# LECTURE NOTES ON CONFORMAL FIELD THEORY

PAVEL MNEV

Contents

## 1. A long introduction: Segal's picture of 2D conformal field theory

### 1.1. Segal's axioms of quantum field theory.

Lecture 1, 8/24/2022

1.1.1. *The definition of QFT.* Segal [35] suggested the following geometrical definition of a quantum field theory. Segal focused mainly on the case of 2D conformal theories; Atiyah in [3] described the case of topological theories.[1]

**Data.** A $D$-dimensional QFT is the following assignment:

- A closed oriented $(D-1)$-manifold $\gamma$ is assigned a vector space $\mathcal{H}_\gamma$ over $\mathbb{C}$ (the "space of states").
- An oriented $D$-manifold $\Sigma$ with boundary split into disjoint in- and out-components such that $\partial\Sigma = -\gamma_{\mathrm{in}} \sqcup \gamma_{\mathrm{out}}$ (minus means orientation reversal),[2] is assigned a linear map $Z_\Sigma \colon \mathcal{H}_{\gamma_{\mathrm{in}}} \to \mathcal{H}_{\gamma_{\mathrm{out}}}$ (the "evolution operator" or "partition function").

[1] Another reference for Segal's viewpoint on QFT, with motivation from quantization of classical field theories, is [33]. In the exposition here I was inspired by Losev's lectures [27].

[2] We will say that $\Sigma$ is a cobordism from $\gamma_{\mathrm{in}}$ to $\gamma_{\mathrm{out}}$ and write $\gamma_{\mathrm{in}} \xrightarrow{\Sigma} \gamma_{\mathrm{out}}$ and think of $\Sigma$ as an arrow in a cobordism category, where objects are oriented closed $(D-1)$-manifolds. See also Remark 1.4 below for a more careful definition of a cobordism.

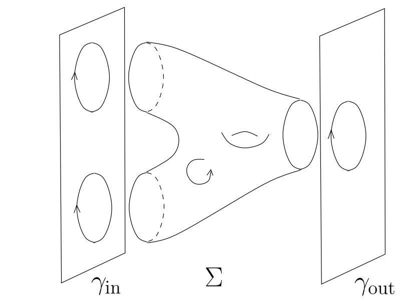



FIGURE 1. Cobordism.

**Axioms.**

• Multiplicativity: "$\sqcup \to \otimes$" (disjoint unions are mapped to tensor products).

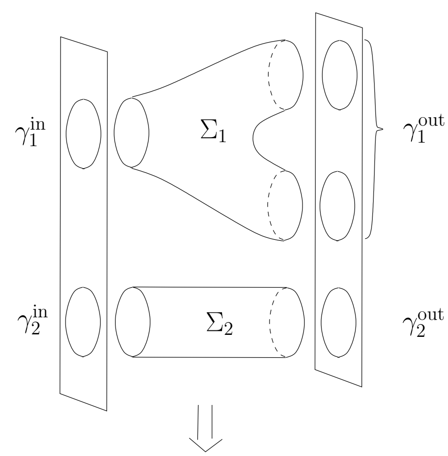
(a) Given two closed $(D-1)$-manifolds $\gamma_1, \gamma_2$, one has

$$\mathcal{H}_{\gamma_1 \sqcup \gamma_2} = \mathcal{H}_{\gamma_1} \otimes \mathcal{H}_{\gamma_2}.$$

(b) Given two $D$-cobordisms $\gamma_1^{\text{in}} \xrightarrow{\Sigma_1} \gamma_1^{\text{out}}$, $\gamma_2^{\text{in}} \xrightarrow{\Sigma_2} \gamma_2^{\text{out}}$, one has

$$Z_{\Sigma_1 \sqcup \Sigma_2} = Z_{\Sigma_1} \otimes Z_{\Sigma_2}$$

where both sides are linear maps $\mathcal{H}_{\gamma_1^{\text{in}}} \otimes \mathcal{H}_{\gamma_2^{\text{in}}} \to \mathcal{H}_{\gamma_1^{\text{out}}} \otimes \mathcal{H}_{\gamma_2^{\text{out}}}$.




$$\mathcal{H}_{\gamma_1^{\text{in}}} \otimes \mathcal{H}_{\gamma_2^{\text{in}}} \xrightarrow{Z_{\Sigma_1} \otimes Z_{\Sigma_2}} \mathcal{H}_{\gamma_1^{\text{out}}} \otimes \mathcal{H}_{\gamma_2^{\text{out}}}$$

FIGURE 2. Multiplicativity with respect to disjoint unions.

• Sewing axiom: "$\cup \to \circ$" (sewing of cobordisms is mapped to composition of linear maps). Given two cobordisms $\gamma_1 \xrightarrow{\Sigma'} \gamma_2$ and $\gamma_2 \xrightarrow{\Sigma''} \gamma_3$ one can sew[3] the

[3] In the case of a topological theory (cobordisms are smooth oriented manifolds with boundary, with no extra geometric structure), one can consider cobordisms modulo diffeomorphisms relative to the boundary, and then sewing is a well-defined operation. In 2d conformal theory, cobordisms are Riemann surfaces with parametrized boundary and the sewing operation, identifying two circles along the parametrization, is also well-defined.

out-boundary of the first one to the in-boundary of the second one, obtaining a sewn cobordism $\Sigma = \Sigma' \cup_{\gamma_2} \Sigma''$.

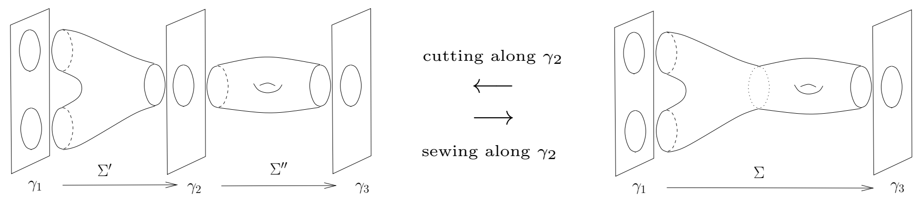



FIGURE 3. Sewing.

Then one has

$$Z_\Sigma = Z_{\Sigma''} \circ Z_{\Sigma'} \tag{1}$$

or, making domains and codomains explicit,

$$\mathcal{H}_{\gamma_3} \xleftarrow{Z_\Sigma} \mathcal{H}_{\gamma_1} = \mathcal{H}_{\gamma_3} \xleftarrow{Z_{\Sigma''}} \mathcal{H}_{\gamma_2} \xleftarrow{Z_{\Sigma'}} \mathcal{H}_{\gamma_1}.$$

- Normalization.

(a) For the empty $(D-1)$-manifold, one has

$$\mathcal{H}_\varnothing = \mathbb{C}.$$

(b) For any closed oriented $(D-1)$ -manifold $\gamma$, the partition function for a "very short" cylinder[4] $\gamma \times [0, \epsilon]$ tends to the identity map on the space of states:

$$\lim_{\epsilon \to 0} Z_{\gamma \times [0,\epsilon]} = \mathrm{id} \colon \mathcal{H}_\gamma \to \mathcal{H}_\gamma$$

**Additional data.**

Action of diffeomorphisms. For $\phi \colon \gamma \to \widetilde{\gamma}$ a diffeomorphism, we have a map

$$\rho(\phi) \colon \mathcal{H}_\gamma \to \mathcal{H}_{\widetilde{\gamma}} \tag{2}$$

which is linear if $\phi$ is orientation-preserving and is antilinear if $\phi$ is orientation-reversing. Moreover, this is an action, i.e., $\rho(\phi_2 \circ \phi_1) = \rho(\phi_2) \circ \rho(\phi_1)$.

Geometric data. Cobordisms $\Sigma$ are equipped with *local geometric data* $\xi_\Sigma \in \mathrm{Geom}_\Sigma$ of type which depends on the particular QFT.[5] Examples of $\xi_\Sigma$:

(1) Riemannian (or pseudo-Riemannian) metric on $\Sigma$. This is the case for many physically relevant QFTs, like, e.g., Yang-Mills theory or electrodynamics.
(2) Conformal structure on $\Sigma$ (metric up to rescaling by a positive function). This is the case relevant to us (especially for $D = 2$).
(3) Nothing. Despite its apparent triviality, actually a very interesting case corresponding to *topological* quantum field theories, in the sense of Atiyah [3].

Boundaries $\gamma$ should also be equipped with geometric data, $\xi_\gamma \in \mathrm{Geom}_\gamma$. E.g., in the cases above the corresponding boundary data is:

[4]The precise meaning of "very short" depends on the type of geometric data we put on cobordisms.

[5]In our notations, $\mathrm{Geom}_\Sigma$ is the space of all geometric data of given type on $\Sigma$, and $\xi_\Sigma$ is a particular choice.

(1) A germ of Riemannian bi-collars on $\gamma$ (a germ of Riemannian metrics on $\gamma\times(-\epsilon,\epsilon)$).
(2) A parametrization of a boundary circle $\gamma$.
(3) Nothing.

The relation between geometric data for cobordisms and for boundaries is that one wants that for a sewn cobordism $\Sigma$, $\mathrm{Geom}_\Sigma$ is the fiber product $\mathrm{Geom}_\Sigma = \mathrm{Geom}_{\Sigma'}\times_{\mathrm{Geom}_\gamma}\mathrm{Geom}_{\Sigma''}$. I.e., when we sew cobordisms in the sewing axiom, we also sew the geometric data.

**Axioms continued.**

Naturality (equivariance under diffeomorphisms).

Given a diffeomorphism between cobordisms, $\phi\colon \Sigma\to\widetilde{\Sigma}$, one has a commutative diagram

$$\begin{tikzcd} \mathcal{H}_{\gamma_{\mathrm{in}}} \arrow[r, "Z_{\Sigma,\xi}"] \arrow[d, "\rho(\phi|_{\mathrm{in}})"'] & \mathcal{H}_{\gamma_{\mathrm{out}}} \arrow[d, "\rho(\phi|_{\mathrm{out}})"] \\ \mathcal{H}_{\widetilde{\gamma}_{\mathrm{in}}} \arrow[r, "Z_{\widetilde{\Sigma},\widetilde{\xi}=\phi_*\xi}"'] & \mathcal{H}_{\widetilde{\gamma}_{\mathrm{out}}} \end{tikzcd} \tag{3}$$

1.1.2. *Remarks.*

*Remark* 1.1. For a closed $D$-manifold $\varnothing\xrightarrow{\Sigma}\varnothing$, the partition function is $Z_\Sigma\colon \mathbb{C}\xrightarrow{\cdot\zeta}\mathbb{C}$ – multiplication by some complex number $\zeta$. By abuse of notations, this number $\zeta$ is also called the partition function (and also denoted $Z_\Sigma$).

*Remark* 1.2. One may summarize the axioms above by saying that a QFT is a functor of symmetric monoidal categories

$$\mathrm{Cob}\xrightarrow{(\mathcal{H},Z)}\mathrm{Vect}_\mathbb{C} \tag{4}$$

where on the left one has the category of spacetimes (a.k.a. geometric cobordism category), where:

- The objects $(\gamma,\xi_\gamma)$ are closed oriented $(D-1)$-manifolds $\gamma$ equipped with geometric structure $\xi_\gamma\in\mathrm{Geom}_\gamma$.
- The morphisms $(\Sigma,\xi_\Sigma)$ are $D$-dimensional oriented cobordisms with geometric structure $\xi_\Sigma\in\mathrm{Geom}_\Sigma$.
- Composition is sewing of cobordisms (accompanied by sewing the geometric data).
- Monoidal tensor product is given by disjoint unions. Monoidal unit is the empty $(D-1)$-manifold.
- Cob is a non-unital category: it does not have identity morphisms. Instead, it has "almost identity" morphisms – short cylinders.[6]

The right hand side of (4) is the category of complex vector spaces and linear maps with obvious monoidal structure given by tensor product.

Naturality axiom says that diffeomorphisms act on the functor $(\mathcal{H},Z)$ by natural transformations.

Another way to understand diffeomorphisms categorically is as an enhancement of Cob to a bicategory, where the second type of 1-morphisms is diffeomorphisms

[6]An exception is the topological case $\mathrm{Geom}=\varnothing$ where finite cylinders $\gamma\times[0,1]$ play the role of identity morphisms on the nose, without having to approximate identity by a family.

of $(D-1)$-manifolds and 2-morphisms are diffeomorphisms of cobordisms. Then naturality says that (4) extends to a functor of bicategories.

*Remark* 1.3. It is very interesting to restrict the naturality axiom (3) to the subgroup $\mathrm{Sym}_{\Sigma,\xi} \subset \mathrm{Diff}_\Sigma$ of diffeomorphisms $\phi\colon \Sigma \to \Sigma$ preserving the chosen geometric data $\xi$, i.e., satisfying $\phi_*\xi = \xi$. Then, (3) yields *symmetries* of $Z_{\Sigma,\xi}$ (the "Ward identities"):

$$Z_{\Sigma,\xi} = \rho(\phi|_{\mathrm{out}}) \circ Z_{\Sigma,\xi} \circ \rho(\phi|_{\mathrm{in}})^{-1}. \tag{5}$$

*Remark* 1.4. A careful definition of a $D$-cobordism is as a quintuple $(\Sigma, \gamma_{\mathrm{in}}, \gamma_{\mathrm{out}}, i_{\mathrm{in}}, i_{\mathrm{out}})$ consisting of the following:

- $\gamma_{\mathrm{in}}$, $\gamma_{\mathrm{out}}$ two closed oriented $(D-1)$-manifolds,
- $\Sigma$ an oriented $D$-manifold with boundary,
- two embeddings $i_{\mathrm{in}}\colon \gamma_{\mathrm{in}} \hookrightarrow \partial\Sigma$, $i_{\mathrm{out}}\colon \gamma_{\mathrm{out}} \to \partial\Sigma$ with disjoint images, such that
  - $\partial\Sigma = i_{\mathrm{in}}(\gamma_{\mathrm{in}}) \sqcup i_{\mathrm{out}}(\gamma_{\mathrm{out}})$,
  - $i_{\mathrm{in}}$ is orientation-reversing and $i_{\mathrm{out}}$ is orientation-preserving.

With this definition, one can say that the data of the action of a diffeomorphism $\phi$ on the spaces of states (2) is redundant, as it is already contained in the data of partition functions assigned to cobordisms, as $Z$ for an infinitesimally short mapping cylinder

(6)
$$M_\phi = \left(\gamma \times [0,\epsilon], \gamma, \gamma, \quad \begin{array}{cccc} i_{\mathrm{in}}\colon & \gamma & \hookrightarrow & \gamma\times[0,\epsilon] \\ & x & \mapsto & (x,0) \end{array}, \quad \begin{array}{cccc} i_{\mathrm{out}}\colon & \gamma & \hookrightarrow & \gamma\times[0,\epsilon] \\ & x & \mapsto & (\phi(x),\epsilon) \end{array} \right).$$

From this viewpoint, the naturality axiom (3) is a special case of the sewing axiom (when one is attaching two short mapping cylinders to the in-/out-ends of a cobordism).

Is this right? One needs then to adjoin conjugation $\rho(r)$ as a separate piece of data?

*Remark* 1.5. One has a natural identification between $\mathcal{H}_{-\gamma}$ and the linear dual of $\mathcal{H}_\gamma$, since the partition function of a short cylinder, seen as a cobordism $\gamma \sqcup (-\gamma) \xrightarrow{\gamma\times[0,\epsilon]} \varnothing$ yields (in $\epsilon \to 0$ limit) a bilinear pairing

$$(,)\colon \mathcal{H}_\gamma \otimes \mathcal{H}_{-\gamma} \to \mathbb{C}, \tag{7}$$

which is nondegenerate.[7]

*Remark* 1.6. Given a cobordism, one can always reassign a connected component of the in-boundary as a component of the out-boundary with reversed orientation. The corresponding partition functions are equal:

$$Z\Big(\gamma_1 \sqcup \gamma_2 \xrightarrow{\Sigma} \gamma_3\Big) = Z\Big(\gamma_1 \xrightarrow{\Sigma} \gamma_3 \sqcup (-\gamma_2)\Big),$$

using the identification $\mathcal{H}_{-\gamma_2} = \mathcal{H}^*_{\gamma_2}$ from Remark 1.5. In [35] this property is called the "crossing axiom."

[7]Nondegeneracy is shown by the following argument. One can consider a second short cylinder $\varnothing \xrightarrow{\gamma\times[0,\epsilon]} (-\gamma)\sqcup\gamma$. Attaching $-\gamma$ from the in-boundary of the first cylinder to the $-\gamma$ from the out-boundary of the second cylinder, we obtain a cylinder $\gamma \xrightarrow{\gamma\times[0,2\epsilon]} \gamma$ whose partition function converges to identity. That implies that the pairing (7) cannot have any kernel vectors.

1.1.3. *Unitarity (and its Euclidean counterpart).* For any $\gamma$ (we are suppressing the geometric data in notation) one has the tautological orientation-reversing mapping $r\colon \gamma \to -\gamma$ mapping each point to itself. By (2), one has a corresponding antilinear map $\rho(r)\colon \mathcal{H}_\gamma \to \mathcal{H}_{-\gamma}$. Combining it with pairing (7), one has a sesquilinear form

$$\langle,\rangle\colon \mathcal{H}_\gamma \otimes \mathcal{H}_\gamma \xrightarrow{\rho(r)\otimes \mathrm{id}} \mathcal{H}_{-\gamma}\otimes \mathcal{H}_\gamma \xrightarrow{(,)} \mathbb{C}. \quad (8)$$

Unitarity is an *optional* collection of assumptions on a QFT which it might satisfy (or not):

mention that it might be good to drop the completeness assumtion and refer to Remark 5.4?

(a) $(\mathcal{H}_\gamma, \langle,\rangle)$ is a Hilbert space for each $\gamma$. In particular, the sesquilinear form $\langle,\rangle$ is positive definite.
(b) For a cylinder $\gamma\times[0,t]$, the partition function $Z_{\gamma\times[0,t]}$ is a *unitary* operator $\mathcal{H}_\gamma\to\mathcal{H}_\gamma$.
(c) The representation of diffeomorphisms on spaces of states (2) is unitary.

We will be studying 2d CFTs in Euclidean signature; they are not unitary theories in the sense above. In fact, properties (a) and (c) may hold for them (in which case one talks about a "unitary CFT"), but (b) fails. Instead, (b) gets replaced by its Euclidean counterpart:

(b') The partition function of a cobordism $\gamma_1 \xrightarrow{\Sigma} \gamma_2$, and of its orientation-reversed copy $\gamma_2 \xrightarrow{-\Sigma} \gamma_1$ are related by

$$Z_{-\Sigma} = \bar{Z}^*_\Sigma,$$

where bar stands for complex conjugation and star is the dual (transpose) map.[8]

Note also that if $\dim \mathcal{H} = +\infty$, (b) is incompatible with the trace-class property that one wants to have in a CFT.

Lecture 2, 8/26/2022

## 1.2. **Examples of Segal's QFTs.**

1.2.1. *TQFTs and a silly example.* A Segal's QFT with no geometric data on cobordisms and boundaries is a topological quantum field theory in the sense of Atiyah [3]. A TQFT assigns to a closed oriented $D$-manifold a complex number $Z_\Sigma\in\mathbb{C}$ – invariant of a $D$-manifold up to diffeomorphism, behaving nicely with respect to cutting/gluing.

There are very interesting examples like e.g. $D = 3$ Chern-Simons theory.

**A silly example.** For any $D$ we can construct a TQFT with $\mathcal{H}_\gamma = \mathbb{C}$ for any $\gamma$ and

$$Z(\Sigma) = e^{\chi(\Sigma)-\chi(\gamma_{\mathrm{in}})}$$

for any cobordism $\gamma_{\mathrm{in}} \xrightarrow{\Sigma} \gamma_{\mathrm{out}}$.[9] Here $\chi$ is the Euler characteristic. It follows from the additivity of Euler characteristic that Segal's axioms are satisfied (in particular, multiplicativity and sewing).

[8] In Osterwalder-Schrader axioms, this property is called "reflection positivity." Segal [35] calls it "*-functor" property.

[9] Slightly more generally, we can set $Z(\Sigma) = e^{\chi(\Sigma)-\alpha\chi(\gamma_{\mathrm{in}})-\beta\chi(\gamma_{\mathrm{out}})}$ where $\alpha,\beta$ are fixed numbers such that $\alpha+\beta = 1$. E.g. one can make a symmetric choice $\alpha=\beta=\frac{1}{2}$.

1.2.2. *$D=1$ Riemannian Segal's QFT – quantum mechanics.* Here objects of the spacetime category (0-manifolds) are collections of points with orientation $\pm$. Fix a vector space $\mathcal{H}$ and let the space of states for $\mathrm{pt}^+$ be $\mathcal{H}_{\mathrm{pt}^+} := \mathcal{H}$. Then $\mathcal{H}_{\mathrm{pt}^-} = \mathcal{H}^*$.

