INTERMEDIATE GEOMETRY AND TOPOLOGY EXERCISES 4, 9/17/2021. THOM ISOMORPHISM IN DE RHAM COHOMOLOGY.

All vector bundles in this exercise sheet will be smooth, real, *oriented*.

Definition. For E an oriented (real, smooth) vector bundle over a manifold M of rank k, we call a *Thom form* any k-form on the total space $\omega \in \Omega^k_{fcs}(E)$ (subscript "fcs" stands for "compactly supported in fibers") such that

(i) ω is closed,
(ii) For each $x \in M$, $\int_{E_x} \omega = 1$.[1]

We call the de Rham cohomology class $[\omega] \in H^k_{fcs}(E;\mathbb{R})$ the *Thom class* of the vector bundle in de Rham cohomology.

1. (Thom isomorphism) Let ω be a Thom form for a vector bundle $E \to M$. Consider the map $\phi\colon H^i(M) \to H^i_{fcs}(E)$ in de Rham cohomology given by mapping the class of an i-form $\alpha \in \Omega^i(M)$ to the class of $\omega \wedge \pi^*\alpha \in \Omega^{i+k}_{fcs}(E)$. Prove that ϕ is well-defined (does not depend on the choice of a representative of the class $[\alpha]$) and is an isomorphism, with the inverse given by the fiber integration map $\pi_*\colon H^{i+k}_{fcs}(E) \to H^i(M)$.[2]

2. (Euler form) Let ω be a Thom form for $E \to M$. Given a section $\sigma\colon M \to E$, define the Euler form as the pullback $\mathsf{e} = \sigma^*\omega \in \Omega^k(M)$. Show that e is a closed form and that its class in de Rham cohomology is independent of the choice of the section σ.

3. (Gaussian Thom form – a toy example) Consider the trivial rank one bundle $E = M \times \mathbb{R}$ over $\mathbb{R}$. We will be calling the ξ the coordinate in the fiber.
   (a) Show that for any $\epsilon > 0$ the 1-form

$$\omega = \frac{1}{\sqrt{\pi\epsilon}}\, d\xi\; e^{-\frac{\xi^2}{\epsilon}} \qquad \in \Omega^1(E)$$

[1] More pedantically: if $i_x\colon E_x \hookrightarrow E$ is the inclusion of the fiber in the total space, $\int_{E_x} \iota_x^*\omega = 1$.

[2] For an oriented vector bundle $\pi\colon E \to M$ of rank k, one has the fiber integration map $\pi_*\colon \Omega^i_{fcs}(E) \to \Omega^{i-k}(M)$. In a local trivialization it is given by

$$\pi_*\colon\ \Omega^i_{fcs}(U \times \mathbb{R}^k) \cong \bigoplus_{j=0}^{k} \Omega^{i-j}(U)\widehat{\otimes}\Omega^j_{cs}(\mathbb{R}^k) \xrightarrow{(-1)^{k(i-k)}\mathrm{id}\otimes\int_{\mathbb{R}^k}} \Omega^{i-k}(U)$$

given by selecting the $j=k$ term on the left and integrating the k-form on $\mathbb{R}^k$ over $\mathbb{R}^k$; $\widehat{\otimes}$ is the completed tensor product; the conventional sign $(-1)^{k(i-k)}$ comes from passing the degree $-k$ operation of integration over $\mathbb{R}^k$ through a $(i-k)$-form on U. Note that $\pi_*\colon \Omega^i_{fcs}(E) \to \Omega^{i-k}(M)$ is a chain map: $d\pi_* = (-1)^k \pi_* d$ (by Stokes' theorem on $\mathbb{R}^k$).

satisfies conditions (i), (ii) of the Definition. Thus ω is a Thom form, which is "Gaussian-shaped" in fibers of E, rather than being compactly supported in fibers.[3]

(b) Show that for a section σ of E given by $x \mapsto (x, f(x))$, for a function $f \in C^\infty(M)$, one has the Euler form

$$\mathsf{e} := \sigma^*\omega = \frac{1}{\sqrt{\pi\epsilon}}\, df\, e^{-\frac{f^2}{\epsilon}} \quad \in \Omega^1(M)$$

Show that e is an exact 1-form and hence represents the zero class in $H^1(M)$.

(c) Fix a small positive number α. Fix some Riemannian metric on M. Show that for $\epsilon \to 0$ and f a fixed function on M with isolated zeros, e is very small outside the union of balls of radius $\epsilon^{\frac{1}{2}-\alpha}$ around each zero of f.

4. (Gaussian Thom form – a nontrivial example) Consider the sphere $S^2 \sim \mathbb{CP}^1$ with standard complex charts $D_+ = \{(z:1) | z \in \mathbb{C}\}$, $D_- = \{(1:w) | w \in \mathbb{C}\}$. We will be writing tangent vectors to S^2 as $v = \alpha\frac{\partial}{\partial z} + \bar\alpha\frac{\partial}{\partial \bar z} \in T_z S^2$ (for $\alpha, \bar\alpha$ two complex conjugated numbers) over a point in D_+ and similarly as $v = \beta\frac{\partial}{\partial w} + \bar\beta\frac{\partial}{\partial \bar w}$ over a point in D_-. Fix $\epsilon > 0$ and consider the 2-form ω on the total space of the tangent bundle $\pi\colon TS^2 \to S^2$ defined on $\pi^{-1}(D_+)$ by

$$\omega_+ = \frac{i}{2\pi\epsilon}(1+z\bar z)^{-2}\left(\left(d\alpha - \frac{2\alpha\bar z dz}{1+z\bar z}\right)\left(d\bar\alpha - \frac{2\bar\alpha z d\bar z}{1+z\bar z}\right) + 2\epsilon dz d\bar z\right) e^{-\frac{\alpha\bar\alpha}{\epsilon(1+z\bar z)^2}} \tag{1}$$

and by the same formula with z replaced by w and α replaced by β on $\pi^{-1}(D_-)$ – let us denote the resulting form on $\pi^{-1}(D_-)$ by ω_-.

(a) Show that forms ω_+ and ω_- agree on $\pi^{-1}(D_+ \cap D_-)$ and thus indeed they together define a smooth form on TS^2.

(b) Show that $d\omega = 0$.[4]

(c) Show that for any $x \in S^2$, $\int_{T_x} \omega = 1$. Thus, ω is a Thom form on TS^2 (again, a Gaussian-shaped one, rather than compactly supported in fibers).

(d) Show that the Euler form $\mathsf{e} = \sigma^*\omega \in \Omega^2(S^2)$ for any section $\sigma\colon S^2 \to TS^2$ satisfies $\int_{S^2} \mathsf{e} = 2$ (which agrees with the Euler characteristic $\chi(S^2) = 2$).[5]

(e) Think what happens if we take the section σ to be a vector field on S^2 with isolated zeroes and take ϵ in (1) to be a very small positive number. What shape will $\sigma^*\omega$ take? How is it related to Poincaré-Hopf theorem?

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[3] One can in fact replace in the discussion of the Thom isomorphism forms compactly supported forms by forms rapidly decreasing in the fiber direction (i.e., decreasing faster than $|\xi|^{-N}$ for any N).

[4] It is a somewhat computationally heavy exercise – skip it if you don't like long computations.

[5] It is easiest to check it for σ the zero-section, and extend to arbitrary σ by the result of problem 2.