in \mathfrak{g}\} \subset \mathfrak{g}^*$. More precisely, show that $T_\xi O$ fits in a short exact sequence

$$\mathrm{stab}(\xi) \hookrightarrow \mathfrak{g} \overset{\sigma}{\twoheadrightarrow} T_\xi O$$

where σ maps X to ad_X^* and $\mathrm{stab}(\xi) = \{X \in \mathfrak{g} | \mathrm{ad}_X^*(\xi) = 0\} \subset \mathfrak{g}$ is the stabilizer of ξ under the coadjoint action of the Lie algebra.

(b) Let $\omega_\xi : T_\xi O \times T_\xi O \to \mathbb{R}$ be the bilinear form on the tangent space to the orbit defined by $\omega_\xi(\sigma(X), \sigma(Y)) = \langle \xi, [X, Y]\rangle$. Show that ω_ξ is well-defined (independent of preimages X, Y of tangent vectors to the orbit in $\mathfrak{g}$), is skew-symmetric and non-degenerate.

(c) Show that the family of bilinear forms ω_ξ for points ξ of a fixed coadjoint orbit O arrange into a symplectic form ω on O (in particular, show that ω is a closed form).

(d) Study the coadjoint orbits in the example $G = SO(3)$. Prove that under the identification of $\mathfrak{so}(3)$ with $\mathbb{R}^3$ with exterior vector product, and under identification of $\mathfrak{g}^*$ with $\mathfrak{g}$ using the inner product, the coadjoint orbits are 2-spheres centered at the origin. Find the symplectic area of a coadjoint orbit passing through a vector $\xi \in \mathfrak{g}^* \simeq \mathbb{R}^3$.

[3] In the context of Poisson algebras, one generally calls central elements for the Poisson bracket the "Casimir elements."