Morphisms of the spacetime category (1-cobordisms) are collections of oriented intervals and circles equipped with Riemannian metric. Note that naturality axiom implies that the partition function for a cobordism depends only on metric modulo diffeomorphisms, i.e., only on lengths of connected components. Denote the partition function for an interval of length $t$ (thought of as a cobordism $\mathrm{pt}^+ \xrightarrow{[0,t]} \mathrm{pt}^+$) by $Z_t \colon \mathcal{H} \to \mathcal{H}$.

Sewing intervals of lengths $t_1$ and $t_2$, we get an interval of length $t_1 + t_2$. Thus, the sewing axiom implies the semi-group law

$$Z_{t_1+t_2} = Z_{t_2} \circ Z_{t_1}. \tag{9}$$

Assume that we have an improved normalization property:

$$Z_\epsilon \underset{\epsilon\to 0}{\sim} \mathrm{id} + A\epsilon + O(\epsilon^2) \tag{10}$$

with $A \in \mathrm{End}(\mathcal{H})$ some linear operator. In physical normalization, one writes $A = -\frac{i}{\hbar}\widehat{H}$, then the operator $\widehat{H} \in \mathrm{End}(\mathcal{H})$ is called the "quantum Hamiltonian" (or "Schrödinger operator"). Together, (9) and (10) imply

$$Z_t = (Z_{\frac{t}{N}})^N = \lim_{N\to\infty} (\mathrm{id} + A\frac{t}{N} + O(\frac{1}{N^2}))^N = e^{At} = e^{-\frac{i}{\hbar}\widehat{H}t}. \tag{11}$$

(I.e., the idea is that we cut a finite interval into $N$ tiny intervals where $Z$ is well-approximated by (10), and then reassemble them using the sewing axiom.)

Formula (11), which we recovered from Segal's axioms, is the standard expression for the evolution operator in time $t$ in quantum mechanics with quantum Hamiltonian $\widehat{H}$. In quantum mechanics, one recovers (11) from Shrödinger equation

$$(i\hbar\,\partial_t + \widehat{H})\psi_t = 0 \tag{12}$$

for a $t$-dependent state $\psi_t \in \mathcal{H}$. Equation (12) implies $\psi_t = Z_t(\psi_0)$, with $Z_t$ given by (11). One may also say that the Schrödinger equation itself (12), seen from Segal's standpoint, expresses the sewing axiom for sewing an infinitesimal interval of length $dt$ to a finite interval of length $t$.

*Remark* 1.7. Recall that $\mathcal{H}$ is automatically equipped with a sesquilinear form (8). The 1D QFT above is unitary if additionally $\mathcal{H}, \langle,\rangle$ is a Hilbert space and if $\widehat{H}$ is a *self-adjoint* operator, which implies that the evolution operator (11) is unitary.

*Remark* 1.8. If we ask $\widehat{H}$ to be self-adjoint, but consider evolution in imaginary time $t = -iT_{\mathrm{Eucl}}$ with $T_{\mathrm{Eucl}} > 0$ (the "Euclidean time"), then (11) becomes a self-adjoint operator

$$Z = e^{-\frac{T_{\mathrm{Eucl}}}{\hbar}\widehat{H}} \tag{13}$$

(instead of unitary) and the theory satisfies (b') of Section 1.1.3 instead of (b). Comment more on Wick rotation?

*Remark* 1.9. It follows from the sewing axiom that the partition function for a circle of length $t$ is given by the trace of the partition function for the interval of length $t$

$$Z(S^1_t) = \mathrm{tr}_{\mathcal{H}} Z_t = \mathrm{tr}_{\mathcal{H}} e^{-\frac{i}{\hbar}\widehat{H}t} \tag{14}$$

This is a shortened write-up of problem 1 from exercise sheet 1

**Example 1.10** (Quantum mechanics of a free particle on a circle)**.** Let $X$ be a circle of length $L$. Free particle on $X$ is described by the quantum Hamiltonian

$$\widehat{H} = -\frac{1}{4\pi}\frac{\partial^2}{\partial x^2} \tag{15}$$

acting on the Hilbert space $\mathcal{H} = L^2(X)$; $x \in \mathbb{R}/L\cdot\mathbb{Z}$ is the coordinate on the circle $X$. Here for simplicity we adopted the units where $\hbar = 1$ and the mass of the particle is $2\pi$ (this normalization of the Hamiltonian is chosen in order to have simpler formulae below).

The partition function for an interval of length $t$ is a unitary integral operator $Z_t = e^{-i\widehat{H}t}$ with integral kernel

$$K_t(x_1, x_0) = \sum_{n=-\infty}^{\infty} (it)^{-\frac{1}{2}} e^{\pi i \frac{(x_1 - x_0 + nL)^2}{t}}. \tag{16}$$

The partition function for $\Sigma$ a circle of length $t$ is then

$$Z(S^1_t) = \mathrm{tr}_{\mathcal{H}} Z_t = \int_X dx\, K_t(x,x) = L(it)^{-\frac{1}{2}} \sum_{n=-\infty}^{\infty} e^{\pi i \frac{L^2}{t} n^2} \tag{17}$$

We note that another way to obtain $\mathrm{tr}_{\mathcal{H}} Z_t$ is via the eigenvalue spectrum of the Hamiltonian (15). The eigenfunctions of $\widehat{H}$ are $\psi_k = e^{\frac{2\pi i k x}{L}}$ and the corresponding eigenvalues are $E_k = \pi\left(\frac{k}{L}\right)^2$. Thus, one has

$$Z(S^1_t) = \mathrm{tr}_{\mathcal{H}} e^{-i\widehat{H}t} = \sum_{k=-\infty}^{\infty} e^{-iE_k t} = \sum_{k=-\infty}^{\infty} e^{-\pi i \frac{t}{L^2} k^2} \tag{18}$$

One can show directly by Poisson summation formula[10] that the right hand sides of (17) and (18) agree; in Poisson summation, the sum over "winding numbers" $n$ is transformed into a sum over the dual summation index – the "momentum" $k$.

We note that one can consider the evolution in Euclidean time $t = -iT_{\mathrm{Eucl}}$ with $T_{\mathrm{Eucl}} > 0$. Then the operator $Z_t$ becomes trace-class and sums (17), (18) become absolutely convergent.

Denoting for convenience $\lambda \colon = \frac{L^2}{T_{\mathrm{Eucl}}}$ and denoting the partition function for a circle (17), (18) by $\zeta(\lambda)$, we have an interesting transformation property under $\lambda \to \lambda^{-1}$:

$$\zeta(\lambda) = \lambda^{-\frac{1}{2}} \zeta(\lambda^{-1}) \tag{19}$$

This property can be regarded as a very simple instance of the so-called $T$-duality (behavior under inversion of the radius of the target circle). Alternatively, if one fixes $L = 1$, (19) becomes a toy 1d model of modular invariance in 2d conformal field theory, see (28) below.

[10] Recall that Poisson summation formula says that for a function $f(x)$ on $\mathbb{R}$ decaying sufficiently fast at $x \to \pm\infty$, with $\widetilde{f}(p) = \int_{\mathbb{R}} f(x) e^{2\pi i p x} dx$ its Fourier transform, one has $\sum_{n\in\mathbb{Z}} f(n) = \sum_{k\in\mathbb{Z}} \widetilde{f}(k)$. One can see this as the equality of distributions $\sum_{n\in\mathbb{Z}} \delta(x-n) = \sum_{k\in\mathbb{Z}} e^{2\pi i k x}$, integrated against a test function $f$.

1.3. **Quantum observables in Segal's language (the idea).** Fix a Segal's QFT. For $\gamma_{\mathrm{in}} \xrightarrow{\Sigma} \gamma_{\mathrm{out}}$ a cobordism, let $\Gamma \subset \Sigma$ be a CW subcomplex disjoint from $\partial\Sigma$.[11] Let consider a family of $\epsilon$-thickenings $U_\epsilon(\Gamma)$ of $\Gamma$ in $\Sigma$, with $\epsilon \in (0, \epsilon_0)$.[12]

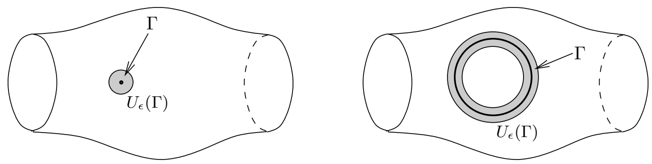



FIGURE 4. $\epsilon$-thickenings.

A quantum observable supported on $\Gamma$ is a family (parametrized by $\epsilon \in (0, \epsilon_0)$) of elements

$$\widehat{O}_{\Gamma,\epsilon} \in \mathcal{H}_{\partial U_\epsilon(\Gamma)} \tag{20}$$

I.e. for each $\epsilon$ we have a state on the boundary of the $\epsilon$-tube around $\Gamma$.

The correlator (or VEV – "vacuum expectation value") of the observable is defined as

$$\langle \widehat{O}_\Gamma \rangle_\Sigma : = \lim_{\epsilon \to 0} Z_{\Sigma - U_\epsilon(\Gamma)} \circ \widehat{O}_{\Gamma,\epsilon} \qquad \in \mathrm{Hom}(\mathcal{H}_{\gamma_{\mathrm{in}}}, \mathcal{H}_{\gamma_{\mathrm{out}}}) \tag{21}$$

The idea here is that $\Sigma$ with the tube around $\Gamma$ cut out has as its boundaries $\gamma_{\mathrm{in}}$, $\gamma_{\mathrm{out}}$ and a new piece of boundary – the boundary of the tube, where we plug the state given by the observable. An important case is when $\Sigma$ is closed (i.e., $\gamma_{\mathrm{in}} = \gamma_{\mathrm{out}} = \varnothing$). Then the correlator (21) is a complex number.

The $\epsilon$-dependence in the family (20) is supposed to be such that the limit in the r.h.s. of (21) exists. One way to arrange it is to require that elements (20) for different $\epsilon$ are related by

$$\widehat{O}_{\Gamma,\epsilon'} = Z_{U_{\epsilon'}(\Gamma) - U_\epsilon(\Gamma)} \circ \widehat{O}_{\Gamma,\epsilon} \tag{22}$$

for $0 < \epsilon < \epsilon' < \epsilon_0$. In this case the expression under the limit in (21) does not depend on $\epsilon \in (0, \epsilon_0)$ (as follows from the sewing axiom).

For us, the most important case would be when $\Gamma$ is a collection of points (correlators of point observables). However, in topological and gauge theories it is natural to consider different $\Gamma$s, e.g., Wilson loop observable in Chern-Simons and Yang-Mills theories corresponds to $\Gamma$ an embedded circle in $\Sigma$; its generalization – Wilson graph – corresponds to $\Gamma$ an embedded graph in $\Sigma$.

1.3.1. *Example: point observables in quantum mechanics.* In the setting of Section 1.2.2 – quantum mechanics as 1d QFT – consider the cobordism $\Sigma = [t_{\mathrm{in}}, t_{\mathrm{out}}]$ and consider an observable supported at a single point $\Gamma = \{t\}$ inside $\Sigma$. As the thickening we can take small intervals

$$U_\epsilon(\Gamma) = [t - \epsilon, t + \epsilon].$$

[11] It is very interesting to allow $\Gamma$ to go to intersect the boundary of $\Sigma$, but that would lead us into QFTs with corners (known in the topological case, as *extended* TQFTs in the sense of Baez-Dolan-Lurie).

[12] E.g. we can equip $\Sigma$ with a metric and define $U_\epsilon(\Gamma)$ as the set of points of distance $\leq \epsilon$ from $\Gamma$.

The boundary of the thickening is a pair of points of opposite orientation

$$\partial U_\epsilon(\Gamma) = \mathrm{pt}^- \sqcup \mathrm{pt}^+.$$

Thus, a quantum observable is an element

$$\widehat{O} \in \mathcal{H}_{\partial U_\epsilon(\Gamma)} = \mathcal{H}_{\mathrm{pt}^- \sqcup \mathrm{pt}^+} = \mathcal{H}^* \otimes \mathcal{H} \cong \mathrm{End}(\mathcal{H}) \tag{23}$$

– an operator on the space of states $\mathcal{H}$.

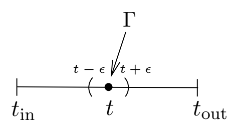



FIGURE 5. Point observable in quantum mechanics.

We can similarly consider several point observables on $\Sigma$, supported at $\Gamma = \{t_1, \ldots, t_n\}$ (we assume that $t_{\mathrm{in}} < t_1 < t_2 < \cdots < t_n < t_{\mathrm{out}}$). The picking a state on the boundary of $\epsilon$-thickening of each point amounts to choosing a collection of operators $\widehat{O}_1, \ldots, \widehat{O}_n \in \mathrm{End}(\mathcal{H})$. The correlator (21) then is

$$\langle \widehat{O}_1(t_1) \cdots \widehat{O}_n(t_n) \rangle_\Sigma = e^{-\frac{i}{\hbar}\widehat{H}(t_{\mathrm{out}}-t_n)} \widehat{O}_n \cdots e^{-\frac{i}{\hbar}\widehat{H}(t_2-t_1)} \widehat{O}_1 e^{-\frac{i}{\hbar}\widehat{H}(t_1-t_{\mathrm{in}})}$$

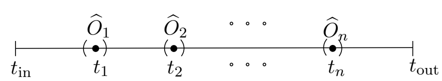



FIGURE 6. Correlator of several point observables in quantum mechanics.

1.4. **2d conformal field theory as a Segal's QFT.** In the main case of interest for us – two-dimensional conformal field theory – the geometric structure on cobordisms is conformal structure (Riemannian metric up to rescaling by a positive function), plus orientation; in two dimensions this data is equivalent[13] to complex structure. Thus, cobordisms are (possibly disconnected) Riemann surfaces with parametrized boundary circles (when sewing in- and out-circles, one should respect the parametrization – points with the same angle parameter are identified).[14] Parametrization of boundaries is needed for the sewn surface to have a well-defined complex structure.[15]

Such a Riemann surface with $n$ in-circles and $m$ out-circles, $\sqcup_{i=1}^n S^1 \xrightarrow{\Sigma} \sqcup_{j=1}^m S^1$, is assigned a linear map $Z(\Sigma) \colon \mathcal{H}_{S^1}^{\otimes n} \to \mathcal{H}_{S^1}^{\otimes m}$.

[13] We will come to this later.

[14] Parametrization of boundary circles can be seen in terms of Remark 1.4 as the embeddings $i_{\mathrm{in}}, i_{\mathrm{out}}$ of unions of standard circles into $\partial\Sigma$.

[15] E.g. sewing the two boundary circles of a cylinder with a twist by angle $\theta$, one obtains non-equivalent complex tori for different $\theta$.

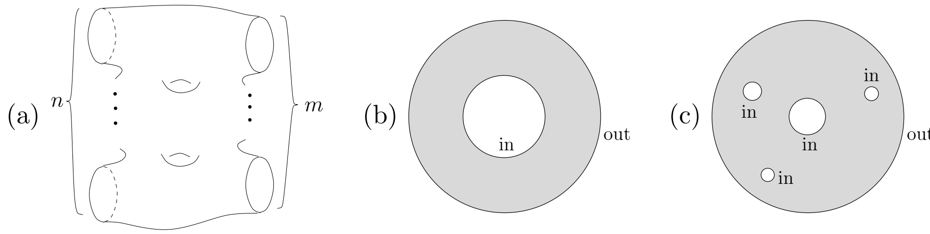



FIGURE 7. (a) a generic cobordism in 2d CFT and some relevand cobordisms embedded in $\mathbb{C}$ – (b) annulus (coformally equivalent to a cylinder) and (c) a 2d equivalent of Figure 6 (corresponding to several point observables).

The space of states for a circle $\mathcal{H}_{S^1}$ is a Hilbert space carrying a representation of the group of diffeomorphisms $\mathrm{Diff}(S^1)$,

$$\rho\colon \mathrm{Diff}(S^1) \to \mathrm{End}(\mathcal{H}_{S^1}). \tag{24}$$

Vacuum vector. The space $\mathcal{H}_{S^1}$ contains a distinguished vector

$$|\mathrm{vac}\rangle \in \mathcal{H}_{S^1} \tag{25}$$

– "vacuum vector" – the partition function of the disk.[16] In (b), (c) of Figure 7, pairing with $|\mathrm{vac}\rangle$ for any of the in-boundaries corresponds to removing (or filling in with the disk) the corresponding hole.

Self-sewing. If the surface $\widetilde{\Sigma}$ is obtained from $\Sigma$ by gluing $i$-th in-circle to $j$-th out-circle, one has

$$Z(\widetilde{\Sigma}) = \mathrm{tr}_{\mathcal{H}} Z(\Sigma) \tag{26}$$

Here on the right hand side we mean a partial trace – the trace taken in the first factor of

$$Z(\Sigma) \in \mathrm{Hom}\left(\mathcal{H}_{S^1_{\mathrm{in},i}}, \mathcal{H}_{S^1_{\mathrm{out},j}}\right) \bigotimes \mathrm{Hom}\left(\bigotimes_{1\leq k\leq n, k\neq i} \mathcal{H}_{S^1_{\mathrm{in},k}}, \bigotimes_{1\leq l\leq m, l\neq j} \mathcal{H}_{S^1_{\mathrm{out},l}}\right).$$

Self-sewing formula (26) is not an extra axiom – it follows from the usual sewing axiom by attaching an infinitesimally short cylinder to $S^1_{\mathrm{in},i}$ and $S^1_{\mathrm{out},j}$.

In particular, traces (26) must exist if we have a full CFT.[17] Segal in [35] imposes a slightly stronger condition that traces exist in the sense of absolute convergence, i.e., that partition functions are *trace-class* operators.

[16]This vector is not invariant under $\mathrm{Diff}(S^1)$. However, as a consequence of naturality, it is invariant under the 3-dimensional subgroup (isomorphic, via identifying the disk with upper half-plane, to $PSL_2(\mathbb{R})$ – real Möbius transformations) consisting of diffeomorphisms of $S^1$ which can be extended as conformal transformations over the whole disk.

[17]One may consider a partial CFT where partition functions are only defined on genus zero cobordisms. In that case one can make do with partition function for which traces do not exist. An example of such a model is massless scalar field with values in $\mathbb{R}$; the variant with values in $S^1$ (a.k.a. "compactified free boson") is a full CFT existing in all genera.

1.4.1. *Genus one partition function, modular invariance.* Given a complex number $\tau \in \mathbb{C}$ with $\mathrm{Im}\,\tau > 0$, one can consider the Riemann surface

$$\mathbb{T}_\tau :\ = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z}) \tag{27}$$

– the quotient of $\mathbb{C}$ equipped with standard complex structure by a lattice; (27) is the complex torus with modular parameter $\tau$.

One can evaluate the CFT on $\mathbb{T}_\tau$. Denote

$$Z(\tau) :\ = Z(\mathbb{T}_\tau) \in \mathbb{C}$$

Then since tori $\mathbb{T}_\tau$ and $\mathbb{T}_{-1/\tau}$ are equivalent as complex manifolds (via the holomorphic map $z \mapsto z/\tau$), $Z_\tau$ as a function of $\tau$ possesses modular invariance property

$$Z(\tau) = Z(-\frac{1}{\tau}) \tag{28}$$

Also, tori $\mathbb{T}_\tau$ and $\mathbb{T}_{\tau+n}$ are equivalent for any $n \in \mathbb{Z}$ (via the tautological map $z \mapsto z$), hence one also has $Z(\tau + n) = Z(\tau)$.

In particular one can consider the torus (27) with $\tau = iT$, $T > 0$, as obtained from a cylider $\Sigma = S^1 \times [0, T]$ (we think of $S^1$ as having length 1) by sewing the out-end to in-end. CFT restricted to cylinders can be regarded as quantum mechanics with partition functions

$$Z(S^1 \times [0, T]) = e^{-2\pi T \widehat{H}} \tag{29}$$

for some self-adjoint operator $\widehat{H} \in \mathrm{End}(\mathcal{H}_{S^1})$ – the Hamiltonian, cf. Section 1.2.2.[18] Then by (26) we have

$$Z(iT) = \mathrm{tr}_{\mathcal{H}_{S^1}} e^{-2\pi T \widehat{H}} \tag{30}$$

As a function of $T$, (30) has to be invariant under inversion $T \leftrightarrow \frac{1}{T}$, as a special case of (28).

The general torus (27), with $\tau = \frac{\theta}{2\pi} + iT$ can be obtained from (29) by identifying boundary circles with a twist by the angle $\theta$:

$$\mathbb{T}_\tau = \frac{S^1 \times [0, T]}{(\sigma, 0) \sim (\sigma + \frac{\theta}{2\pi}, T),\ \sigma \in S^1}$$

By sewing and naturality axioms, the corresponding partition function is

$$Z(\tau) = \mathrm{tr}_{\mathcal{H}_{S^1}} e^{-2\pi T \widehat{H} + i\theta \widehat{P}} \tag{31}$$

where $\widehat{P} \in \mathrm{End}(\mathcal{H})$ is the infinitesimal generator of the action of the subgroup of rigid rotations $S^1 \subset \mathrm{Diff}(S^1)$ on $\mathcal{H}_{S^1}$ (in particular, $\widehat{P}$ is a self-adjoint operator with integer eigenvalues).

