**INTERMEDIATE GEOMETRY AND TOPOLOGY EXERCISES 12, 12/3/2021.**

1. Let $(M, \omega)$ be a compact symplectic manifold equipped with a hamiltonian action of a torus $\mathbb{T}^m = \underbrace{S^1 \times \cdots \times S^1}_{m}$, for some $m \geq 1$, with moment map $\mu \colon M \to \mathbb{R}^n$. Show that $\mu$ is automatically equivariant.

2. Consider the two versions of the condition of equivariance of the moment map $\mu$ for a hamiltonian group action $G \curvearrowright (M, \omega)$ – "finite" and "infinitesimal" versions:
   (i) $\mu(g \cdot p) = \mathrm{Ad}_g^* \mu(p)$ for any $p \in M$, $g \in G$;
   (ii) the comoment map $\mu^* : \mathfrak{g} \to C^\infty(M)$ is a Lie algebra morphism, i.e., $\mu^*([X, Y]) = \{\mu^*(X), \mu^*(Y)\}$ for any $X, Y \in \mathfrak{g}$.

   We know (from the lectures) that (i) implies (ii). Show that for $G$ connected, (i) and (ii) are equivalent, but for $G$ disconnected (ii) does not imply (i) (show this by presenting a counterexample).

3. Calculate from the definition the Lie algebra cohomology with coefficients in $\mathbb{R}$ for
   (a) $\mathfrak{g} = \mathfrak{so}(3)$
   (b) $\mathfrak{g} = \mathbb{R}^n$ (abelian Lie algebra)

4. Let $G$ be a compact Lie group and $\mathfrak{g}$ the corresponding Lie algebra. Let $\Omega^\bullet_{inv}(G) = \{\alpha \in \Omega^\bullet(G) | L_g^* \alpha = \alpha \ \ \forall g \in G\}$ be the space of left-invariant forms on $G$. Here $L_g : G \to G$, $h \mapsto gh$ is the left multiplication by $g$.
   (a) Show that $\Omega^\bullet_{inv}(G)$ is a subcomplex of $\Omega^\bullet(G)$ with respect to de Ram operator and a subalgebra with respect to the wedge product.
   (b) Show that one has an isomorphism (of algebras) of invariant forms with the exterior algebra $\Omega^\bullet_{inv}(G) \simeq \wedge^\bullet \mathfrak{g}^*$. Show that under this isomorphism, de Rham differential on forms becomes the Chevalley-Eilenberg differential on Lie algebra cochains $C^\bullet_{CE}(\mathfrak{g}) = \wedge^\bullet \mathfrak{g}^*$.
   (c) Show that de Rham cohomology of $G$ is isomorphic to the Lie algebra cohomology of $\mathfrak{g}$.

5. Let $(M, \omega)$ be a connected symplectic manifold with a Hamiltonian action of a connected Lie group $G$ with a (not necessarily equivariant) moment map $\mu : M \to \mathfrak{g}^*$. Show that if the second Lie algebra cohomology vanishes $H^2_{CE}(\mathfrak{g}, \mathbb{R}) = 0$, then $\mu$ can be corrected to an equivariant moment map $\tilde{\mu}$ (for the same group action).

6. Consider the symplectic manifold $\mathbb{C}^3$ with symplectic structure $\omega_0 = \frac{1}{2\pi i}\sum_{k=0}^{2} dz_k \wedge d\bar{z}_k$. Consider the circle action

$$\Psi_1(e^{2\pi i\theta}, (z_0, z_1, z_2)) = (e^{2\pi i\theta} z_0, e^{2\pi i\theta} z_1, e^{2\pi i\theta} z_2)$$

and the 2-torus action

$$\Psi_2((e^{2\pi i\theta_1}, e^{2\pi i\theta_2}), (z_0, z_1, z_2)) = (z_0, e^{2\pi i\theta_1} z_1, e^{2\pi i\theta_2} z_2)$$

(a) Check that the following are moment maps for $\Psi_1$ and $\Psi_2$:

$$\mu_1(z_0, z_1, z_2) = \sum_{k=0}^{2} |z_k|^2 - 1, \qquad \mu_2(z_0, z_1, z_2) = (|z_1|^2, |z_2|^2)$$

(b) Passing to the symplectic quotient of $\mathbb{C}^3$ by $S^1$ (with respect to the action $\Psi_1$ and moment map $\mu_1$), we obtain $\mu_1^{-1}(0)/S^1 = \mathbb{CP}^2$ as the reduced manifold. Calculate explicitly the symplectic structure $\omega$ on $\mathbb{CP}^2$ arising from the symplectic quotient construction.

(c) Show that the action $\Psi_2$ of $\mathbb{T}^2$ on $\mathbb{C}^3$ descends to an action $\Psi$ of $\mathbb{T}^2$ on $\mathbb{CP}^2$ with moment map

$$\mu(z_0 : z_1 : z_2) = \left(\frac{|z_1|^2}{||z||^2}, \frac{|z_2|^2}{||z||^2}\right)$$

where $(z_0 : z_1 : z_2)$ are the homogeneous coordinates on $\mathbb{CP}^2$ and $||z|| = \sqrt{\sum_{k=0}^{2} |z_k|^2}$ (note that $||z||$ and $|z_k|$ are function on $\mathbb{C}^3$, not on $\mathbb{CP}^2$, but the ratios do descend to the quotient $\mathbb{CP}^2 = \mathbb{C}^3 - \{0\}/\mathbb{C}^*$).

(d) Show that the image of the moment map $\mu(\mathbb{CP}^2)$ in $\mathbb{R}^2$ is given by the triangle

$$\{(t_1, t_2) \in \mathbb{R}^2 \mid t_1 \geq 0, t_2 \geq 0, t_1 + t_2 \leq 1\}$$

(e) Consider a circle $K \subset \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ given by $K = \{n_1\theta, n_2\theta) \in \mathbb{T}^2 \mid \theta \in \mathbb{R}/\mathbb{Z}\}$ where $n_1, n_2$ are two integers. Action $\Psi$ induces an action $\Psi_K$ of $K$ on $\mathbb{CP}^2$. Find the corresponding moment map $\mu_K$. Describe the set of fixed points of the action $K \curvearrowright \mathbb{CP}^2$ – what does it look like depending on the values of $n_1, n_2$?