INTERMEDIATE GEOMETRY AND TOPOLOGY EXERCISES 8, 10/29/2021.

1. Show that the manifold $S^2 \times S^4$ does not admit a symplectic structure.

2. Let (V, B) be a symplectic vector space of dimension $2n$. Let $\Lambda(V, B)$ (the "Lagrangian Grassmannian") be the set of all Lagrangian subspaces of (V, B). Show that $\Lambda(V, B)$ has a natural structure of a compact smooth manifold. Find its dimension.

3. Let X_1 and X_2 be two n-dimensional manifolds. Prove that for a map $F \colon T^*X_1 \to T^*X_2$ the following two properties are equivalent:[1]
   (i) $F^*\alpha_2 = \alpha_1$, where $\alpha_{1,2}$ are the tautological 1-forms on $T^*X_{1,2}$. (In particular, F is a symplectomorphism.)
   (ii) F is the contangent lift of a diffeomorphism between the bases, $f \colon X_1 \to X_2$.

4. Consider $S^3 = \{(z_1, z_2) \mid |z_1|^2 + |z_2|^2 = 1\}$ as unit sphere in $\mathbb{C}^2$ equipped with standard symplectic structure $\omega = \frac{i}{2}\sum_{k=1}^{2} dz_k \wedge d\bar{z}_k$.
   (a) Show that S^3 is a coisotropic submanifold of $\mathbb{C}^2$.
   (b) Show that for any point $p \in S^3$, the symplectic orthogonal of T_pS^3 (as a subspace of $T_p\mathbb{C}^2$) satisfies
   $$(T_pS^3)^\perp = \mathrm{span}(v_p)$$
   where v is the vector field of the $U(1)$-action on S^3 given by
   $$\begin{array}{ccc} U(1) \times S^3 & \to & S^3 \\ (e^{i\theta}, (z_1, z_2)) & \mapsto & (e^{i\theta}z_1, e^{i\theta}z_2) \end{array}$$

5. Let (V, B) be a $2n$-dimensional symplectic vector space. Prove that any 1-dimensional subspace of V is isotropic, while any $(2n-1)$-dimensional subspace of V is coisotropic.

6. Let (V, B) be a symplectic vector space and let $C \subset V$ be a coisotropic subspace.
   (a) Show that the quotient space $\underline{C} \colon = C/C^\perp$ inherits a symplectic form from B. I.e., show that the bilinear skew-symmetric form on $\underline{C}$ defined by
   $$\underline{B} \colon \quad \begin{array}{ccc} \underline{C} \times \underline{C} & \to & \mathbb{R} \\ ([u], [v]) & \mapsto & B(u, v) \end{array}$$
   for any $u, v \in C$ (with $[u], [v]$ the classes of u, v in the quotient) is well-defined (independent of the choice of representatives of equivalence classes $[u], [v]$) and nondegenerate.
   (b) Show that the subspace
   $$W = \{(u, v) \in C \times C \mid u - v \in C^\perp\} \qquad \subset \quad V \oplus V$$

[1] For hints, see Ana Cannas da Silva, "Lectures on symplectic geometry," pp 20-21.

is a Lagrangian subspace of $V \oplus V$ equipped with symplectic form $(-B) \oplus B$.

7. (a) Give an example of a compact Lagrangian submanifold of $\mathbb{R}^{2n}$ equipped with standard symplectic structure, for any $n \geq 1$.
   (b) Using Darboux theorem, show that any symplectic manifold has a compact Lagrangian submanifold.