1.4.2. *Correction to the picture: conformal anomaly.* Conformal field theories one constructs in reality satisfy Segal's axioms in a weakened – "*projective*" – sense:

- The representation of $\mathrm{Diff}(S^1)$ on $\mathcal{H}_{S^1}$ is projective. Put another way, there is an honest representation of a central extension $\widehat{\mathrm{Diff}(S^1)}$ of the group of diffeomorphisms on $\mathcal{H}_{S^1}$. This central extension is known as the Virasoro group.

Q: Naturality – strict or projective? A: strict.

[18] Here we are considering evolution in "Euclidean time" $T$, cf. (13). We also set $\hbar = 1$. The factor $2\pi$ in the exponential is a choice of normalization of the Hamiltonian and is put there for compatibility with standard CFT conventions.

- Sewing axiom (1) holds up to a factor in $\mathbb{C}^*$. – One says that (4) is a *projective functor*. Equivalently, one can say that partition functions are operators in $\mathrm{Hom}(\mathcal{H}_{\mathrm{in}}, \mathcal{H}_{\mathrm{out}})$ defined up to scaling by a factor in $\mathbb{C}^*$.

As another viewpoint, one can understand a projective functor as a strict functor out of a central extension of the cobordism category (see [35]). This is equivalent to saying that $Z_\Sigma$ is not a function on $\mathrm{Geom}_\Sigma$ but rather a section of a line bundle on it.[19]

Yet another viewpoint on the projectivity phenomenon is that CFT partition functions are well-defined as operators for a given Riemannian metric $g$ on the surface $\Sigma$, but if one changes the metric within its conformal class, $g \mapsto \Omega \cdot g$, with Weyl factor $\Omega = e^{2\sigma}$, then the partition function scales by a complex factor:

$$Z(\Sigma, e^{2\sigma} g) = e^{icS_{\mathrm{Liouville}}(\sigma,g)} \cdot Z(\Sigma, g). \tag{32}$$

Here $c$ is a number (the "strength" of the projectivity effect), known as the *central charge* of the CFT;

$$S_{\mathrm{Liouville}}(\sigma, g) = \int_\Sigma \frac{1}{2}(d\sigma \wedge *d\sigma + 4\sigma R_g\, dvol_g)$$

is the "Liouville action," $R_g$ is the scalar curvature.

Lecture 3, 8/29/2022

### 1.5. Heuristic motivation for Segal's axioms from path integral quantization.

A classical (Lagrangian) field theory on a cobordism $\gamma_{\mathrm{in}} \xrightarrow{\Sigma} \gamma_{\mathrm{out}}$ is determined by the following data:

(a) The space of fields on $\Sigma$,

$$\mathrm{Fields}_\Sigma = \Gamma(\Sigma, E)$$

– the space of smooth sections of a fiber bundle $E$ over $\Sigma$ – the bundle of fields. (For instance, fields could be maps from $\Sigma$ to some target manifold $X$, or fields could be differential forms on $E$.)

(b) The action functional – a real-valued function on the space of fields of the form

$$S_\Sigma(\phi) = \int_\Sigma L(\phi, \partial\phi, \cdots) \qquad \in \mathbb{R} \tag{33}$$

where $\phi \in \mathrm{Fields}_\Sigma$ is a field. Here $L$ (the Lagrangian) is a $D$-form (or density) on $\Sigma$, depending on the field $\phi$ in a local way: the value of $L$ at a point $x \in \Sigma$ can depend only on the value of $\phi$ at $x$ and its derivatives up to a finite order at $x$.[20]

Given the data above, at the classical level one is interested in the solutions $\phi \in \mathrm{Fields}_\Sigma$ of the "equations of motion" – the critical point equation

$$\delta S = 0 \tag{34}$$

---

[19] More explicitly, in CFT this line bundle is $\mathcal{L}^{\otimes c} \otimes \bar{\mathcal{L}}^{\otimes \bar{c}}$ as a bundle over the moduli space of complex structures on $\Sigma$. Thus, $Z_\Sigma \in \mathrm{Hom}(\mathcal{H}_{\mathrm{in}}, \mathcal{H}_{\mathrm{out}}) \otimes \mathcal{L}^{\otimes c} \otimes \bar{\mathcal{L}}^{\otimes \bar{c}}$. Here $\mathcal{L} = \mathrm{Det}(\bar\partial)$ is the Quillen line bundle – the determinant line bundle of the Dolbeault operator; $\bar{\mathcal{L}}$ is the complex conjugate one; $c$ and $\bar{c}$ are numbers – holomorphic and anti-holomorphic central charges. Usually one has $c = \bar{c}$ (this is the case assumed in (32) below).

[20] It is convenient (see [1] for details) to consider the "variational bicomplex" $\Omega^{p,q}_{\mathrm{loc}}(\Sigma \times \mathrm{Fields}_\Sigma)$ of $(p,q)$-forms on $\Sigma \times \mathrm{Fields}_\Sigma$ local in the same sense. In this terms, the Lagrangian $L$ is in $\Omega^{D,0}_{\mathrm{loc}}(\Sigma \times \mathrm{Fields}_\Sigma)$.

(with $\delta$ the de Rham operator on $\mathrm{Fields}_\Sigma$). One considers the equation (34) (which is the Euler-Lagrange PDE) with boundary conditions on the field at $\gamma_{\mathrm{in}}, \gamma_{\mathrm{out}}$. In a range of cases (Lagangians of second order in derivatives), one can consider the boundary conditions of the form

$$\phi|_{\gamma_{\mathrm{in}}} = \phi_{\mathrm{in}}, \quad \phi|_{\gamma_{\mathrm{out}}} = \phi_{\mathrm{out}} \tag{35}$$

where $\phi_{\mathrm{in}} = \mathrm{Fields}_{\gamma_{\mathrm{in}}}$ and $\phi_{\mathrm{out}} \in \mathrm{Fields}_{\gamma_{\mathrm{out}}}$ – fixed sections of the bundle $E$ over the boundaries $\gamma_{\mathrm{in}}$ and $\gamma_{\mathrm{out}}$, respectively.

*Remark* 1.11. One can consider more general boundary conditions on $\gamma$ of the form

$$\pi(\mathrm{Jet}(\phi)|_\gamma) = b_\gamma \tag{36}$$

where $\mathrm{Jet}(\phi)|_\gamma$ is the normal $\infty$-jet of $\phi$ at $\gamma$;

$$\pi\colon \{\text{normal jets of fields at } \gamma\} \to B_\gamma$$

is some fibration and $b_\gamma \in B_\gamma$ a point in the base. The desired scenario is when the solution of (34) with boundary condition (36) exists and is locally-unique (non-deformable).

**Example 1.12** (Classical mechanics of a particle on a Riemannian manifold )**.** Let $D = 1$. Fix a Riemannian manifold $(M, g)$ (target), a positive number $m$ (mass) and a function $V \in C^\infty(M)$ (the force potential). Consider as the cobordism the interval $\Sigma = [0, t]$ and set

$$\mathrm{Fields}_\Sigma = \mathrm{Map}([0, t], M) \tag{37}$$

– the space of paths in $M$ parametrized by the interval $[0, t]$. We set the action $S\colon \mathrm{Fields} \to \mathbb{R}$ to be defined by

$$S_\Sigma(\phi) = \int_0^t d\tau \left( \frac{m}{2} g_{\phi(\tau)}(\dot\phi(\tau), \dot\phi(\tau)) - V(\phi(\tau)) \right) \tag{38}$$

for $\phi\colon [0, t] \to M$ a field (a path).[21]

Setting for simplicity $(M, g) = \mathbb{R}^N$ with standard Euclidean metric, the critical point equation $\delta S = 0$ is equivalent to the ODE

$$m\ddot\phi(\tau) + \mathrm{grad} V(\phi(\tau)) = 0 \tag{39}$$

– the Newtonian equation of motion of a particle in $\mathbb{R}^N$ in the force field with potential $V$. One can consider this equation with Dirichlet boundary conditions $\phi(0) = \phi_{\mathrm{in}}$, $\phi(t) = \phi_{\mathrm{out}}$ where $\phi_{\mathrm{in}}$, $\phi_{\mathrm{out}}$ – two given points in $\mathbb{R}^N$. Thus, we are considering parametrized paths in $\mathbb{R}^N$ satisfying the equation (39) with *fixed endpoints*. E.g. if $V = 0$, there is a unique solution – the straight interval connecting $\phi_{\mathrm{in}}$ and $\phi_{\mathrm{out}}$ with constant-velocity parametrization by $[0, t]$.

If we take a general Riemannian manifold $(M, g)$ and set $V = 0$, then $\delta S = 0$ is equivalent to the geodesic equation. So, solutions of the boundary problem (34), (35) are the geodesics in $M$ connecting the two given points.

[21] Note that the Riemannian metric on the source cobordism $\Sigma$ is implicitly used in (38): the action (38) is not invariant under reparametrization of a path. One can also write a $\mathrm{Diff}(\Sigma)$-invariant version of the action (38):

$$S_{\Sigma,\xi}(\phi) = \int d\tau \sqrt{\xi(\tau)} \left( \xi(\tau)^{-1} \frac{m}{2} g_{\phi(\tau)}(\dot\phi(\tau), \dot\phi(\tau)) - V(\phi(\tau)) \right)$$

Here $\xi(\tau) d\tau^2$ is the metric on $\Sigma$. Then for $\psi\colon \Sigma \to \Sigma$ a diffeomorphism, one has $S_{\Sigma,\xi}(\phi) = S_{\Sigma,\psi^*\xi}(\psi^*\phi)$.

For a general Riemannian manifold $(M, g)$ and general potential $V$, the Euler-Lagrange equation for the action (38) (the critical point equation $\delta S = 0$) written in local coordinates on $M$ takes the form

$$m\left(\frac{d^2\phi^i(\tau)}{d\tau^2} + \Gamma^i_{jk}(\phi(\tau))\frac{d\phi^j(\tau)}{d\tau}\frac{d\phi^k(\tau)}{d\tau}\right) + g^{ij}(\phi(\tau))\partial_i V(\phi(\tau)) = 0 \tag{40}$$

where $\Gamma^i_{jk}$ are the Christoffel symbols.

**Example 1.13** (Scalar field)**.** Let $D \geq 1$ be any, fix $m \geq 0$ (the mass) and fix some polynomial function $V$ on $\mathbb{R}$ (interaction potential). Consider a cobordism $\Sigma$ equipped with Riemannian metric and set

$$\mathrm{Fields}_\Sigma = \mathrm{Map}(\Sigma, \mathbb{R}) \tag{41}$$

and

$$S_\Sigma = \int_\Sigma \frac{1}{2} d\phi \wedge *d\phi + \frac{m^2}{2}\phi^2 d\mathrm{vol} + V(\phi) d\mathrm{vol} \tag{42}$$

with $\phi\colon \Sigma \to \mathbb{R}$ the scalar field on $\Sigma$. The corresponding equation of motion $\delta S = 0$ is equivalent to the PDE

$$\Delta\phi + m^2\phi + V'(\phi) = 0 \tag{43}$$

with $\Delta$ the Laplacian on functions on $\Sigma$. Equation (43) is the Laplace equation if $m = 0$, $V = 0$, Helmholtz equation if $m \neq 0$, $V = 0$; for general $V$, it is a nonlinear PDE. Equation (43) can be considered with Dirichlet boundary conditions (35) where $\phi_{\mathrm{in,out}}$ are fixed functions on $\gamma_{\mathrm{in,out}}$.

1.5.1. *Path integral quantization.* Given a classical field theory on $\Sigma$, we want to define a corresponding QFT. Consider the following expression depending on $\phi_{\mathrm{in}}$, $\phi_{\mathrm{out}}$ – sections of $E$ over $\gamma_{\mathrm{in,out}}$:

$$K_\Sigma(\phi_{\mathrm{out}}, \phi_{\mathrm{in}})\colon = \int\limits_{\substack{\phi \in \mathrm{Fields}_\Sigma \text{ s.t.} \\ \phi|_{\gamma_{\mathrm{in}}} = \phi_{\mathrm{in}}, \\ \phi|_{\gamma_{\mathrm{out}}} = \phi_{\mathrm{out}}}} \mathcal{D}\phi\, e^{-S_\Sigma(\phi)} \tag{44}$$

The right hand side is a formal expression – the integral over the (infinite-dimensional) space of fields on $\Sigma$ subject to boundary conditions; the "measure" $\mathcal{D}\phi$ on fields is a formal symbol.

*Remark* 1.14. Depending on the context, there are different normalizations of the exponential in (44):

- In unitary (or "relativistic") quantum field theory on a Lorentzian space-time manifold, one writes the integrand of (44) as $e^{\frac{i}{\hbar}S(\phi)}$.
- In statistical mechanics one writes the integrand as $e^{-\beta E(\phi)}$ (the Gibbs measure on states of the statistical system), with $\phi$ a state of the system on $\Sigma$, $E(\phi)$ the energy of the state and $\beta = \frac{1}{T}$ the inverse temperature. Summarizing the comparison between QFT and statistical mechanics, we have the following.

| QFT | statistical mechanics |
|---|---|
| field $\phi$ on $\Sigma$<br>action functional $S$ | state $\phi$ of the system on $\Sigma$<br>energy functional $E$ |
| path integral $\int \mathcal{D}\phi\, e^{\frac{i}{\hbar}S(\phi)}$<br>at $\hbar \to 0$: fast oscillating integrand<br>stationary phase point = classical solution | sum over states $\int \mathcal{D}\phi\, e^{-\beta E(\phi)}$<br>at temperature$\to 0$: integrand is supported near<br>the state with minimal energy |

- In Euclidean field theory (which will be our setting for 2d CFT), on a Riemannian (as opposed to pseudo-Riemannian) spacetime manifold $\Sigma$, one considers the path integral with the integrand $e^{-\beta S(\phi)}$ where $\beta = \frac{1}{\hbar}$ and – unless we want to do perturbation theory yielding a power series in $\hbar$ – we can choose to set $\beta = \hbar = 1$.

One can transition a unitary QFT on a cobordisms of cylinder type $\gamma \times [0, t]$ to a Euclidean field theory on $\gamma \times [0, T_{\mathrm{Eucl}}]$ by "Wick rotation" – analytical continuation in $t$ to $t = -iT_{\mathrm{Eucl}}$.

*Remark* 1.15. There are ways to make mathematical sense of the path integral (a.k.a. functional integral or Feynman integral) (44), like e.g.

(a) perturbative approach – expansion in Feynman diagrams (replacing the path integral by its stationary phase or Laplace approximation), or
(b) lattice approach – replacing $\Sigma$ with a lattice with the field defined at the nodes – then (44) is replaced by a finite-dimensional integral; after that one needs to take the limit of the lattice spacing going to zero (one should think of this procedure as an analog of a Riemannian sum for an ordinary integral).

We define the space of states of the QFT on $\gamma$ as

$$\mathcal{H}_\gamma = \mathrm{Fun}_{\mathbb{C}}(\mathrm{Fields}_\gamma) \tag{45}$$

the space of complex-valued functions on $\mathrm{Fields}_\gamma$ (the space parametrizing the possible boundary conditions in (35)).[22]

For instance, in Example 1.12, one would set $\mathcal{H}_{\mathrm{pt}} = \mathrm{Fun}_{\mathbb{C}}(M)$. If we want to have unitarity, then we should be more specific about regularity of allowed functions and ask that it is of $L^2$ class:

$$\mathcal{H}_{\mathrm{pt}} = L^2(M)$$

– the standard Hilbert space in the quantum mechanical system consisting of a particle moving on $M$. By extension, it is tempting to write (45) as $\mathcal{H}_\gamma = L^2(\mathrm{Fields}_\gamma)$.

We define the partition function of the cobordism $\Sigma$ using the path integral (44) as follows: for $\Psi_{\mathrm{in}} \in \mathrm{Fun}_{\mathbb{C}}(\mathrm{Fields}_{\gamma_{\mathrm{in}}})$, we set

$$(Z_\Sigma \Psi_{\mathrm{in}})(\phi_{\mathrm{out}}) := \int_{\mathrm{Fields}_{\gamma_{\mathrm{in}}}} \mathcal{D}\phi_{\mathrm{in}}\, K_\Sigma(\phi_{\mathrm{out}}, \phi_{\mathrm{in}}) \Psi_{\mathrm{in}}(\phi_{\mathrm{in}}) \tag{46}$$

In other words, $Z_\Sigma$ is an integral operator, determined by the *integral kernel* $K_\Sigma$ defined by the path integral (44).

*Remark* 1.16 (Dirac's bra- and ket-notations). One can consider a basis in $\mathcal{H}_\gamma$ consisting of vectors $\{|\phi_0\rangle\}_{\phi_0 \in \mathrm{Fields}_\gamma}$. The vector $|\phi_0\rangle$ is understood as corresponding

[22] For more general boundary conditions of the type (36), instead of (45) we should write $\mathrm{Fun}_{\mathbb{C}}(B_\gamma)$. Occurrences on $\mathrm{Fields}_\gamma$ as integration space throughout this subsection (such as e.g. in (46)) should then also be swapped for $B_\gamma$.

to the delta-function on the space $\mathrm{Fields}_\gamma$ centered at $\phi = \phi_0$. In particular, a vector $\Psi \in \mathcal{H}_\gamma$ can be written tautologically as

$$\Psi = \int_{\mathrm{Fields}_\gamma} \mathcal{D}\phi_0\, \Psi(\phi_0)|\phi_0\rangle$$

Likewise, one has a dual basis in $\mathcal{H}^*$ consisting of covectors $\{\langle\phi_0|\}_{\phi_0 \in \mathrm{Fields}_\gamma}$. In terms of these notations, it is natural to denote the integral kernel (44) by

$$\langle\phi_{\mathrm{out}}|Z_\Sigma|\phi_{\mathrm{in}}\rangle :\,= K_\Sigma(\phi_{\mathrm{out}}, \phi_{\mathrm{in}}) \tag{47}$$

One also calls this expression the "matrix element" of $Z_\Sigma$ (corresponding to "row" $\phi_{\mathrm{out}}$ and "column" $\phi_{\mathrm{in}}$).

1.5.2. *Sewing as Fubini theorem for path integrals.* Let $\gamma_1 \xrightarrow{\Sigma'} \gamma_2$ and $\gamma_2 \xrightarrow{\Sigma''} \gamma_3$ be two cobordisms and $\gamma_1 \xrightarrow{\Sigma} \gamma_3$ the corresponding sewn cobordism. Then we have

$$\begin{aligned}
\langle\phi_3|Z_\Sigma|\phi_1\rangle &= \int_{\substack{\phi\,\in\,\mathrm{Fields}_\Sigma \text{ s.t.}\\ \phi|_{\gamma_1} = \phi_1,\\ \phi|_{\gamma_3} = \phi_3}} \mathcal{D}\phi\; e^{-S_\Sigma(\phi)} \\
&\underset{\mathrm{Fubini}}{=} \int_{\phi_2 \in \mathrm{Fields}_{\gamma_2}} \mathcal{D}\phi_2 \int_{\substack{\phi'\,\in\,\mathrm{Fields}_{\Sigma'} \text{ s.t.}\\ \phi|_{\gamma_1} = \phi_1,\\ \phi|_{\gamma_2} = \phi_2}} \mathcal{D}\phi' \int_{\substack{\phi''\,\in\,\mathrm{Fields}_{\Sigma''} \text{ s.t.}\\ \phi|_{\gamma_2} = \phi_2,\\ \phi|_{\gamma_3} = \phi_3}} \mathcal{D}\phi'' \; \underbrace{e^{-S_\Sigma(\phi)}}_{e^{-S_{\Sigma'}(\phi')} e^{-S_{\Sigma''}(\phi'')}} \\
&= \int_{\phi_2 \in \mathrm{Fields}_{\gamma_2}} \mathcal{D}\phi_2 \int_{\substack{\phi'\,\in\,\mathrm{Fields}_{\Sigma'} \text{ s.t.}\\ \phi|_{\gamma_1} = \phi_1,\\ \phi|_{\gamma_2} = \phi_2}} \mathcal{D}\phi'\; e^{-S_{\Sigma'}(\phi')} \int_{\substack{\phi''\,\in\,\mathrm{Fields}_{\Sigma''} \text{ s.t.}\\ \phi|_{\gamma_2} = \phi_2,\\ \phi|_{\gamma_3} = \phi_3}} \mathcal{D}\phi''\; e^{-S_{\Sigma''}(\phi'')} \\
&= \int_{\phi_2 \in \mathrm{Fields}_{\gamma_2}} \mathcal{D}\phi_2 \;\; \langle\phi_3|Z_{\Sigma''}|\phi_2\rangle\, \langle\phi_2|Z_{\Sigma''}|\phi_1\rangle
\end{aligned} \tag{48}$$

This is the convolution property of integral kernels equivalent to the relation

$$Z_\Sigma = Z_{\Sigma''} \circ Z_{\Sigma'}$$

between the corresponding integral operators, i.e. the sewing property.

