INTERMEDIATE GEOMETRY AND TOPOLOGY EXERCISES 9, 11/5/2021.

1. (**Transporting Lagrangians to coisotropic reduction.**) Let (V, B) be a symplectic vector space, $C \subset V$ a coisotropic subspace and $L \subset V$ a Lagrangian subspace. Let $\underline{C} = C/C^\perp$ the "coisotropic reduction" with its induced symplectic structure $\underline{B}$ (cf. previous exercise sheet) and denote $p\colon C \to \underline{C}$ the quotient map. Show that $\underline{L}\colon = p(L \cap C)$ is a Lagrangian subspace of $(\underline{C}, \underline{B})$.

2. (**Composition of canonical relations.**) A canonical relation between symplectic vector spaces (V, B) and (V', B') is a subspace $L \subset \overline{V} \times V'$ which is Lagrangian with respect to the "twisted" product symplectic structure, $(-B) \oplus B'$. Given a triple of symplectic vector spaces (V, B), (V', B'), (V'', B''), a canonical relation L_1 between (V, B) and (V', B') and a canonical relation L_2 between (V', B') and (V'', B''), consider the "composition" $L_1 \circ L_2$ defined as

$$L_1 \circ L_2 = \{(x, z) \in V \times V'' \mid \exists y \in V' \text{ s.t. } (x, y) \in L_1 \text{ and } (y, z) \in L_2\}$$

Put another way, $L_1 \circ L_2 = p((L_1 \times L_2) \cap (V \times \Delta \times V''))$ where $p\colon \overline{V} \times V' \times \overline{V'} \times V'' \to \overline{V} \times V''$ the projection to the first and last factor and $\Delta = \{(y, y) \mid y \in V'\} \subset V' \times \overline{V'}$ the diagonal in V'.

   (a) Prove that $L_1 \circ L_2$ is an isotropic subspace of $\overline{V} \times V''$ with respect to $(-B) \oplus B''$.

   (b) Prove that in fact $L_1 \circ L_2$ is Lagrangian (and hence is a canonical relation between (V, B) and (V'', B'')).[1]

   (c) In differential-geometric setting, one defines canonical relations and their compositions similarly, replacing symplectic vector spaces with symplectic manifolds and Lagrangian subspaces with Lagrangian submanifolds. Prove that the composition of canonical relations is isotropic (if it is a submanifold of $\overline{M} \times M''$).

3. (**Generating functions.**) Consider a family of symplectomorphisms

$$\begin{array}{rccc} \phi_\theta\colon & T^*\mathbb{R} & \to & T^*\mathbb{R} \\ & (x, \xi) & \mapsto & (y = x\cos\theta - \xi\sin\theta, \eta = x\sin\theta + \xi\cos\theta) \end{array}$$

with $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ (i.e. ϕ_θ is a counterclockwise rotation of the plane $T^*\mathbb{R} = \mathbb{R}^2$ by angle θ). Here $T^*\mathbb{R}$ is understood as coming with its canonical symplectic form $dx \wedge d\xi$. Find the generating function $f_\theta(x, y)$ of the symplectomorphism ϕ_θ.

4. (**Legendre transform.**) Let V be a vector space, then $M = V \oplus V^*$ can be seen as a cotangent bundle T^*V with its canonical symplectic structure ω_0, or as the cotangent bundle $\overline{T^*(V^*)}$ with the negative of its canonical symplectic structure, which we denote by the overline (negative ensures that it coincides with ω_0).

[1] Hint: use the result of problem 1 for the symplectic space $\overline{V} \times V' \times \overline{V'} \times V''$, a coisotropic $C = V \times \Delta \times V''$ (why is it coisotropic?) and Lagrangian $L_1 \times L_2$.

Assume that $L \subset M$ is a Lagrangian submanifold which is projectable both to V and thus given as a graph, $L = \mathrm{graph}(df)$ for some function $f \in C^\infty(V)$. Assume additionally that L is also projectable to V^*, thus it can also be written as $L = \mathrm{graph}(dh)$ for some other function $h \in C^\infty(V^*)$.

(a) Prove that h is related to f by the *Legendre transform*:

$$h(\xi) = \langle \xi, x \rangle - f(x)$$

for $\xi \in V^*$, where $x = x(\xi) \in V$ is a solution of the equation

$$\xi = df_x$$

(b) (Example.) Let $V = \mathbb{R}$ and $f(x) = \frac{1}{2}ax^2 + bx$ with $a \neq 0$ and b two real numbers. Find the corresponding $h(\xi)$.

(c) (Example.) Let again $V = \mathbb{R}$ and $f(x) = e^x$. Find the corresponding $h(\xi)$.

5. (**A version of Poincaré lemma.**) Let M be a smooth manifold, $i\colon X \hookrightarrow M$ a submanifold and $\alpha \in \Omega^k(M)$ a closed k-form on M satisfying $i^*\alpha = 0$. Prove that then one can find an open neighborhood U of X in M and a $(k-1)$-form $\beta \in \Omega^{k-1}(U)$ such that $\beta|_X = 0$ and $\alpha = d\beta$ on U.[2]

[2]Hint – use the *tubular neighborhood theorem*: given a manifold M and a submanifold $i\colon X \hookrightarrow M$, one can find a neighborhood U_0 of the zero-section $i_0\colon X \to NX$ of the normal bundle NX (the normal bundle $NX = i^*TM/TX$ of X in M), a neighborhood U of X in M, and a diffeomorphism $\phi\colon U_0 \to U$ satisfying $\phi \circ i_0 = i$. (You can accept this statement without proof. For the proof, see e.g. section 6.2 in Cannas da Silva's notes.)