The idea in (48) is to treat the integration over fields on $\Sigma$ in the following way:

(i) Fix the value $\phi_2$ of the field on the sewing interface $\gamma_2$.
(ii) Integrate over fields on the two sub-cobordisms $\Sigma', \Sigma''$ with $\phi_2$ becoming a boundary condition – this gives the matrix elements of partition functions for the sub-cobordisms.
(iii) Integrate out the field $\phi_2$ on the interface.

adjust the font size in the picture

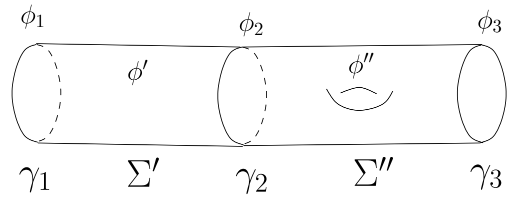



FIGURE 8. Sewing: integrating over the field $\phi$ everywhere is equivalent to integrating over $\phi', \phi''$ and then over $\phi_2$.

In particular, we think of the space of fields on $\Sigma$ (with boundary conditions on $\gamma_{1,3}$) as fibered over fields on $\gamma_2$, and we write this integral using "Fubini theorem for path integrals" as an intergral over the fiber followed by integral over the base.[23]

In the computation (48) we also used additivity of action (which is automatic from the local ansatz (33)): $S_\Sigma(\phi) = S_{\Sigma'}(\phi') + S_{\Sigma''}(\phi'')$ if $\phi', \phi''$ are the restrictions of the field $\phi$ on $\Sigma$ to $\Sigma'$, $\Sigma''$.

1.5.3. *Observables in path integral formalism.* Suppose we are given a classical field theory on a cobordism $\Sigma$ and also given $i\colon \mathbb{\Gamma} \hookrightarrow \Sigma$ a CW complex embedded into $\Sigma$ (with the image disjoint from the boundary). We define a classical observable $O_{\mathbb{\Gamma}}$ supported on $\mathbb{\Gamma}$ as some function on $\Gamma(\mathbb{\Gamma}, i^*\mathrm{Jet}_\infty E)$, i.e., a function of jets of fields on $\mathbb{\Gamma}$.

For instance, if $\mathbb{\Gamma} = \{x\}$ is a single point, then a classical observable at $x$ is just a function of the jet of the field at $x$, $O_x = f(\phi(x), \partial\phi(x), \ldots)$.

The expectation value of $O_{\mathbb{\Gamma}}$ is formally defined in the path integral formalism as

$$\langle O_{\mathbb{\Gamma}} \rangle = \int_{\phi \in \mathrm{Fields}_\Sigma} \mathcal{D}\phi\; e^{-S_\Sigma(\phi)} O_{\mathbb{\Gamma}}(\mathrm{Jet}_\infty(\phi)|_{\mathbb{\Gamma}}) \qquad \in \mathbb{C} \tag{49}$$

Here we assumed for simplicity that $\Sigma$ is closed.

If $\Sigma$ has a boundary, then we should include boundary conditions in the r.h.s., as in (44), thus obtaining the "matrix element," between states $|\phi_{\mathrm{in}}\rangle$ and $\langle\phi_{\mathrm{out}}|$, of the theory on $\Sigma$ enriched by the observable $O_{\mathbb{\Gamma}}$:

$$\langle \phi_{\mathrm{out}} | Z_{\Sigma, O_{\mathbb{\Gamma}}} | \phi_{\mathrm{in}} \rangle = \int_{\substack{\phi \in \mathrm{Fields}_\Sigma \text{ s.t.} \\ \phi|_{\gamma_{\mathrm{in}}} = \phi_{\mathrm{in}}, \\ \phi|_{\gamma_{\mathrm{out}}} = \phi_{\mathrm{out}}}} \mathcal{D}\phi\; e^{-S_\Sigma(\phi)} O_{\mathbb{\Gamma}}(\mathrm{Jet}_\infty(\phi)|_{\mathbb{\Gamma}}) \qquad \in \mathbb{C} \tag{50}$$

In quantization, a classical observable $O_{\mathbb{\Gamma}}$ is mapped to a quantum observable $\widehat{O}_{\mathbb{\Gamma}}$ such that the expectation value (21) of $\widehat{O}_{\mathbb{\Gamma}}$ defined withing Segal (quantum) language agrees with the path integral expression (49), (50). This can be arranged by defining $\widehat{O}_\Gamma$ to be the state on the boundary of a thickening $U_\epsilon(\mathbb{\Gamma})$ given by the

[23]This Fubini theorem is heuristically clear if the path integral measure is thought of as a continuum product of measures $d\phi(x)$ over points $x$ of $M$. However, when one defines path integrals mathematically, e.g., as perturbative integrals (via Feynman diagrams), this statement requires an independent proof. For special cases studied in detail, see e.g. [19] (quantum mechanics), [22] (2d scalar theory with polynomial potential), [7] (topological field theories of AKSZ type), [18] (2d Yang-Mills).

expression (50) where instead of the cobordism $\Sigma$ we take the "tube" $U_\epsilon(\mathbb{\Gamma})$ (seen as a cobordism from $\varnothing$ to $\partial U_\epsilon(\mathbb{\Gamma})$).

Lecture 4 8/31/2022

1.6. **Why care about CFT?.** Here we list some of the points of motivation, why one might be interested in 2d conformal field theory.

1.6.1. *CFT description of 2d Ising model.* This is the historical point of motivation, and it was the point of the seminal paper on CFT by Belavin-Polyakov-Zamolodchikov [6].

One considers the Ising model – a lattice model of statistical physics. On a graph $\Xi$, a state of the system is an assignment of spins $\pm 1$ (or "spin up/spin down") to vertices of $\Xi$. In particular, there are $2^{\#V(\Xi)}$ states in total where $V(\Xi)$ is the set of vertices of $\Xi$. The energy of a state is defined as

$$E(s) = -\sum_{\text{edges }(u,v)} s_u s_v - h \sum_{\text{vertices }v} s_v \tag{51}$$

where $h \in \mathbb{R}$ is a parameter ("external magnetic field"). Then one has the Gibbs probability measure on the set of states

$$\text{Probability}(s) = \frac{1}{Z(T,h)} e^{-\frac{1}{T}E(s)} \tag{52}$$

where $T > 0$ is the temperature and

$$Z(T,h) = \sum_{\text{states }s} e^{-\frac{1}{T}E(s)} \tag{53}$$

is the partition function (the normalization factor in the Gibbs measure (52), needed to normalize it to total mass 1).

Then one considers the continuum (or "thermodynamical") limit, taking $\Xi$ to be a very fine square lattice on a large square on $\mathbb{R}^2$ and sending the spacing of the lattice to zero (while appropriately rescaling the energy function (51)).

In the continuum limit, the system has a phase transition: the partition function $Z(T,h)$ and the $n$-point correlation functions of spins become real-analytic functions of $(T,h)$ on $\mathbb{R}_{>0} \times \mathbb{R}$ except on the interval $(0, T_{\text{crit}}]$ with some positive critical temperature $T_{\text{crit}}$. The partition function and correlation functions are discontinuous (have a finite jump) across the interval $(0, T_{\text{crit}})$, when going from small negative $h$ to small positive $h$. Points $(0 < T < T_{\text{crit}}, h = 0)$ are points of first-order phase transition of the system and $(T = T_{\text{crit}}, h = 0)$ is the point of second order phase transition.

From now on, set $h = 0$. If $T > T_{\text{crit}}$, the two-point correlation function behaves as

$$\langle s(x)s(y)\rangle \sim e^{-\frac{||x-y||}{r_{\text{corr}}}} \tag{54}$$

where $r_{\text{corr}}$ is the "correlation radius," depending on $T$. In the limit $T \to T_{\text{crit}}$, the correlation radius goes to $+\infty$ and the system loses the "characteristic scale" – becomes scaling invariant. In particular, the two-point function (54) at $T = T_{\text{crit}}$ becomes a power law

$$\langle s(x)s(y)\rangle \sim \frac{1}{||x-y||^{\frac{1}{4}}} \tag{55}$$

The power $\frac{1}{4}$ here is a result from the explicit solution of 2d Ising model (at any $T$) by Onsager [31].

Thus, at the point $(T_{\mathrm{crit}}, h=0)$ of second-order phase transition, the system becomes scaling invariant. Put another way, its symmetry gets enhanced from Euclidean motions (translations+rotations) to include scaling. At this point it is natural to conjecture the system on $\mathbb{R}^2$, at the point of second-order phase transition, can be described by some model of conformal field theory (which would also mean that the symmetry is further enhanced from rotations+translations+scaling to all conformal transformations). This was proven – and the corresponding CFT was identified as the *free Weyl fermion* – in [6].

It turns out that a much wider class of statistical systems exhibiting phase transitions at the point of phase transition can be described by a CFT, which eventually leads to explanation of the interesting rational exponents ("critical exponents") one encounters in these systems – such as the power $\frac{1}{4}$ in (54).[24]

1.6.2. *Bosonic string theory.* Classically, bosonic string theory can be though of as a classical field theory of maps from a surface $\Sigma$ ("worldsheet") to the target $\mathbb{R}^N$ ("spacetime" in string theory terminology), with action

$$S(\Phi; b, c, \bar{b}, \bar{c}) = \int_\Sigma \sum_{i=1}^{N} \frac{1}{2} d\Phi_i \wedge * d\Phi_i + b\bar{\partial}c + \bar{b}\partial\bar{c}. \tag{56}$$

Here $\Phi\colon \Sigma \to \mathbb{R}^N$ is the bosonic field describing the string in $\mathbb{R}^N$, $\Phi_i$ are components corresponding to coordinates on $\mathbb{R}^N$, so that each $\Phi_i$ can be seen as a scalar field on $\Sigma$. The last two terms in (56) (the "reparametrization ghost system") are auxiliary anticommuting fields ("Faddeev-Popov ghosts") that appear in the action through Faddeev-Popov mechanism, because one wants to consider the path integral over $\mathrm{Map}(\Sigma, \mathbb{R}^N)/\mathrm{Diff}(\Sigma)$ – they appear in essence from homological resolution of this quotient. The fields $c, \bar{c}$ are sections of $T^{1,0}\Sigma$, $T^{0,1}\Sigma$ – holomorphic/antiholomorphic tangent bundle; fields $b, \bar{b}$ are quadratic differentials – sections of $((T^{(1,0)})^*\Sigma)^{\otimes 2}$, $((T^{(0,1)})^*\Sigma)^{\otimes 2}$, respectively.

Upon quantization, (56) becomes a particular CFT on $\Sigma$ – the "sum" of several mutually non-interacting theories – $N$ free massless scalar fields and the ghost system. The central charge of this CFT (measuring the "strength" of projectivity effect/conformal anomaly, see Section 1.4.2) turns out to be

$$c = N - 26 \tag{57}$$

– each free scalar contributes 1 to the central charge and the ghost system contributes $-26$. In particular, the central charge (and thus the conformal anomaly) vanishes iff $N = 26$. Which is the reason why dimension 26 of the target is distinguished in bosonic string theory.

1.6.3. *Invariants of 3-manifolds.* There are interesting connections between 3d topological quantum field theories and 2d conformal field theories on the boundary of a 3-manifold.

Notably, there is a relation between 3d Chern-Simons theory (which is topological) and 2d Wess-Zumino-Witten theory (which is a CFT). This relation was very fruitfully exploited in [40] to construct invariants of knots and 3-manifolds.

One relation is that Chern-Simons correlator of a tangle in a 3-ball can be interpreted as a correlator of point observables in WZW theory on the boundary

[24] Ultimately, $\frac{1}{4}$ comes from the fact that Ising spin field can be identified with a primary field of conformal weight $(\frac{1}{16}, \frac{1}{16})$ in the free fermion field.

2-sphere. This fact was explained and used in [40] to explain why the correlators of Wilson loop observables in 3d theory have to satisfy certain skein relation (which is ultimately a move performed on the portion of a knot contained in a small ball).

Put differently, the relation between Chern-Simons theory on a 3-manifold $M$ and 2d WZW on the boundary $\Sigma = \partial M$ is that the space of states that Chern-Simons assigns to $\Sigma$ is the "space of conformal blocks" (holomorphic building blocks of correlators) that WZW assigns to $\Sigma$, see e.g. [15].

1.6.4. *A zoo of computable QFTs.* Part of motivation to study CFTs is that they give examples quantum field theories with explicit and nontrivial answers.

For instance in a typical CFT situation,

- two-point functions are often given by power laws with interesting rational exponents,
- four-point functions can be expressed in terms of the hypergeometric function,
- genus 1 partition function can be expressed in terms of such objects as Jacobi theta functions and Dedekind eta function.

The zoo of well-known examples of CFTs includes among others:

- Free theories:
  - free massless scalar field (or "free boson"),
  - free massless scalar with values in $S^1$,
  - free fermion,
  - $bc$-system (and a very similar $\beta\gamma$-system).
- Minimal models $M(p, q)$ of CFT.
- Wess-Zumino-Witten model.

1.6.5. *Motivation from representation theory.*

Representations of loop groups/Lie algebras. CFT is naturally linked to representation theory of loop groups and loop Lie algebras (or rather their central extensions). E.g., the space of states $\mathcal{H}_{S^1}$ always carries a representation of the Virasoro algebra. In the case of WZW models, $\mathcal{H}_{S^1}$ also carries a representation of a Kac-Moody algebra $\widehat{\mathfrak{g}}$ (which gives in a sense a "refinement"[25] of the Virasoro representation).

Representations of the mapping class group. Additionally, a part of the data of CFT (the space of conformal blocks) naturally carries a representation of the mapping class group of the surface.

1.6.6. *Motivation from topology of $\mathcal{M}_{g,n}$ and enumerative geometry.* In topological conformal field theories (such as Witten's A-model), special correlators define closed differential forms on the moduli space of algebraic curves $\overline{\mathcal{M}}_{g,n}$ (with Deligne-Mumford compactification) yielding interesting elements in de Rham cohomology of the moduli space. Periods of these forms over compactification cycles satisfy certain quadratic relations (equivalently, the corresponding generating functions satisfy the so-called Witten-Dijkgraaf-Verlinde-Verlinde equation).

In the A-model, such periods are the Gromov-Witten invariants – counts of holomorphic curves in the target Kähler manifold $X$ intersecting a given collection of cycles.

[25] In the sense that Virasoro generators act as quadratic expressions in Kac-Moody generators, via the so-called Sugawara construction.

1.7. **CFT as a system of correlators.** CFT is often studied in a simplified setting (as compared to Segal's picture): instead of surfaces with boundary, one considers surfaces with punctures (marked points).

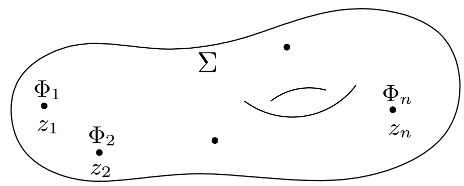



FIGURE 9. Surface with punctures decorated by fields.

One can think of punctures as "infinitesimally small circles." Instead of partition function on surfaces with boundary, one studies $n$-point correlation functions

$$\langle \Phi_1(z_1)\cdots\Phi_n(z_n)\rangle \quad \in \mathbb{C} \tag{58}$$

depending on a configuration of $n$ distinct ordered points on the Riemann surface $\Sigma$ and on a choice of vectors $\Phi_1,\ldots,\Phi_n$ in the vector space $V$ (the space of states $\mathcal{H}_{S^1}$ in Segal's language). There are different possible names for elements of $V$:

- Fields (or "composite fields") at a point $z$.[26]
- Point observables.
- Operators.

In the path integral language, (58) corresponds to the expression

$$\int \mathcal{D}\phi\ e^{-S(\phi)}\Phi_1(z_1)\cdots\Phi_n(z_n) \tag{59}$$

where expressions $\Phi_i$ under the path integral are point classical observables – functions of the jet of the classical field $\phi$ at $z_i$ (in the notations we are blurring the distinction between classical observables and corresponding quantum observables).

Subtlety: to make sense of a correlator (58) as a number, one needs to fix a complex coordinate chart around each point $z_1,\ldots,z_k$.[27]

For particularly nice elements of $V$ – so-called "primary" fields (see below), one doesn't need the full data of coordinate charts – it is sufficient to have a trivialization of tangent spaces $T_{z_i}\Sigma$, thus the correlators of primary fields can be regarded as a section

$$\langle \Phi_1\cdots\Phi_n\rangle \in \Gamma(\mathrm{Conf}_n(\Sigma),\mathcal{L}) \tag{60}$$

over the open configuration space $\mathrm{Conf}_n(\Sigma)=\{(z_1,\ldots,z_n)\in\mathbb{C}^n | z_i\neq z_j \text{ if } i\neq j\}$ of $n$ ordered points on $\Sigma$, of a certain complex line bundle $\mathcal{L}$ depending of the fields $\Phi_i$. In (60) we allow points $z_1,\ldots,z_n$ to move around on a fixed Riemann surface $\Sigma$ (i.e. the complex structure is fixed).

We can also allow the complex structure to change (then movement of points is absorbed into changes of complex structure). Then the correlator of primary

[26] Not to be confused with the fields of the Lagrangian formulation of the underlying classical field theory.

[27] Or at least one needs to fix an $\infty$-jet of complex coordinate charts centered at each $z_i$ – a "formal" complex coordinate chart at $z_i$.

fields becomes a section of certain complex line bundle $\widetilde{\mathcal{L}}$ over the moduli space of complex structures on $\Sigma$ with $n$ punctures:

$$\langle \Phi_1 \cdots \Phi_n \rangle \in \Gamma(\mathcal{M}_{\Sigma,n}, \widetilde{\mathcal{L}}) \tag{61}$$

*Remark* 1.17. For general (possibly non-primary) $\Phi_i$, one needs to replace $\mathcal{M}_{\Sigma,n}$ in (61) with an enhanced version $\mathcal{M}^{\mathrm{coor}}_{\Sigma,n}$ of the moduli space where each puncture carries a formal coordinate system. Put another way, when defining $\mathcal{M}^{\mathrm{coor}}_{\Sigma,n}$ as complex surfaces modulo diffeomorphisms, one should only quotient by diffeomorphisms which have the $\infty$-jet of identity at each $z_i$. In this setup the line bundle over $\mathcal{M}^{\mathrm{coor}}_{\Sigma,n}$ is trivial and the general $n$-point correlator is a function on $\mathcal{M}^{\mathrm{coor}}_{\Sigma,n}$ with values in $\mathrm{Hom}(V^{\otimes n}, \mathbb{C})$, invariant under formal conformal vector fields at the punctures $z_i$ (acting both on $V$ at $z_i$ and on the formal coordinate system):

$$\langle \cdots \rangle \in C^\infty(\mathcal{M}^{\mathrm{coor}}_{\Sigma,n}, \mathrm{Hom}(V^{\otimes n}, \mathbb{C}))^{\text{formal c.v.f. at punctures}}. \tag{62}$$

1.7.1. *The action of conformal vector fields on $V$.* The space $V$ comes equipped with a projective representation of the Lie algebra $\mathcal{A}^{\mathrm{loc}}$ of conformal vector fields on $\mathbb{C}^*$ (real parts of meromorphic vector fields with only pole at zero allowed),

$$\rho\colon \mathcal{A}^{\mathrm{loc}} \to \mathrm{End}(V) \tag{63}$$

This representation can be thought of as the complexified (in a certain sense) infinitesimal version of the representation (24) in Segal's picture, see Section 1.7.2 below.[28]

In the common nomenclature, the standard generators of $\mathcal{A}^{\mathrm{loc}}_{\mathbb{C}}$ – the complexified Lie algebra of conformal vector fields on $\mathbb{C}^*$ – are denoted

$$l_n\colon = -z^{n+1}\frac{\partial}{\partial z}, \quad \bar{l}_n\colon = -\bar{z}^{n+1}\frac{\partial}{\partial \bar{z}}, \qquad n \in \mathbb{Z} \tag{64}$$

The corresponding operators acting on $V$ are denoted

$$L_n\colon = \rho(l_n), \quad \bar{L}_n\colon = \rho(\bar{l}_n) \tag{65}$$

Did not mention in the lecture. Should mention later on.

1.7.2. *The "double complexification".* The Lie algebra $\mathcal{A}^{\mathrm{loc}}_{\mathbb{C}} = \mathcal{A}^{\mathrm{loc}} \otimes \mathbb{C}$ conveniently splits into holomorphic and antiholomorphic copies of complex Witt[29] algebra and its central extension splits similarly into two copies of complex Virasoro algebras. The Lie algebra $\mathcal{A}^{\mathrm{loc}}_{\mathbb{C}}$ can be seen, in a sense, as "double complexification" of the Lie algebra of diffeomorphisms of a circle:

$$\begin{tikzcd} \mathfrak{X}(S^1) \arrow[r, "\text{complexification}"] & \mathcal{A}^{\mathrm{loc}} \arrow[r, "\text{complexification}", "\otimes\mathbb{C}"'] & \underset{\simeq \mathrm{Witt}\oplus\overline{\mathrm{Witt}}}{\mathcal{A}^{\mathrm{loc}}_{\mathbb{C}}} \\ \mathrm{Diff}(S^1) \arrow[u] \arrow[r, "\text{``complexification''}"'] & \mathrm{Ann} \arrow[u] & \end{tikzcd} \tag{66}$$

[28]Remark: representation (63) contains strictly more information (morally, "twice more") than the action of diffeomorphisms (24). For instance, the difference of conformal weights $h - \bar{h}$ of a field (see Section 1.7.3 below) corresponds to the action of rotation around the origin and is a part of the data of (24), while $h + \bar{h}$ corresponds to the action of dilation, which infinitesimally is a vector field on $S^1$ not tangential to $S^1$, and it is not a part of the data (24) but is a part of the data (63).

[29]Witt algebra is the Lie algebra of meromorphic vector fields on $\mathbb{C}$ with only pole at 0 allowed, see Section 2.5.1. In terms of (64), it is $\mathrm{Span}_{\mathbb{C}}(\{l_n\}_{n\in\mathbb{Z}})$.

Here $\mathfrak{X}(S^1)$ is the Lie algebra of real vector fields on a circle, Ann is the Segal's semigroup of annuli [35] – the full subcategory of Segal's cobordism category consisting of cobordisms $S^1 \xrightarrow{\Sigma} S^1$ (with conformal structure on $\Sigma$ and parametrization of boundary circles). The vertical arrows are the transitions from a Lie group or semigroup to its Lie algebra. The first complexification in the top row of (66) allows vector fields on $S^1$ that are not necessarily tangential to $S^1$ and then extends them to real conformal vector fields (which are special sections of the *non-complexified* tangent bundle $T\mathbb{C}^*$ of $\mathbb{C}^*$ seen as a smooth 2-manifold) on $\mathbb{C}^*$. The second complexification allows complex-valued conformal vector fields on $\mathbb{C}^*$ – special sections of the complexified tangent bundle $T_\mathbb{C}\mathbb{C}^*$. Explicitly, one has

(67)
$$\begin{aligned}\mathfrak{X}(S^1) &= \mathrm{Span}_\mathbb{R}(\{\cos n\theta\,\partial_\theta\}_{n\geq 0}, \{\sin n\theta\,\partial_\theta\}_{n\geq 1}) \\ &= \mathrm{Span}_\mathbb{R}\left(\left\{-\frac{i}{2}(l_n + l_{-n} - \bar{l}_n - \bar{l}_{-n})\right\}_{n\geq 0}, \left\{-\frac{1}{2}(l_n - l_{-n} + \bar{l}_n - \bar{l}_{-n})\right\}_{n\geq 1}\right), \\ \mathcal{A}^{\mathrm{loc}} &= \mathrm{Span}_\mathbb{R}\left(\left\{\frac{l_n+\bar{l}_n}{2}\right\}_{n\in\mathbb{Z}}, \left\{\frac{l_n-\bar{l}_n}{2i}\right\}_{n\in\mathbb{Z}}\right), \\ \mathcal{A}^{\mathrm{loc}}_\mathbb{C} &= \mathrm{Span}_\mathbb{C}\Big(\{l_n, \bar{l}_n\}_{n\in\mathbb{Z}}\Big).\end{aligned}$$

The bottom horizontal arrow in (66) is explained in [35].

1.7.3. *Grading on $V$ by conformal weights.* The complexified Lie algebra $\mathcal{A}^{\mathrm{loc}}_\mathbb{C}$ is naturally graded by elements of $\mathbb{Z}\oplus\mathbb{Z}$. In particular, the meromorphic vector field $z^{n+1}\frac{\partial}{\partial z}$ on $\mathbb{C}^*$ has degree $(n,0)$ and the antimeromorphic vector field $\bar{z}^{n+1}\frac{\partial}{\partial \bar{z}}$ has degree $(0,n)$. Accordingly, $V$ carries a grading by "conformal weight" $(h,\bar{h}) \in \mathbb{R}\oplus\mathbb{R}$. A field $\Phi\in V$ is said to have conformal weight $(h,\bar{h})$ if

$$\rho\left(-z\frac{\partial}{\partial z}\right)\circ\Phi = h\Phi, \quad \rho\left(-\bar{z}\frac{\partial}{\partial \bar{z}}\right)\circ\Phi = \bar{h}\Phi. \tag{68}$$

The grading on the Lie algebra is compatible with the grading on the module: acting by an element of $\mathcal{A}^{\mathrm{loc}}_\mathbb{C}$ of degree $(n,\bar{n})$ shifts the conformal weight of a vector in $V$ as $(h,\bar{h}) \to (h-n,\bar{h}-\bar{n})$.[30] One can split $V$ into graded components:

$$V = \bigoplus_{(h,\bar{h})\in\Lambda} V^{(h,\bar{h})}.$$

Here $\Lambda\subset\mathbb{R}\oplus\mathbb{R}$ is the set of admissible conformal weights (dependent on a particular CFT model); $\Lambda$ is necessarily a $\mathbb{Z}\oplus\mathbb{Z}$-module.

*Remark* 1.18. The condition that the representation $\rho$ of $\mathcal{A}^{\mathrm{loc}}_\mathbb{C}$ comes from a representation of the group $\mathrm{Diff}(S^1)$ implies in particular that rotation by the angle $2\pi$ should act on a field as identity (or, in the notations (65), one should have $e^{2\pi i(L_0-\bar{L}_0)} = \mathrm{id}$). That implies

$$h-\bar{h}\in\mathbb{Z} \tag{69}$$

for any element of $V$.[31]

[30] We emphasize that in $\bar{h}$, $\bar{n}$, the bar does not mean complex conjugation.

[31] One can consider models where (69) is violated, but in this case correlators are multivalued. In other words, correlators are functions (or sections of a line bundle) not on the configuration space of $n$-points but rather on its covering space.

1.7.4. *Conformal Ward identity.* Conformal Ward identity is the following symmetry property of correlators. Fix a Riemann surface $\Sigma$ with punctures $z_1, \ldots, z_n$ and fix fields $\Phi_1, \ldots, \Phi_n \in V$. Let $v$ be a conformal vector field on $\Sigma$ with singularities allowed at $\{z_i\}$ – the real part of a meromorphic vector field with poles allowed at $z_1, \ldots, z_n$ (we will denote the Lie algebra of such vector fields $\mathcal{A}_{\Sigma,\{z_i\}}$). Then we have the Ward identity

$$\underbrace{\sum_{i=1}^{n} \langle \Phi_1(z_1) \cdots \rho(\mathrm{Laurent}_{z_i}(v)) \circ \Phi_i(z_i) \cdots \Phi_n(z_n) \rangle}_{\mathcal{L}_v \langle \Phi_1(z_1) \cdots \Phi_n(z_n) \rangle} = 0. \tag{70}$$

Here the left hand side can be thought of as the "Lie derivative of the correlator along $v$;"

$$\mathrm{Laurent}_{z_i} : \mathcal{A}_{\Sigma,\{z_i\}} \to \mathcal{A}^{\mathrm{loc}}$$

is the Laurent expansion of a (real part of the) meromorphic vector field at the point $z_i$.

One can think of (70) as a version of naturality (5) in Segal's setting.[32]

1.7.5. *The "$L_{-1}$ axiom".* Representation (63) is supposed to satisfy the following natural property: Did not explain this in the class. signs?

$$\langle \Phi_1(z_1) \cdots \rho\left(\frac{\partial}{\partial w}\right) \circ \Phi_i(z_i) \cdots \Phi_n(z_n) \rangle = \frac{\partial}{\partial z_i} \langle \Phi_1(z_1) \cdots \Phi_i(z_i) \cdots \Phi_n(z_n) \rangle \tag{71}$$

$$\langle \Phi_1(z_1) \cdots \rho\left(\frac{\partial}{\partial \bar{w}}\right) \circ \Phi_i(z_i) \cdots \Phi_n(z_n) \rangle = \frac{\partial}{\partial \bar{z}_i} \langle \Phi_1(z_1) \cdots \Phi_i(z_i) \cdots \Phi_n(z_n) \rangle \tag{72}$$

for any surface with any collection of punctures and fields; $w$ is a local complex coordinate centered at $z_i$.

Thus, (71) says that acting by $L_{-1}$ on a field under the correlator is tantamount to taking the holomorphic derivative in the position of the corresponding puncture (up to a sign). Similarly, (72) says that acting by $\bar{L}_{-1}$ is tantamount to taking the antiholomorphic derivative in the position.

1.7.6. *Some special fields.*

Identity field. The identity field $\mathbb{1} \in V^{(0,0)}$ corresponds in Segal's picture to the vacuum vector $|\mathrm{vac}\rangle \in \mathcal{H}_{S^1}$ – the partition function of a disk. The field $\mathbb{1}$ is characterized by the property that for any fields $\Phi_1, \ldots, \Phi_n$ and any points $z_0, z_1, \ldots, z_n$ on $\Sigma$, one has

$$\langle \mathbb{1}(z_0)\Phi_1(z_1) \cdots \Phi_n(z_n) \rangle = \langle \Phi_1(z_1) \cdots \Phi_n(z_n) \rangle \tag{73}$$

Put another way, putting the field $\mathbb{1}$ at a puncture effectively forgets that puncture.

Stress-energy tensor. The stress-energy tensor $T \in V^{(2,0)} \oplus V^{(0,2)}$ is defined as

$$T := \rho\left(\mathrm{Re}\left(\frac{-2}{z}\partial_z\right)\right) \circ \mathbb{1} \tag{74}$$

Or in terms of standard notations (65) introduced above,

$$T = (L_{-2} + \bar{L}_{-2}) \circ \mathbb{1} \tag{75}$$

Lecture 5 9/2/2022

[32]In this version, one passes (a) from finite boundaries to infinitesimal ones (punctures), (b) from Lie group action to the associated Lie algebra action, (c) one complexifies the Lie algebra, which corresponds to allowing vector fields not tangential to the boundary.

Primary fields. A field $\Phi \in V^{(h,\bar{h})}$ is said to be primary if it is a highest weight vector under the action of $\mathcal{A}^{\mathrm{loc}}_{\mathbb{C}}$, i.e., if

$$L_n\Phi = 0,\ \bar{L}_n\Phi = 0 \quad \text{for any } n > 0. \tag{76}$$

Equivalently, field $\Phi$ is primary if it is annihilated by conformal vector fields which vanish to second order at the origin (the point of insertion of $\Phi$).

It is natural to assign to a primary field of conformal weight $(h,\bar{h})$ a complex line bundle

$$\mathcal{L}^{h,\bar{h}} = K^{\otimes h} \otimes \bar{K}^{\otimes \bar{h}} \tag{77}$$

over $\Sigma$ where

$$K = (T^{1,0})^*\Sigma, \quad \bar{K} = (T^{0,1})^*\Sigma$$

are the holomorphic and antiholomorphic cotangent bundles of $\Sigma$, respectively.

Then the correlator (60) of primary fields $\Phi_i \in V^{h_i,\bar{h}_i}$ is a section over $\mathrm{Conf}_n(\Sigma)$ of the line bundle

$$\mathcal{L} = \iota^* \boxtimes_{i=1}^n \mathcal{L}^{h_i,\bar{h}_i} \tag{78}$$

where $\iota\colon \mathrm{Conf}_n(\Sigma) \to \Sigma^{\times n}$ is the natural inclusion.

From the standpoint of the moduli space of complex structures, the correlator of primary fields (61) is a section of the line bundle

$$\widetilde{\mathcal{L}} = \left(\bigotimes_{i=1}^n \mathcal{L}_i^{h_i,\bar{h}_i}\right) \otimes \mathcal{L}_{\mathrm{anomaly}} \tag{79}$$

over the moduli space $\mathcal{M}_{\Sigma,n}$. Here $\mathcal{L}_i^{h_i,\bar{h}_i}$ is the line bundle (77) assoicated to $i$-th puncture on $\Sigma$;

$$\mathcal{L}_{\mathrm{anomaly}} = (\mathrm{Det}\,\bar{\partial})^{\otimes c} \otimes (\mathrm{Det}\,\partial)^{\otimes \bar{c}} \tag{80}$$

is the effect of conformal anomaly, with $(c,\bar{c})$ the central charge (see Section 1.4.2 and footnote 19).

1.7.7. *Operator product expansions.* When studying CFT as a system of correlators, instead of sewing along boundaries, one studies OPEs ("operator product expansions") governing the singularities of correlators of fields (60) as the point of insertion of one field approaches another, $z_i \to z_j$.

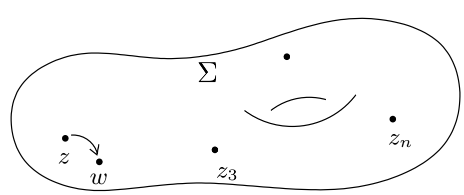



Figure 10. One puncture approaching another.

An OPE is an expression of the form

$$\Phi_1(z)\Phi_2(w) \underset{z\to w}{\sim} \sum_{\widetilde{\Phi}} f^{\widetilde{\Phi}}_{\Phi_1\Phi_2}(z,w)\widetilde{\Phi}(w) + \mathrm{reg} \tag{81}$$

Here on the right hand side:

- The sum is over a basis $\{\widetilde{\Phi}\}$ in $V$.
- Coefficient functions $f^{\widetilde{\Phi}}_{\Phi_1\Phi_2}(z,w)$ are some real-analytic functions on a neighborhood of $\mathrm{Diag} \subset \Sigma \times \Sigma$, singular on $\Sigma$.
- reg stands for terms that are continuous (in special cases, even holomorphic) on the diagonal $z = w$.

In (81) we could have chosen instead to express the operator product in terms of fields $\widetilde{\Phi}$ at $z$ rather than $w$ (or even, say, at some point between $z$ and $w$); this choice affects the coefficients in the OPE.

The expression (81) is understood as a substitution that one can perform under the correlator of $\Phi_1(z)$, $\Phi_2(w)$, and any collection of other fields away from $z$ and $w$, in the asymptotics $z \to w$:

(82)
$$\langle \Phi_1(z)\Phi_2(w) \underbrace{\Phi_3(z_3)\cdots\Phi_n(z_n)}_{\text{away from } z,w} \rangle \underset{z\to w}{\sim} \sum_{\widetilde{\Phi}} f^{\widetilde{\Phi}}_{\Phi_1\Phi_2}(z,w) \langle \widetilde{\Phi}(w)\Phi_3(z_3)\cdots\Phi_n(z_n)\rangle + \mathrm{reg}$$

Thus, singularities of $n$-point correlators are governed by $(n-1)$-point correlators.

Note: the OPE (81) does not depend on the collection of "test fields" $\Phi_3, \ldots, \Phi_n$ in the correlator (82).

Idea. One wants to recover $n$-point correlators functions from $(n-1)$-point correlators using the OPEs (82), ultimately reducing everything to 3-point correlators. The idea is similar to recovering a meromorphic function from knowing the principal part of its Laurent expansion at each pole.

The idea that all correlators can be derived from 3-point correlators is close to the idea in Segal's approach, that one can cut any surface into "pairs of pants" (spheres with three holes). Edit/remove?

Another form of that thought: an $n$-point correlator on a plane can be seen as a sewing of a collection of annuli with one hole.

*Remark* 1.19. The asymptotic of two punctures on $\Sigma$ approaching one another from the standpoint of the moduli space of curves $\mathcal{M}_{\Sigma,n}$ corresponds to approaching a nodal curve, where punctures $z, w$ are in one component, connected by a "neck" to the other component, where the remaining punctures $z_3, \ldots, z_n$ are (where we put the "test fields").

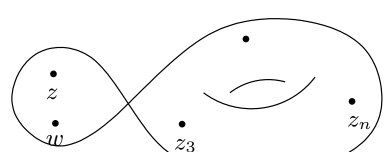



FIGURE 11. Nodal curve.

## 2. Bits of conformal geometry

2.1. **Conformal maps.** Reference: [34].

Let $(M, g)$ be a Riemannian (or pseudo-Riemannian) manifold.

**Definition 2.1.** A *Weyl transformation* is a change of metric on a (pseudo-)Riemannian manifold $(M, g) \to (M, g' = \Omega \cdot g)$ consisting in multiplying the metric by an everywhere positive function $\Omega \in C^\infty_{>0}(M)$ (the "Weyl factor").

Two metrics on $M$ differing by a Weyl transformation are said to be *conformally equivalent*. A metric on $M$ modulo conformal equivalence is called a *conformal structure* on $M$.

**Definition 2.2.** A smooth map of (pseudo-)Riemannian manifolds $\phi\colon (M, g) \to (M', g')$ is a *conformal map* if

$$\phi^* g' = \Omega \cdot g$$

For some positive function $\Omega \in C^\infty_{>0}(M)$ (the *conformal factor* associated to $\phi$).

One says that two (pseudo-)Riemannian manifolds $(M, g)$ and $(M', g')$ are conformally equivalent if there exists a conformal diffeomorphism

$$\phi\colon (M, g) \to (M', g') \tag{83}$$

Some immediate properties of conformal maps:

(a) If $\phi_1\colon (M, g) \to (M', g')$ and $\phi_2\colon (M', g') \to (M'', g'')$ are two conformal maps with conformal factors $\Omega', \Omega''$, then $\phi_2 \circ \phi_1\colon (M, g) \to (M'', g'')$ is a conformal map with $\Omega = \phi_1^* \Omega_2 \cdot \Omega_1$.

(b) If $\phi\colon (M, g) \to (M', g')$ is a conformal diffeomorphism with conformal factor $\Omega$, then $\phi^{-1}\colon (M', g') \to (M, g)$ is also a conformal diffeomorphism with conformal factor $(\phi^{-1})^* \Omega^{-1}$.

(c) If $\phi\colon (M, g) \to (M', g')$ is a conformal map with conformal factor $\Omega$ and $\Lambda \in C^\infty_{>0}(M)$, $\Lambda' \in C^\infty_{>0}(M')$ are positive functions, then $\phi\colon (M, \Lambda \cdot g) \to (M', \Lambda' \cdot g')$ is also a conformal map, with conformal factor $\frac{\phi^* \Lambda'}{\Lambda} \cdot \Omega$.

In particular, the notion of a conformal map between manifolds equipped with just conformal structure (rather than metric) is well-defined, but the conformal factor of such a map is not well-defined.

**Definition 2.3.** Conformal automorphisms $\phi\colon (M, g) \to (M, g)$ form a group – the *conformal group* $\mathrm{Conf}(M, g)$. By (c) above, this group depends only on the conformal class of $g$.

### 2.2. **Examples of conformal maps.**

**Example 2.4.** Isometries of $(M, g)$ form a subgroup of $\mathrm{Conf}(M, g)$ (characterized by the property $\Omega = 1$).

**Example 2.5.** Translations and $O(n)$-rotations of Euclidean space $\mathbb{R}^n$ (with the standard metric $g = (dx^1)^2 + \cdots + (dx^n)^2$) are conformal automorphisms:

$$ISO(n) = O(n) \ltimes \mathbb{R}^n \subset \mathrm{Conf}(\mathbb{R}^n)$$

(This is a special case of Example 2.4.)

More generally, one can consider the space $\mathbb{R}^{p,q}$ with metric $g = (dx^1)^2 + \cdots + (dx^p)^2 - (dx^{p+1})^2 - \cdots - (dx^{p+q})^2$. Then one has translations and $O(p, q)$-rotations as isometries (and in particular, conformal automorphisms) of $\mathbb{R}^{p,q}$.

**Example 2.6** (Dilations)**.** Fix a nonzero real number $\lambda$. The dilation (or scaling) map

$$\begin{array}{ccc} \mathbb{R}^n & \to & \mathbb{R}^n \\ \vec{x} & \mapsto & \lambda \vec{x} \end{array} \tag{84}$$

is a conformal map with $\Omega = \lambda^2$. (One can replace $\mathbb{R}^n$ with $\mathbb{R}^{p,q}$ in this example.)

Lecture 6, 9/5/2022

**Example 2.7** (Stereographic projection)**.** Let

$$S^n = \{(x^0,\ldots,x^n)\in\mathbb{R}^n \mid \sum_{i=0}^n (x^i)^2 = 1\}$$

be the unit sphere in $\mathbb{R}^{n+1}$ with $N=(1,0,\ldots,0)$ the North pole. Consider the stereographic projection

$$\begin{array}{cccc} \phi\colon & S^n\backslash\{N\} & \to & \mathbb{R}^n \\ & (x^0,x^1,\ldots,x^n) & \mapsto & \frac{1}{1-x^0}(x^1,\ldots,x^n) \end{array} \tag{85}$$

The map $\phi$ is a conformal diffeomorphism (w.r.t. the round metric on $S^n$ – induced from the standard flat metric on the ambient $\mathbb{R}^{n+1}$ – and w.r.t. the standard metric on $\mathbb{R}^n$). The conformal factor is $\Omega = \frac{1}{(1-x^0)^2}$.

**Example 2.8.** Any diffeomorphism $\phi\colon \mathbb{R}\to\mathbb{R}$ is a a conformal map (w.r.t. the metric $g=(dx)^2$ on the source and the target), with $\Omega = \left(\frac{d\phi}{dx}\right)^2$.

**Example 2.9** (Inversion)**.** The map

$$\begin{array}{cccc} \phi\colon & \mathbb{R}^n\backslash\{0\} & \to & \mathbb{R}^n\backslash\{0\} \\ & \vec{x} & \mapsto & \frac{\vec{x}}{||\vec{x}||^2} \end{array} \tag{86}$$

is an orientation-reversing diffeomorphism. It is a conformal map (w.r.t. the metric induced from the standard one on $\mathbb{R}^n$), with $\Omega = \frac{1}{||\vec{x}||^4}$.

The following lemma gives a full classification of local holomorphic maps on $\mathbb{R}^2$.

**Lemma 2.10.** *Let $D\subset\mathbb{R}^2$ be an open set. For a smooth map $\phi\colon D\to\mathbb{R}^2$ the following statements are equivalent:*

*(i) $\phi$ is conformal (w.r.t. the standard metric on source and target)*
*(ii) $\phi$ is either holomorphic or antiholomorphic (we are identifying $\mathbb{R}^2$ with $\mathbb{C}$) and has no critical points in $D$.*

*Proof.* Let $x,y$ be the real coordinates on $D$ and let $u,v$ be the coordinates on the target $\mathbb{R}^2$. Let $z=x+iy$ be the complex coordinate on $D$ and let $w=u+iv$ be the complex coordinate on the target $\mathbb{R}^2=\mathbb{C}$. The pullback of the target metric $g=du^2+dv^2=dw\,d\bar{w}$ is then

$$\phi^*g = \phi^*(dw\,d\bar{w}) = \partial_z\phi\,\partial_z\bar{\phi}(dz)^2 + \partial_{\bar{z}}\phi\,\partial_{\bar{z}}\bar{\phi}(d\bar{z})^2 + (\partial_z\phi\,\partial_{\bar{z}}\bar{\phi} + \partial_{\bar{z}}\phi\,\partial_z\bar{\phi})dzd\bar{z} \tag{87}$$

We are using the standard notations for holmorphic/antiholomorphic derivatives:

$$\partial_z = \frac{\partial}{\partial z} = \frac{1}{2}(\partial_x - i\partial_y), \quad \partial_{\bar{z}} = \frac{\partial}{\partial\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y).$$

(i)⇒(ii): If we know that $\phi$ is conformal, then

$$\phi^*g = \Omega g_D = \Omega dzd\bar{z} \tag{88}$$

for some positive function $\Omega$, thus coefficients of $(dz)^2$ and $(d\bar{z})^2$ must vanish. For this there are two possibilites:

(a) $\partial_{\bar{z}}\bar{\phi}=0$ (and thus also $\partial_z\bar{\phi}=0$), i.e., $\phi$ is holomorphic. In this case, comparing the $dzd\bar{z}$ term in (87) with (88), we have

$$\Omega = |\partial_z\phi|^2. \tag{89}$$

(b) $\partial_z \phi = 0$ (and thus also $\partial_{\bar z}\bar\phi = 0$), i.e., $\phi$ is antiholomorphic. In this case we have

$$\Omega = |\partial_{\bar z}\phi|^2. \tag{90}$$

Note that in both cases $\phi$ cannot have critical points, since there $\Omega$ would vanish (by (89), (90)).

(ii)$\Rightarrow$ (i): Assume $\phi$ is holomorphic with no critical points. Then $\partial_{\bar z}\phi = \partial_z\bar\phi = 0$, thus by (87), $\phi^* g = |\partial_z\phi|^2 dz d\bar z$. Hence, $\phi$ is conformal with $\Omega = |\partial_z \phi|^2$ which is positive, since $\phi$ has no critical points. The antiholomorphic case is similar. $\square$

**Example 2.11** (Möbius transformations)**.** The Lie group

$$PSL_2(\mathbb{C}) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array}\right) \,\middle|\, a,b,c,d \in \mathbb{C},\ ad-bc=1 \right\} / \mathbb{Z}_2, \tag{91}$$

where quotient by $\mathbb{Z}_2$ identifies a matrix and its negative, acts on the Riemann sphere $\bar{\mathbb{C}} = \mathbb{CP}^1$ by fractional-linear transformations (or "Möbius transformations")

$$\left( \begin{array}{cc} a & b \\ c & d \end{array}\right): \quad z \mapsto z' = \frac{az+b}{cz+d} \tag{92}$$

For any element of $PSL_2(\mathbb{C})$, (92) is a conformal map with conformal factor (w.r.t. the standard metric on $\mathbb{R}^2$)[33]

$$\Omega = \left| \frac{dz'}{dz} \right|^2 = \frac{1}{|cz+d|^4} \tag{93}$$

For instance, one has the following interesting classes of Möbius transformations:

(a) Element $\left( \begin{array}{cc} 1 & b \\ 0 & 1 \end{array}\right)$, with $b \in \mathbb{C}$, acts by translation $z \mapsto z+b$.
(b) $\left( \begin{array}{cc} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{array}\right)$ acts by rotation by angle $\phi$, $z \mapsto e^{i\phi} z$.
(c) $\left( \begin{array}{cc} \lambda^{1/2} & 0 \\ 0 & \lambda^{-1/2} \end{array}\right)$ with $\lambda > 0$ acts by dilation $z \mapsto \lambda z$.
(d) $\left( \begin{array}{cc} 1 & 0 \\ c & 1 \end{array}\right)$ with $c \in \mathbb{C}$ yields a *special conformal transformation* (SCT), $z \mapsto \frac{z}{cz+1} = \frac{1}{c+z^{-1}}$. In particular, it maps $-c^{-1} \mapsto \infty$ and $\infty \mapsto c^{-1}$.

Note that translations, rotations and dilations are conformal automorphisms of $\mathbb{C} \subset \bar{\mathbb{C}}$, but SCTs are not – they have a pole on $\mathbb{C}$.

**Example 2.12.** Consider the exponential map

$$\mathbb{C}/2\pi i \mathbb{Z} \xrightarrow{\exp} \mathbb{C}\backslash\{0\}$$

from the cylinder to the punctured plane. By Lemma 2.10, it is a conformal diffeomorphism, with $\Omega = e^{z+\bar z}$ (w.r.t. to the standard Euclidean metric on the source and target).

[33] Note that if $c \neq 0$, then (93) vanishes at the point $\{\infty\} \in \bar{\mathbb{C}}$ (and also explodes at $z = -\frac{d}{c}$) which seems to contradict that $\Omega$ should be a positive (and everywhere defined) function. This is to do with the fact that we chose a metric on $\mathbb{C}$ which does not extend to the point $\{\infty\}$. One can choose another metric in the same conformal class which extends to $\{\infty\}$ (e.g. the round metric on $\bar{\mathbb{C}}$ seen as $S^2$), then $\Omega$ relative to that metric will be truly everywhere positive and everywhere defined.

2.3. **Conformal vector fields.** One can think of conformal vector fields as "infinitesimal conformal maps."

**Definition 2.13.** A conformal vector field on a (pseudo-)Riemannian manifold $(M,g)$ is a vector field $v \in \mathfrak{X}(M)$ satisfying

$$\mathcal{L}_v g = \omega\, g \tag{94}$$

for some function $\omega \in C^\infty(M)$ (the inifitesimal conformal factor); $\mathcal{L}_v$ stands for the Lie derivative along $v$.[34] We denote the set of all conformal vector fields on $(M,g)$ by $\mathrm{conf}(M,g)$.

Conformal vector fields form a Lie subalgebra in the Lie algebra of all vector fields w.r.t. the standard Lie bracket of vector fields:

$$\mathrm{conf}(M,g) \subset \mathfrak{X}(M). \tag{95}$$

One has a natural inclusion

$$\iota\colon \mathrm{Lie}\,(\mathrm{Conf}(M,g)) \hookrightarrow \mathrm{conf}(M,g) \tag{96}$$

of the Lie algebra of the group of conformal automorphisms into the Lie algebra of conformal vector fields (by taking derivative the at $t=0$ of a 1-parametric subgroup). If $M$ is compact, $\iota$ is an isomorphism (one can construct the flow of a conformal vector field $v \mapsto \mathrm{Flow}_t(v)$ yielding a 1-parametric subgroup of $\mathrm{Conf}(M,g)$). However, for $M$ noncompact, conformal vector fields can fail to be complete, so only a part of elements of $\mathrm{conf}(M,g)$ can be exponentiated.

2.4. **Conformal symmetry of $\mathbb{R}^{p,q}$ with $p+q>2$.**

2.4.1. *Conformal vector fields on $\mathbb{R}^{p,q}$.* Consider the space $\mathbb{R}^{p,q}$ with its standard metric $g = \eta_{ij}dx^i dx^j$ with the matrix $\eta_{ij}$ being

$$\eta_{ij} = \mathrm{diag}(\underbrace{+1,\ldots,+1}_{p},\underbrace{-1,\ldots,-1}_{q}).$$

We denote $n = p+q$.

We are looking for conformal vector fields $v = v^k(x)\partial_k$ on $\mathbb{R}^{p,q}$. (Summation over repeated indices is implied everywhere in this section.) The defining equation (94) for them takes the form

$$\partial_i v_j + \partial_j v_i = \omega \eta_{ij} \tag{97}$$

with $v_i :\ = \eta_{ij}v^j$. (97) is a system of $n^2$ (dependent) differential equations on $n+1$ unknown functions – components $v_i$ of the conformal vector field and $\omega$ – the infinitesimal conformal factor. Solving (97) is a well-known exercise [34, 8, 17]; for reader's convenience, we reproduce the argument.[35] locate the right reference

(i) Contracting (97) with $\eta^{ij}$, we get

$$2\,\underbrace{\partial_i v^i}_{\mathrm{div}\, v} = n\omega. \tag{98}$$

[34] Note that there is no positivity constraint on $\omega$.

[35] Part of the value of the explicit argument here is that it gives an explanation (albeit a technical one) of why the cases $n=1,2$ and $n>2$ are so vastly different.

(ii) Applying $\partial^j$ to (97), we get

$$\partial_i(\operatorname{div} v) + \Delta v_i = \partial_i \omega, \tag{99}$$

where $\Delta = \partial_j \partial^j$. By (98), this implies

$$\Delta v_i = \left(1 - \frac{n}{2}\right) \partial_i \omega. \tag{100}$$

(iii) Applying $\partial_j$ to (100), symmetrizing in $i \leftrightarrow j$ and using (97), we get

$$\frac{1}{2}\eta_{ij}\Delta\omega = \left(1 - \frac{n}{2}\right) \partial_i \partial_j \omega. \tag{101}$$

(iv) Applying $\partial^i$ to (100), we get

$$\Delta(\underbrace{\operatorname{div} v}_{\underset{(98)}{=} \frac{n}{2}\omega}) = \left(1 - \frac{n}{2}\right) \Delta\omega, \tag{102}$$

which implies

$$(n-1)\Delta\omega = 0 \tag{103}$$

(v) Equations (101) and (103) imply that for $n \neq 1, 2$ one has

$$\partial_i \partial_j \omega = 0. \tag{104}$$

I.e., $\omega$ is at most linear in coordinates.

(vi) Taking a derivative of (97), we have

$$\partial_i \partial_j v_k + \partial_i \partial_k v_j = \partial_i \omega\, \eta_{jk} \tag{105}$$

The equation $(105) + (105)_{(ijk)\to(jik)} - (105)_{(ijk)\to(kij)}$ then reads

$$2\partial_i \partial_j v_k = \partial_i \omega\, \eta_{jk} + \partial_j \omega\, \eta_{ik} - \partial_k \omega\, \eta_{ij} \tag{106}$$

(vii) Equation (104) and (106) together imply, for $n \neq 1, 2$, that

$$\partial_i \partial_j \partial_k v_l = 0. \tag{107}$$

I.e., $v$ is at most quadratic in coordinates.

Now, specializing to the case $n > 2$, we have an ansatz

$$v_i(x) = a_i + b_{ij}x^j + c_{ijk}x^j x^k, \quad \omega(x) = 2\mu + 4\nu_i x^i \tag{108}$$

with $a_i, b_{ij}, c_{ijk}, \mu, \nu_i$ some coefficients. Substituting this ansatz into (97), we find that (108) is a conformal vector field and its conformal factor if the coefficients satisfy the following:

(a) No restriction on $a_i$.
(b) $b_{ij} + b_{ji} = 2\mu\eta_{ij}$ which implies

$$b_{ij} = \mu\eta_{ij} + \beta_{ij}$$

with some anti-symmetric tensor $\beta_{ij} = -\beta_{ji}$.
(c) $c_{ijk} + c_{jik} = 2\nu_k \eta_{ij}$ which implies, similarly to the derivation of (106) above,

$$c_{ijk} = \nu_j \eta_{ik} + \nu_k \eta_{ij} - \nu_i \eta_{jk}.$$

This proves the following.

**Theorem 2.14** (Liouville)**.** *For $n = p+q > 2$, the Lie algebra of conformal vector fields on $\mathbb{R}^{p,q}$ splits into the following subspaces:*

$$\mathrm{conf}(\mathbb{R}^{p,q}) = \underbrace{\{\text{translations}\}}_{\simeq\mathbb{R}^n} \oplus \underbrace{\{\text{rotations}\}}_{\simeq\mathfrak{so}(p,q)} \oplus \underbrace{\{\text{dilations}\}}_{\simeq\mathbb{R}} \oplus \underbrace{\{\text{SCTs}\}}_{\simeq\mathbb{R}^n} \tag{109}$$

*where SCTs stands for "special linear transformations." Explicitly, these conformal vector fields are as follows.*

| | *conf. vector field* | $\omega$ |
|---|---|---|
| *translation* | $v^i(x) = a^i$ | $0$ |
| *rotation* | $v^i(x) = \beta^i_j x^j$ *with* $\beta_{ij} = -\beta_{ji}$ | $0$ |
| *dilation* | $v^i(x) = \mu x^i$ | $2\mu$ |
| *SCT* | $v^i(x) = 2(\vec{x},\vec{\nu})x^i - \nu^i\vert\vert\vec{x}\vert\vert^2$ | $4(\vec{\nu},\vec{x})$ |

2.4.2. *Finite conformal automorphisms of $\mathbb{R}^{p,q}$ with $p+q>2$.* Here are the finite[36] conformal maps exponentiating (via constructing the flow in time 1) the conformal vector fields of Theorem 2.14.[37]

| | conf. map | $\Omega$ |
|---|---|---|
| translation | $x^i \mapsto x^i + a^i$, $\vec{a} \in \mathbb{R}^n$ | $1$ |
| rotation | $x^i \mapsto O^i_j x^j$, $O^i_j \in SO(p,q)$ | $1$ |
| dilation | $x^i \mapsto \lambda x^i$, $\lambda > 0$ | $\lambda^2$ |
| SCT | $x^i \mapsto \frac{x^i - \vert\vert\vec{x}\vert\vert^2 b^i}{1-2(\vec{b},\vec{x})+\vert\vert\vec{b}\vert\vert^2\vert\vert\vec{x}\vert\vert^2}$, $\vec{b}\in\mathbb{R}^n$ | $(1-2(\vec{b},\vec{x})+\vert\vert\vec{b}\vert\vert^2\vert\vert\vec{x}\vert\vert^2)^{-2}$ |

**Definition 2.15.** Given a manifold $M$ equipped with a conformal structure $\gamma_N$ (a choice of metric modulo Weyl transformations), we say that a *compact* manifold $N$ equipped with conformal structure, is a *conformal compactification of $M$*, if the following holds:

- One has an embedding $M \hookrightarrow N$ with open dense image.
- All conformal vector fields on $M$ extend to $N$. (And $N$ they can automatically be integrated to conformal automorphisms.)

*Remark* 2.16 (On finite SCTs). (a) A finite SCT can be written as

$$(\text{inversion}) \circ (\text{translation by } -\vec{b}) \circ (\text{inversion}).$$

I.e., it maps $\vec{x} \mapsto \vec{x}'$ with image and preimage related by

$$\frac{\vec{x}'}{||\vec{x}'||^2} = \frac{\vec{x}}{||\vec{x}||^2} - \vec{b}.$$

(b) A finite SCT is not everywhere defined as a map $\mathbb{R}^{p,q} \to \mathbb{R}^{p,q}$ (the denominator in the formula for SCT may vanish). This corresponds to the quadratic vector field describing the infinitesimal SCT not being complete on $\mathbb{R}^{p,q}$.

(c) In Section 2.4.3 we will construct a conformal compactification $N^{p,q}$ of $\mathbb{R}^{p,q}$, such that SCTs are everywhere well-defined on $N^{p,q}$.

We also remark that in the exceptional dimensions $n = 1,2$, the r.h.s. of (109) is a (small) subspace of the l.h.s., while the l.h.s is an $\infty$-dimensional Lie algebra. Lecture 7, 9/7/2022

**Theorem 2.17.** *Assume $p+q>2$.*

[36] "Finite" conformal maps are just conformal maps. We use the adjective "finite" to emphasize the difference from "infinitesimal conformal maps," i.e., conformal vector fields.

[37] Under the flow-in-time-one map, the parameters of the finite conformal maps are related to the parameters of the conformal vector fields by $\vec{a} = \vec{a}$, $O = \exp(\beta)$, $\lambda = e^{\mu}$, $\vec{b} = \vec{\nu}$.

(i) *One has an isomorphism of Lie algebras*

$$\mathrm{conf}(\mathbb{R}^{p,q}) \cong \mathfrak{so}(p+1,q+1). \tag{110}$$

(ii) *For the group* $\mathrm{Conf}^{\mathrm{sing}}$ *of almost everywhere defined conformal automorphisms of* $\mathbb{R}^{p,q}$*, one has:*
- *If* $-1$ *and* $1$ *are in different connected components of* $SO(p+1,q+1)$*, then*

$$\mathrm{Conf}_0^{\mathrm{sing}}(\mathbb{R}^{p,q}) \cong SO_0(p+1,q+1) \tag{111}$$

*Subscript* $0$ *on both sides stands for "connected component of* $1$*."*
- *Otherwise,*

$$\mathrm{Conf}_0^{\mathrm{sing}}(\mathbb{R}^{p,q}) \cong SO_0(p+1,q+1)/\mathbb{Z}_2 \tag{112}$$

(iii) *The conformal manifold* $\mathbb{R}^{p,q}$ *possesses a conformal compactification* $N^{p,q}$ *in the sense of Definition 2.15.*

For the proof, see [34].

As a sanity check of (110), let us check that the dimensions of both sides match:

(113)
$$\dim \mathrm{conf}(\mathbb{R}^{p,q}) \underset{(109)}{=} \dim\{\text{translations}\}+\dim\{\text{rotations}\}+\dim\{\text{dilations}\}+\dim\{\text{SCTs}\}$$
$$= n + \frac{n(n-1)}{2} + 1 + n = \frac{(n+1)(n+2)}{2} = \dim \mathfrak{so}(p+1,q+1)$$

2.4.3. *Sketch of proof of Theorem 2.17: action of* $SO(p+1,q+1)$ *on* $\mathbb{R}^{p,q}$ *and the conformal compactification of* $\mathbb{R}^{p,q}$. For the following construction, we also follow [34].

<u>Case of $\mathbb{R}^n$</u>. Consider first the case $(p,q)=(n,0)$.

- The group $SO(n+1,1)$ acts on $\mathbb{R}^{n+1,1}$ by linear isometries and preserves the light cone

$$LC = \{(x^0,\ldots,x^n,y)\in\mathbb{R}^{n+1,1} \mid (x^0)^2+\cdots+(x^n)^2-y^2=0\} \quad \subset \mathbb{R}^{n+1,1} \tag{114}$$

- We have two commuting actions

$$SO(n+1,1) \ \circlearrowright\ LC \ \underset{\text{dilations}}{\circlearrowleft}\ \mathbb{R}^* \tag{115}$$

- In particular, $SO(n+1,1)$ acts on $LC-\{0\}/\mathbb{R}^*$.
- $LC-\{0\}$ inherits a *degenerate* metric from $\mathbb{R}^{n+1,1}$. Its kernel is the fundamental vector field of the $\mathbb{R}^*$-action and thus is killed by quotienting over $\mathbb{R}^*$.
- By the previous, $LC-\{0\}/\mathbb{R}^*$ inherits a conformal structure and $SO(n+1,1)$ acts on $LC-\{0\}/\mathbb{R}^*$ by conformal maps.
- Note: $LC-\{0\}/\mathbb{R}^*$ can be identified with the unit sphere $S^n\subset\mathbb{R}^{n+1}$: intersecting $LC$ with the hyperplane $y=1$ in $\mathbb{R}^{n+1,1}$, we a re selecting a single point from each $\mathbb{R}^*$-orbit.

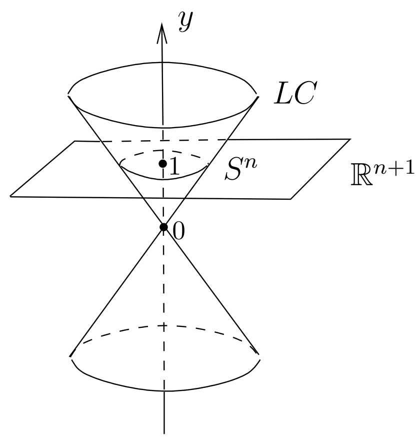



FIGURE 12. Light cone and its section by $y = 1$ hyperplane.

- One has a stereographic projection
$$S^n - \{\underbrace{(1, 0, \ldots, 0)}_{\text{North pole}}\} \to \mathbb{R}^n$$
(which is a conformal diffeomorphism). Thus we identify $S^n$ as a conformal compactification of $\mathbb{R}^n$: conformal vector fields on $\mathbb{R}^n$ extend to $S^n$ and finite conformal maps are everywhere defined on $S^n$.

Case of general $\mathbb{R}^{p,q}$.

- We have the light cone

$$LC = \left\{ (x^0, \ldots, x^p, y^0, \ldots, y^q) \;\middle|\; \sum_{i=0}^{p} (x^i)^2 - \sum_{j=0}^{q} (y^j)^2 = 0 \right\} \quad \subset \mathbb{R}^{p+1,q+1}. \tag{116}$$

- We have two commuting actions

$$SO(p+1, q+1) \underset{\text{lin. isometries}}{\circlearrowleft} LC - \{0\} \underset{\text{dilations}}{\circlearrowright} \mathbb{R}^*. \tag{117}$$

- We have a projection

$$\pi \colon LC - \{0\} \;\to\; \mathbb{RP}^{n+1}. \tag{118}$$

Denote its image

$$N^{p,q} \colon = \mathrm{im}(\pi) \simeq (LC - \{0\})/\mathbb{R}^* \tag{119}$$

Being a submanifold of a compact manifold $\mathbb{RP}^{n+1}$, $N^{p,q}$ is compact.

- Consider the map $\iota \colon \mathbb{R}^{p,q} \to N^{p,q}$ defined by

$$\iota(x^1, \ldots, x^p, y^1, \ldots, y^q) = \tag{120}$$
$$= \left( \frac{1}{2}\Big(1 - \sum_{i=1}^{p} (x^i)^2 + \sum_{j=1}^{q} (y^j)^2\Big) : x^1 : \cdots : x^p : \frac{1}{2}\Big(1 + \sum_{i=1}^{p} (x^i)^2 - \sum_{j=1}^{q} (y^j)^2\Big) : y^1 : \cdots : y^q \right)$$

where $(- : - \cdots : -)$ stands for the homogeneous coordinates on the projective space. The map $\iota$ is injective and has open dense image.

*Sketch of proof of Theorem 2.17.*

1. We have constructed a compact manifold $N^{p,q}$ equipped with an inclusion $\mathbb{R}^{p,q} \hookrightarrow N^{p,q}$ (compatible with conformal structures) as an open dense subset.
2. We have constructed an action of $SO(p+1,q+1)$ on $N^{p,q}$ by conformal diffeomorphisms. The only elements acting trivially are multiples of identity, i.e., 1 and $-1$ (in the case when $-1$ belongs to $SO(p+1,q+1)$).
3. The differential of the action of $SO(p+1,q+1)$ gives an injective Lie algebra map $\mathfrak{so}(p+1,q+1) \hookrightarrow \mathrm{conf}(N^{p,q})$ (and by restriction to $\mathbb{R}^{p,q}$, an inclusion $\mathfrak{so}(p+1,q+1) \hookrightarrow \mathrm{conf}(\mathbb{R}^{p,q})$). By the dimension count (113), these inclusions are in fact isomorphisms. This proves (i) and (iii) of Theorem 2.17, identifying (119) as the desired conformal compactification.
4. The previous two points imply that the Lie group $\mathrm{Conf}(N^{p,q})$ contains $SO(p,q)/\mathbb{Z}_2$ and both groups have the same Lie algebra. That implies that the connected components of 1 in both groups coincide. That proves (ii) of Theorem 2.17.

$\square$

*Remark* 2.18. The product of unit spheres

$$S^p \times S^q = \{(x^0,\ldots,x^p,y^0,\ldots,y^q) \mid \sum_{i=0}^{p}(x^i)^2 = 1,\ \sum_{j=0}^{q}(y^j)^2 = 1\} \tag{121}$$

is a submanifold of $LC - \{0\}$ and intersect each $\mathbb{R}^*$-orbit twice ($(x,y)$ and $(-x,-y)$ are in the same $\mathbb{R}^*$-orbit). Thus, one has a twofold covering map

$$S^p \times S^q \to N^{p,q} \tag{122}$$

given by the projection (118) restricted to $S^p \times S^q$. In particular, we can identify $N^{p,q}$ with the quotient

$$N^{p,q} \simeq S^p \times S^q/\mathbb{Z}_2 \tag{123}$$

where $\mathbb{Z}_2$ acts by the diagonal antipodal map, $(x,y) \mapsto (-x,-y)$.

2.5. **Conformal symmetry of $\mathbb{R}^2$.** A vector field $v = v_i(x,y)\partial_i$ (with $x = x^1$, $y = x^2$) on $\mathbb{R}^2$ equipped with the standard Euclidean metric is conformal if the equation (94) holds:

$$\partial_i v_j + \partial_j v_i = \omega \delta_{ij} \quad \Leftrightarrow \left\{ \begin{array}{l} \partial_x v_x = \partial_y v_y = \frac{1}{2}\omega \\ \partial_x v_y = -\partial_y v_x \end{array} \right. \tag{124}$$

for some function (conformal factor) $\omega$. On the right side we can recognize the Cauchy-Riemann equations. Thus, the vector field $v = v_i\partial_i$ is conformal if and only if the function

$$u{:}\ = v_x + i v_y \tag{125}$$

is holomorphic. Note that the vector field $v$ can be written in terms of the holomorphic function $u$ and its complex conjugate $\bar{u}$ as

$$v = u(z)\partial_z + \bar{u}(\bar{z})\partial_{\bar{z}} = 2\,\mathrm{Re}(u(z)\partial_z) \tag{126}$$

The corresponding conformal factor is $\omega = \partial_z u + \partial_{\bar{z}}\bar{u}$.

In (126) we use the complex coordinate $z = x + iy$, its conjugate $\bar{z} = x - iy$ and the corresponding derivatives $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$, $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$.

To summarize, we have the following.

**Lemma 2.19.** *One has an isomorphism of Lie algebras*

$$\psi\colon \mathrm{conf}(\mathbb{R}^2) \xrightarrow{\sim} \{\text{holomorphic vector fields on } \mathbb{C}\}. \tag{127}$$

*It maps a conformal vector field $v_x\partial_x + v_y\partial_y$ to the holomorphic vector field $u(z)\partial_z$ where $u(z) = v_x + iv_y$. The inverse map $\psi^{-1}$ assigns to a holomorphic vector field $u(z)\partial_z$ a conformal vector field $2\,\mathrm{Re}(u(z)\partial_z) = u(z)\partial_z + \bar{u}(\bar{z})\partial_{\bar{z}}$.*

The fact that $\psi$ intertwines the Lie brackets on the two sides of (127) is a straightforward check.

*Remark* 2.20. In the isomorphism (127), we are thinking of both sides as Lie algebras over $\mathbb{R}$. However, the right hand side is also a Lie algebra over $\mathbb{C}$. Multiplication by $i$ on the right side translates in the left side to acting on a conformal vector field by pointwise rotation by $\pi/2$ (in the tangent space at each point of $\mathbb{R}^2$).

Lecture 8, 9/9/2022

Lemma 2.19 classifes infinitesimal confromal maps; its counterpart for finite conformal maps is Lemma 2.10 above, or its rephrasing:

**Lemma 2.21.** *Let $D, D'$ be two open sets in $\mathbb{C}$. A map $\phi\colon D \to D'$ is a conformal diffeomorphism if and only if $\phi$ is either biholomorphic or biantiholomorphic (i.e., the complex conjugate map $\bar{\phi}\colon D \to \bar{D}'$ is bihomolomorphic).*

2.5.1. *Conformal vector fields on $\mathbb{C}^*$, Witt algebra.*

**Definition 2.22.** We define the Witt algebra $\mathcal{W}$ as the Lie algebra of meromorphic vector fields on $\mathbb{C}$ with a pole (of finite order) allowed only at 0. The Lie algebra $\mathcal{W}$ has a standard basis of meromorphic vector fields

$$l_n = -z^{n+1}\frac{\partial}{\partial z}, \quad n \in \mathbb{Z}. \tag{128}$$

Thus, the Witt algebra is

$$\mathcal{W} = \{\sum_{n=-n_0}^{\infty} c_n l_n \mid c_n \in \mathbb{C},\ \text{the sum converges on } \mathbb{C}^*\}. \tag{129}$$

The generators $l_n$ of $\mathcal{W}$ satisfy the commutation relations

$$[l_n, l_m] = (n-m)l_{n+m}. \tag{130}$$

Indeed:

$$\begin{aligned}&[-z^{n+1}\partial_z, -z^{m+1}\partial_z] = z^{n+1}[\partial_z, z^{m+1}\partial_z] - z^{m+1}[\partial_z, z^{n+1}\partial_z] = \\ &= ((m+1)z^{n+m+1} - (n+1)z^{n+m+1})\partial_z = (m-n)z^{n+m+1}\partial_z = (n-m)l_{n+m}.\end{aligned}$$

There are several relevant variants of the Lie algebra $\mathcal{W}$, all with the same collection of generators $\{l_n\}$ but with different asymptotic conditions on the coefficients $c_n$ as $n \to \pm\infty$:

(i) Holomorphic vector fields on the punctured formal disk:

$$\mathbb{C}[[z, z^{-1}]\partial_z = \{\sum_{n=-n_0}^{\infty} c_n l_n \mid c_n \in \mathbb{C}\}. \tag{131}$$

– This is a good model for the local conformal algebra $\mathcal{A}^{\mathrm{loc}}$ of Section 1.7.1.

(ii) Meromorphic vector fields on $\mathbb{CP}^1$ with finite-order poles allowed only at 0 and $\infty$:

$$\{\sum_{n=-n_0}^{n_1} c_n l_n \mid c_n \in \mathbb{C}\}. \tag{132}$$

– This model has the benefit that it is symmetric under the involution $z \to 1/z$ on $\mathbb{CP}^1$.

We remark that the space of vector fields with coefficients in all formal Laurent power series $\{\sum_{n=-\infty}^{\infty} c_n l_n\}$ does not form a Lie algebra, since coefficients of the Lie bracket of two elements involves infinite sums that do not have to converge.

By abuse of notations and terminology, we will call all complex Lie algebras spanned by $\{l_n\}_{n\in\mathbb{Z}}$ with different decay conditions on coefficients, the Witt algebra and denote them $\mathcal{W}$.

By Lemma 2.19, conformal vector fields on $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ are the real parts of meromorphic vector fields on $\mathbb{C}^*$:

$$\mathrm{conf}(\mathbb{C}^*) \simeq \mathcal{W} = \mathrm{span}_{\mathbb{C}}\{l_n\}_{n\in\mathbb{Z}} \tag{133}$$

(When we write "span," we are being noncommittal about the decay conditions on coefficients.) Thus, one may also write

$$\mathrm{conf}(\mathbb{C}^*) = \mathrm{span}_{\mathbb{R}}\{l_n + \bar{l}_n, i(l_n - \bar{l}_n)\}_{n\in\mathbb{Z}}. \tag{134}$$

Thus, $\mathrm{conf}(\mathbb{C}^*)$ embeds as a real slice into its complexification

$$\mathrm{conf}(\mathbb{C}^*) \otimes_{\mathbb{R}} \mathbb{C} = \underbrace{\mathcal{W}}_{\mathrm{span}_{\mathbb{C}}\{l_n\}} \oplus \underbrace{\overline{\mathcal{W}}}_{\mathrm{span}_{\mathbb{C}}\{\bar{l}_n\}}.$$

Here

$$\bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}} \tag{135}$$

are the antimeromorphic vector fields on $\mathbb{C}^*$ complex-conjugate to $l_n$. They satisfy the commutation relation similar to (130),

$$[\bar{l}_n, \bar{l}_m] = (n-m)\bar{l}_{n+m}. \tag{136}$$

Also, one has

$$[l_n, \bar{l}_m] = 0.$$

<u>Some interesting Lie subalgebras of $\mathrm{conf}(\mathbb{C}^*)$</u>:

(a) Conformal vector fields on $\mathbb{C}$:

$$\mathrm{span}_{\mathbb{R}}\{l_n + \bar{l}_n, i(l_n - \bar{l}_n)\}_{n\geq -1} \tag{137}$$

Indeed, vector fields $l_n, \bar{l}_n$ are holomorphic at 0 iff $n \geq -1$.

(b) Conformal vector fields on $\mathbb{C}$ vanishing at 0:

$$\mathrm{span}_{\mathbb{R}}\{l_n + \bar{l}_n, i(l_n - \bar{l}_n)\}_{n\geq 0} \tag{138}$$

Indeed, $l_n, \bar{l}_n$ vanish at 0 iff $n \geq 0$.

(c) Conformal vector fields on $\mathbb{CP}^1\backslash\{0\}$:

$$\mathrm{span}_{\mathbb{R}}\{l_n + \bar{l}_n, i(l_n - \bar{l}_n)\}_{n\leq 1} \tag{139}$$

Indeed in the local coordinate $w = z^{-1}$ on $\mathbb{CP}^1\backslash\{0\}$ one has $l_n = w^{-n+1}\frac{\partial}{\partial w}$. Thus, $l_n$ is regular at the point $z = \infty$ (or $w = 0$) iff $-n+1 \geq 0$. (And similarly for $\bar{l}_n$.)

*Remark* 2.23. Naively, the punctured plane $\mathbb{C}^*$, the punctured unit disk $\{z \in \mathbb{C}|0 < |z| < 1\}$ and annulus $\mathrm{Ann}_r^R = \{z \in \mathbb{C}|r < |z| < R\}$ all have the same Lie algebra $\mathrm{conf}(-) \simeq \mathcal{W} = \mathrm{span}_{\mathbb{C}}\{l_n\}_{n\in\mathbb{Z}}$. But in fact, for all these domains, the decay conditions on the coefficients $c_n$ in (129) are different. In the case of the annulus, the decay conditions depend on the inner and outer radii,[38] so that e.g. if one has $r' < r < R < R'$, then one has a proper inclusion $\mathrm{conf}(\mathrm{Ann}_{r'}^{R'}) \hookrightarrow \mathrm{conf}(\mathrm{Ann}_r^R)$ (so that the thinner annulus has a bigger Lie algebra of conformal vector fields).

2.5.2. *Conformal symmetry of* $\mathbb{CP}^1$. Conformal vector fields on $\mathbb{CP}^1$ are:

$$\mathrm{conf}(\mathbb{CP}^1) = \mathrm{span}_{\mathbb{R}}\{l_n + \bar{l}_n, i(l_n - \bar{l}_n)\}_{n\in\{-1,0,1\}} \tag{140}$$

This is the subalgebra of $\mathrm{conf}(\mathbb{C}^*)$ comprised of vector fields which are regular at 0 and at $\infty$, i.e, it is the intersection of (137) and (139). The Lie algebra $\mathrm{conf}(CP^1)$ is also isomorphic to $\mathfrak{sl}_2(\mathbb{C})$ and to $\mathfrak{so}(3,1)$.[39] We can identify the generators of $\mathrm{conf}(CP^1)$ explicitly as infinitesimal translations, rotation, dilation, and special canonical transformations:

| | | |
|---|---|---|
| $-(l_{-1} + \bar{l}_{-1})$ | $= \partial_x$ | translation |
| $-i(l_{-1} - \bar{l}_{-1})$ | $= \partial_y$ | translation |
| $-(l_0 + \bar{l}_0)$ | $= x\partial_x + y\partial_y$ | dilation |
| $-i(l_0 - \bar{l}_0)$ | $= -y\partial_x + x\partial_y$ | rotation |
| $-(l_1 + \bar{l}_1)$ | $= (x^2 - y^2)\partial_x + 2xy\partial_y$ | SCT |
| $-i(l_1 - \bar{l}_1)$ | $= -2xy\partial_x + (x^2 - y^2)\partial_y$ | SCT |

The orientation-preserving part of the group of conformal automorphisms of $\mathbb{CP}^1$ is given by Möbius transformations (92):

$$\mathrm{Conf}_+(\mathbb{CP}^1) = PSL_2(\mathbb{C}) \simeq SO_+(3,1) \tag{141}$$

Where $SO_+(3,1)$ is the *othrochronous* component of $SO(3,1)$, consisting of the elements preserving the positive $(y > 0)$ half of the light-cone.

*Remark* 2.24. Note that while $\mathrm{conf}(\mathbb{C})$ is an infinite-dimensional Lie algebra, passing to the one-point compactification $\mathbb{C} \to \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ reduces this algebra to a finite-dimensional one (140). In fact, $\mathbb{C}$ does not have a conformal compactification (see Definition 2.15), unlike $\mathbb{R}^{p,q}$ with $p + q > 2$.

2.5.3. *The group of conformal automorphisms of a simply-connected domain in* $\mathbb{C}$.

**Lemma 2.25.** *1. The group of conformal automorphisms of the upper half-plane* $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ *is*

$$\mathrm{Conf}(\mathbb{H}) = PSL_2(\mathbb{R}) \tag{142}$$

*where the elements of* $PSL_2(\mathbb{R})$ *are acting by Möbius transformations (92) with* $a, b, c, d \in \mathbb{R}$.

---

[38]Explicitly, the decay conditions for the annulus $\mathrm{Ann}_r^R$ are: $c_n\rho^n \underset{n\to+\infty}{=} O(n^{-\infty})$ for any $0 < \rho < R$ and $c_n\rho^n \underset{n\to-\infty}{=} O(|n|^{-\infty})$ for any $\rho > r$.

[39]One has an action of $\mathfrak{so}(3,1)$ on $\mathbb{CP}^1$ by conformal vector fields by the construction of Section 2.4.3. Also, in the last isomorphism in (141) we are referring to the finite version of that action.

*2. The group of conformal automorphisms of the unit disk $D = \{z \in \mathbb{C} \mid |z| < 1\}$ is*

$$\mathrm{Conf}(D) = PSU(1,1) \tag{143}$$

*– the group of Möbius transformations of the form*

$$z \mapsto e^{i\phi} \frac{z-a}{\bar{a}z - 1} \tag{144}$$

*where $\phi \in \mathbb{R}/2\pi\mathbb{Z}$, $a \in \mathbb{C}$ with $|a|$ are parameters.*

This is proven straightforwardly, by finding the part of the $PSL_2(\mathbb{C})$ which preserves the boundary of the domain (the real line or the unit circle) and does not swap the domain with its complement in $\mathbb{CP}^1$.

*Remark* 2.26. The groups $PSL_2(\mathbb{R})$ and $PSU(1,1)$ are conjugate subgroups $PSL_2(\mathbb{C})$, with conjugating element corresponding to the map $z \mapsto \frac{z-i}{z+i}$ – a conformal diffeomorphism $\mathbb{H} \to D$.

Recall the key result of complex analysis:

**Theorem 2.27** (Riemann mapping theorem)**.** *For any simply-connected open set $U \subset \mathbb{C}$, there exists a biholomorphic map $\phi\colon U \to D$ with $D$ the open unit disk.*

**Corollary 2.28.** *For any simply-connected open set $U$, the group of conformal automorphisms is*

$$\mathrm{Conf}(U) = \phi^* PSL_2(\mathbb{R}) \tag{145}$$

*where $\phi\colon U \to D$ is the map from the Riemann mapping theorem.*

2.5.4. *Vector fields on $S^1$ vs. Witt algebra.* A real vector field tangent to the unit circle $S^1 \subset \mathbb{C}$ can be written as

$$v = f(\theta)\partial_\theta = \sum_{n\in\mathbb{Z}} a_n e^{in\theta}\partial_\theta \tag{146}$$

with the Fourier coefficients $a_n$ satisfying the reality condition

$$a_{-n} = \bar{a}_n. \tag{147}$$

Here $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ is the angle coordinate on $S^1$. We denote the Lie algebra of such vector fields $\mathfrak{X}(S^1)$.

One can express the basis tangent vector fields on $S^1$ in terms of Witt generators restricted to $S^1$:[40]

$$e^{in\theta}\partial_\theta = -i(l_n - \bar{l}_{-n})\Big|_{S^1} \tag{148}$$

Likewise, basis normal vector field to $S^1$ are:

$$e^{in\theta}\partial_r = -(l_n + \bar{l}_{-n})\Big|_{S^1} \tag{149}$$

Re vs 2Re?

We have a map

$$\begin{array}{ccc} \mathcal{W} & \to & \Gamma(S^1, T\mathbb{C}|_{S^1}) \\ \sum_{n=-\infty}^{\infty} c_n l_n & \mapsto & 2\mathrm{Re}\sum_n c_n l_n\Big|_{S^1} \end{array} \tag{150}$$

[40]A related point: consider the inversion map $\mathbb{I}\colon \mathbb{C}^* \to \mathbb{C}^*$, mapping $z \mapsto \frac{1}{\bar{z}}$. The pushforward of $l_n$ by the inversion is $\mathbb{I}_* l_n = -\bar{l}_{-n}$. Vector fields tangent to $S^1$ appearing in the r.h.s. of (148) are invariant under $\mathbb{I}_*$.

In fact, it is an isomorphism, under appropriate decay assumptions on $c_n$. The r.h.s. of (150) consist of vector fields on $S^1$ that are allowed to have both tangent and normal component. The part of $\mathcal{W}$ that maps to vector fields *tangent* to $S^1$ is the real Lie subalgebra

$$\underbrace{\{\sum_n c_n l_n \mid c_{-n} = -\bar{c}_n\}}_{\simeq \mathfrak{X}(S^1)} \subset \mathcal{W} \tag{151}$$

Thus, one has the following.

**Lemma 2.29.** *The Witt algebra $\mathcal{W}$ (with decay conditions on coefficients as above) is a complexification of $\mathfrak{X}(S^1)$.*

One might ask: which vector fields on $S^1$ extend into the unit disk $D$ (cobounding $S^1$) as conformal vector fields? The answer depends drastically on whether the vector fields are required to be tangent to $S^1$ or are allowed to have a normal component on $S^1$.

**Lemma 2.30.** *(i) The subalgebra of $\mathfrak{X}(S^1)$ given by vector fields extending as conformal vector fields into the unit disk $D$ is*

$$\{\mathrm{Re} \sum_{n=-1}^{1} c_n l_n \mid c_n = -\bar{c}_n\} \simeq \mathfrak{sl}_2(\mathbb{R}) \tag{152}$$

*(ii) The subalgebra of $\Gamma(S^1, T\mathbb{C}|_{S^1})$ given by vector fields on $S^1$ (with normal component allowed) extending as conformal vector fields into the unit disk $D$ is*

$$\{\mathrm{Re} \sum_{n\geq -1} c_n l_n \mid c_n = -\bar{c}_n\} \tag{153}$$

In particular, we have a finite-dimensional Lie algebra in one case and an infinite-dimensional one in the other case.

*Proof.* Immediate consequence of (150), (151) and the fact that $l_n$ is regular at 0 iff $n \geq -1$. □

2.6. **Conformal symmetry of $\mathbb{R}^1$ (trivial case).** Recall from Example 2.8 that on $\mathbb{R}^1$ any diffeomorphism is conformal, $\mathrm{Conf}(\mathbb{R}^1) = \mathrm{Diff}(\mathbb{R}^1)$. Likewise, any vector field on $\mathbb{R}^1$ is conformal, $\mathrm{conf}(\mathbb{R}^1) = \mathfrak{X}(\mathbb{R}^1)$.

Also, one can replace $\mathbb{R}^1$ with $S^1$ (thought of as a one-point compactification of $\mathbb{R}^1$). Here one has as a distinguished subgroup the Möbius transformations of $S^1$:

$$\mathrm{Conf}(S^1) = \mathrm{Diff}(S^1) \supset \underbrace{PSL_2(\mathbb{R}) \simeq SO_+(2,1)}_{\text{“restricted conformal group''}} \tag{154}$$

The action of $SO(2,1)$ on $S^1$ by conformal automorphisms is by the construction of Section 2.4.3.

2.7. **Conformal symmetry of $\mathbb{R}^{1,1}$.** Consider Minkowski plane $\mathbb{R}^{1,1}$ with coordinates $x, y$ and metric $g = (dx)^2 - (dy)^2$. Introduce the “light-cone coordinates”

$$x^+ = x + y, \quad x^- = x - y \tag{155}$$

(they are Minkowski analogs of the complex coordinates $z, \bar{z}$ in the Euclidean case $\mathbb{R}^2$).

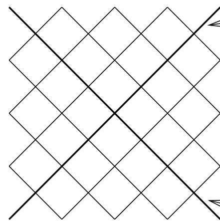



FIGURE 13. Light cone coordinates on $\mathbb{R}^{1,1}$.

In terms of the light-cone coordinates, the metric is: $g = dx^+ dx^-$. Let us write a vector field on $\mathbb{R}^{1,1}$ as

$$v = v^+(x^+, x^-)\partial_+ + v^-(x^+, x^-)\partial_-$$

with $v^\pm$ some functions on $\mathbb{R}^{1,1}$; we denoted $\partial_\pm = \frac{1}{2}(\partial_x \pm \partial_y)$. The condition that $v$ is conformal (94) becomes

$$\partial_- v^+ = 0, \quad \partial_+ v^- = 0, \quad \partial_+ v^+ + \partial_- v^- = \omega \tag{156}$$

Thus, a general conformal vector field on $\mathbb{R}^{1,1}$ is of the form

$$v = v^+(x^+)\partial_+ + v^-(x^-)\partial_- \tag{157}$$

Note that coefficient functions now depend on a single light-cone variable; this is an analog of holomorphic/antiholomorphic coefficient functions in the $\mathbb{R}^2$ case. The conformal factor of $v$ is:

$$\omega = \partial_+ v_+ + \partial_- v_- \tag{158}$$

Thus we have the following.

**Lemma 2.31.** *The Lie algebra of conformal vector fields on $\mathbb{R}^{1,1}$ splits into two copies of the Lie algebra of vector fields on the line:*

$$\mathrm{conf}(\mathbb{R}^{1,1}) = \underbrace{\mathfrak{X}(\mathbb{R}^1)}_{v_+\partial_+} \oplus \underbrace{\mathfrak{X}(\mathbb{R}^1)}_{v_-\partial_-}$$

One can similarly classify (finite) conformal automorphisms of $\mathbb{R}^{1,1}$ – one has the following analog of Lemma 2.21:

**Lemma 2.32.** *A map $\phi\colon \mathbb{R}^{1,1} \to \mathbb{R}^{1,1}$ with components $\phi^+(x^+, x^-)$, $\phi^-(x^+, x^-)$ is a conformal automorphism of $\mathbb{R}^{1,1}$ if and only if one of the two following options holds:*

1. *$\phi^+ = \phi^+(x^+)$, $\phi^- = \phi^-(x^-)$.*
   *I.e., $\phi \in \mathrm{Diff}(\mathbb{R}) \times \mathrm{Diff}(\mathbb{R})$ – a reparametrization of $x^+$ and of $x^-$. The conformal factor in this case is $\Omega = (\partial_+\phi^+)(\partial_-\phi^-)$.*
2. *$\phi^+ = \phi^+(x^-)$, $\phi^- = \phi^-(x^+)$.*
   *I.e., $\phi$ is a composition of a reparametrization of $x^+$ and $x^-$ with a reflection $(x, y) \mapsto (x, -y)$. The conformal factor in this case is $\Omega = (\partial_-\phi^+)(\partial_+\phi^-)$.*

In particular, we have

$$\mathrm{Conf}_0(\mathbb{R}^{1,1}) = \mathrm{Diff}_+(\mathbb{R}) \times \mathrm{Diff}_+(\mathbb{R}) \tag{159}$$

Subscript in $\mathrm{Diff}_+$ stands for orientation-preserving diffeomorphisms. Note that the whole group $\mathrm{Conf}(\mathbb{R}^{1,1})$ has $8 = 2 \times 2 \times 2$ connected components: one can choose to preserve or reverse the orientation along $x_+$ and $x_-$ and whether or not to compose with the reflection $x_+ \leftrightarrow x_-$.

*Remark* 2.33. One can consider $\overline{\mathbb{R}^{1,1}} := S^1 \times S^1$ as a (partial) conformal compactification of $\mathbb{R}^{1,1}$, with respect to a (large) subalgebra of $\mathrm{conf}(\mathbb{R}^{1,1})$ consisting of pairs of vector fields on $\mathbb{R}$ which extend to $S^1 = \mathbb{R} \cup \{\infty\}$. Then, in analogy with (154), one has

(160)
$$\mathrm{Conf}_0(\overline{\mathbb{R}^{1,1}}) = \mathrm{Diff}_+(S^1) \times \mathrm{Diff}_+(S^1) \supset \underbrace{PSL_2(\mathbb{R})}_{\text{Möbius}_+} \times \underbrace{PSL_2(\mathbb{R})}_{\text{Möbius}_-} \simeq \underbrace{SO(2,2)}_{\text{restricted conformal group}}$$

Lecture 9, 9/12/2022

## 2.8. **Moduli space of conformal structures.**

**Definition 2.34.** A (pseudo-)Riemannian manifold $(M, g)$ with metric of signature $(p, q)$ is said to be *conformally flat* if one can find an atlas of coordinate neighborhoods $U_\alpha \subset M$ with local coordinates $\{x^i_\alpha\}$, such that in each chart the metric has the form

$$g|_{U_\alpha} = \Omega_\alpha(x) \cdot ((dx^1_\alpha)^2 + \cdots + (dx^p_\alpha)^2 - (dx^{p+1}_\alpha)^2 - \cdots - (dx^{p+q}_\alpha)^2) \tag{161}$$

with some positive functions $\Omega_\alpha$. Coordinate charts in which the metric satisfies the ansatz (161) are called "isothermal coordinates" on $(M, g)$.

Note that being conformally flat is a local property.

The situation with conformal flatness of manifolds depends on the dimension.

- If $\dim M = 1$ any Riemannian manifold admits local coordinates in which $g = (dx)^2$. I.e. any 1-dimensional Riemannian manifold is flat and, a fortiori, is conformally flat.
- If $\dim M = 2$ (case of main interest for us), any (pseudo-)Riemannian manifold is conformally flat.[41]
- If $\dim M = 3$ a (pseudo-)Riemannian manifold is conformally flat if and only if its Cotton tensor vanishes at every point – this is a certain tensor $C \in \Omega^2(M, TM)$ constructed in terms of derivatives of the Ricci tensor of the metric.
- If $\dim M \geq 4$, a (pseudo-)Riemannian manifold is conformally flat if and only if the Weyl curvature tensor vanishes at every point – this is a certain tensor $W \in \Omega^2(M, \wedge^2 T^*M)$ expressed in terms of the Riemann curvature tensor of $g$.

In particular, (pseudo-)Rimeannian manifolds of dimension $\geq 2$ are conformally flat, while in dimension $\geq 3$ there are local obstructions for conformal flatness.

Given a smooth manifold $M$, one has an action of the Lie group of diffeomorphisms of $M$ on the space of conformal structures:

$$\mathrm{Diff}(M) \circlearrowleft \{\text{conformal structures on } M\} \tag{162}$$

**Definition 2.35.** We call the orbit space $\mathcal{M}_M$ of the action (162) the *moduli space of conformal structures.*

[41] This is not a trivial fact. It can be proven from existence of a solution of the Beltrami equation for the change of coordinates from generic starting coordinates to isothermal coordinates. Originally this statement was proven by Gauss.

Note that the action (162) is not free: for $\xi$ a conformal structure on $M$ there can be a nontrivial stabilizer subgroup

$$\mathrm{Stab}_\xi = \{\phi\colon M \to M \mid \phi^*\xi = \xi\} = \mathrm{Conf}(M,\xi) \quad \subset \mathrm{Diff}(M) \tag{163}$$

– the group of conformal automorphisms of $(M,\xi)$. Also, if $\psi\colon M \to M$ is a diffeomorphism, then $\mathrm{Stab}_\xi$ and $\mathrm{Stab}_{\psi^*\xi}$ are conjugate subgroups of $\mathrm{Diff}(M)$.

*Remark* 2.36. In which sense $\mathcal{M}_M$ is a "space"? There are several ways to understand this object:

(i) As a topological space, with quotient topology.
(ii) As an orbifold – a manifold with "nice" singularities (of the local form $\mathbb{R}^N/\Gamma$, with $\Gamma$ a finite group acting on $\mathbb{R}^N$ properly).
(iii) As a "stack." This is the correct way to talk about $\mathcal{M}_M$, but we will be a bit simple-minded about it and just remember a part of the "stacky data" – that points $[\xi] \in \mathcal{M}_M$ come equipped with stabilizers – subgroups $\mathrm{Stab}_\xi \subset \mathrm{Diff}(M)$.

*Remark* 2.37. The discussion below Definition 2.34 suggests that the moduli space of conformal structures on a manifold of dimension $\geq 3$ is infinite-dimensional, due to the presence of local moduli (Cotton and Weyl tensors). In dimension 2, there are no local moduli: all metric are locally conformally equivalent to the standard flat metric, and only global moduli remain. So, one would expect the $\mathcal{M}_M$ to be "small" (finite-dimensional) in this case. This indeed turns out to be the case, as we discuss below.

2.8.1. *Reminder: almost complex structures and complex structures.*

**Definition 2.38.** An *almost complex structure* on a smooth manifold $M$ is smooth family over $M$ of endomorphisms of (real) tangent spaces that square to $-\mathrm{id}$:

$$J \in \Gamma(M, \mathrm{End}(TM)), \quad \text{s.t. } J_x^2 = -\mathrm{id} \quad \text{for all } x \in M. \tag{164}$$

Consider the matrix of $J_x$ with respect to some basis in $T_xM$. Note that the eigenvalues of a real matrix with square $-\mathrm{id}$ must be $+i$ and $-i$, moreover $+i$ and $-i$ must have the same multiplicity. In particular, if $M$ has an almost complex structure, $\dim M = 2m$ must be even.

Also note that an almost complex structure induces an orientation on $M$: for $(v_1, \ldots, v_m)$ an $m$-tuple of generic vectors in $T_xM$, we say that the $(2m)$-tuple $(v_1, Jv_1, v_2, Jv_2, \ldots, v_m, Jv_m)$ is positively oriented in $T_xM$ (it is a straightforward check that this orientation is independent of the choice of the initial $m$-tuple).

Given an almost complex structure, we have a splitting of the complexified tangent bundle into "holomorphic" and "antiholomorphic" parts:

$$\underbrace{T_\mathbb{C}M}_{\mathbb{C}\otimes TM} = T^{1,0}M \oplus T^{0,1}M. \tag{165}$$

On the right, for each $x \in M$, the complex vector spaces $T_x^{1,0}M$, $T_x^{0,1}M$ are defined as $+i$- and $-i$-eigenspaces of $J_x$, respectively. The splitting (165) induces a dual splitting of the complexified cotangent bundle

$$T^*_\mathbb{C}M = \underbrace{(T^{1,0})^*M}_{K} \oplus \underbrace{(T^{0,1})^*M}_{\bar{K}} \tag{166}$$

we will denote the holomorphic/antiholomorphic cotangent bundles on the right by $K$, $\bar{K}$. Furthermore, the splitting (166) of $k$-forms on $M$ (with complex coefficients) as

$$\Omega^k_{\mathbb{C}}(M) = \bigoplus_{p\geq 0, q\geq 0, p+q=k} \underbrace{\Omega^{p,q}(M)}_{\Gamma(M,\wedge^p K\otimes \wedge^q \bar{K})} \tag{167}$$

We refer to elements of $\Omega^{p,q}$ as $(p,q)$-forms on $M$.

Note that if the (real) dimension of the manifold $M$ is $2m$, then $T^{1,0}_x M$, $T^{0,1}_x M$ have complex dimension $m$ – then we say that $M$ has *complex dimension*

$$\dim_{\mathbb{C}} M = m = \frac{1}{2}\dim M.$$

In particular, one has $\Omega^{p,q}(M) = 0$ if either $p > m$ or $n > m$.

The de Rham differential $d\colon \Omega^\bullet(M) \to \Omega^{\bullet+1}(M)$ splits into two parts: Is this correct for an a.c.s. $J$? no other components?

$$d = \partial + \bar\partial \tag{168}$$

where for $\alpha \in \Omega^{p,q}$,

$$\partial\alpha \colon = \pi_{p+1,q}(d\alpha),\ \bar\partial\alpha \colon = \pi_{p,q+1}(d\alpha) \tag{169}$$

where $\pi_{p,q}$ is the projection of $\Omega(M)$ onto its component $\Omega^{p,q}(M)$. One calls $\partial, \bar\partial$ the holomorphic/antiholomorphic Dolbeault operators. By default, just "Dolbeault operator" is $\bar\partial$.

**Definition 2.39.** An almost complex structure $J$ on a manifold $M$ is *integrable* if one can find an atlas of complex coordinates $(z^j_\alpha, \bar{z}^{\bar{j}}_\alpha)$ on coordinate neighborhoods $U_\alpha$ such that

- $J\partial_{z^j} = i\partial_{z^j}, \quad J\partial_{\bar{z}^{\bar{j}}} = -i\partial_{\bar{z}^{\bar{j}}}$,
- The transition functions between charts are holomorphic: $\frac{\partial z^j_\beta}{\partial \bar{z}^{\bar{j}}_\alpha} = 0$, $\frac{\partial \bar{z}^{\bar{j}}_\beta}{\partial z^j_\alpha} = 0$ for any $j, \bar{j}$ and any two overlapping neighborhoods $U_\alpha$, $U_\beta$ from the atlas.

An integrable almost complex structure $J$ is called a complex structure (not "almost"). A manifold $M$ with a complex structure $J$ is called a complex manifold.

Equivalent characterizations of integrability of $J$ are:

(i) An almost complex structure $J$ is integrable if and only if its Nijenhuis tensor $N_J \in \Omega^2(M, TM)$ vanishes:

$$N_J(X,Y) \colon = -J^2[X,Y] + J[JX,Y] + J[X,JY] - [JX,JY] = 0 \tag{170}$$

for $X, Y \in \mathfrak{X}(M)$. An equivalent restatement of (170) is: for $X^{1,0}, Y^{1,0} \in \Gamma(M, T^{1,0}M)$ two sections of the holomorphic tangent bundle, their Lie bracket is also a section of the holomorphic tangent bundle (the antiholomorphic component vanishes):

$$[X^{1,0}, Y^{1,0}]^{0,1} = 0. \tag{171}$$

(ii) An almost complex structure $J$ is integrable if and only if one has

$$\bar\partial^2 = 0 \tag{172}$$

Equivalently $\partial^2 = 0$ and equivalently $[\partial, \bar\partial] = 0$.

Equivalence of Definition 2.39 and (170) is known as the Newlander-Nirenberg theorem.[42]

On a complex manifold $(M, J)$, the Dolbeault operators written locally in terms of complex coordinates are

$$\partial = \sum_j dz^j \frac{\partial}{\partial z^j}, \quad \bar\partial = \sum_{\bar j} d\bar z^{\bar j} \frac{\partial}{\partial \bar z^{\bar j}} \tag{173}$$

**Lemma 2.40.** *Any almost complex structure $J$ on a manifold $M$ of dimension $\dim M = 2$ is integrable.*

*Proof.* This follows e.g. from (172): $\bar\partial^2$ maps $(p, q)$-forms to $(p, q + 2)$-forms. But there are no forms of degree $(*, \geq 2)$ on a 2-manifold. □

2.8.2. *2d conformal structures (of Riemannian signature) = complex structures.* We will reserve the letter $\Sigma$ for 2-dimensional surfaces, while manifolds of general dimension we denote by $M$.

**Lemma 2.41.** *Fix an oriented 2-dimensional surface $\Sigma$. One has a natural bijection between the following two sets:*

*(i) the set of conformal structures on $\Sigma$ of signature $(2, 0)$ (i.e.* Riemannian *metrics modulo Weyl transformations),*
*(ii) the set of complex structures $J$ on $\Sigma$, compatible with orientation.*

*Proof.* Given a conformal structure $\xi = g/\sim$ on $\Sigma$, we assign to it the complex structure $J\colon T_x\Sigma \to T_x\Sigma$ which maps a tangent vector $u \in T_x\Sigma$ to the vector $v \in T_x\Sigma$ uniquely characterized by the following properties:

- $v$ is orthogonal $u$ (according to any metric $g$ representing $\xi$),
- $v$ and $u$ have the same length (according to any metric $g$ representing $\xi$),
- $(u, v)$ is a positively oriented pair in $T_x\Sigma$.

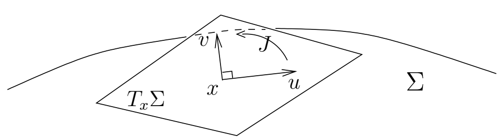



FIGURE 14. Complex structure on a surface.

Here is the inverse construction. Given a complex structure $J$ on $\Sigma$, we assign to it a conformal structure $\xi$ on $\Sigma$, defined as follows: Choose some volume form $\sigma \in \Omega^2(\Sigma)$ compatible with the orientation. Set $g_x(u, v) := \sigma(u, Jv)$. It is a straightforward check that $g_x$ is positive symmetric bilinear form on $T_x\Sigma$, i.e., a metric. The conformal class of $g$ does not depend on a choice of the volume form $\sigma$ (changing $\sigma \to \Omega\sigma$ with $\Omega$ a positive function, induces a change of $g$ by a Weyl transformation). This construction $J \to \xi$ inverts the construction $\xi \to J$ above. □

[42]One can think of it as a complex analog of Frobenius theorem saying that a tangential distribution is involutive if and only if it integrates locally to a foliation. In the case of Newlander-Nirenberg theorem, the distribution in question is complex, $T^{1,0}M \subset T_{\mathbb{C}}M$. In this analogy, a foliation corresponds to local complex coordinates and involutivity is the property (171).

*Remark* 2.42. Under the correspondence between conformal and complex structure of Lemma 2.41, equivalences of conformal and complex surfaces also go into one another: $\phi\colon (\Sigma,\xi)\to(\Sigma',\xi')$ is a conformal diffeomorphism of surfaces equipped with conformal structures if and only if $\phi$ is a biholomorphic map of the corresponding complex surfaces $\phi\colon(\Sigma,J)\to(\Sigma',J')$.

In particular, the correspondence of Lemma 2.41 gives an equivalence of categories, between

(a) the category of surfaces equipped with conformal structure, with morphisms being conformal diffeomorphisms on one side and
(b) the category of complex surfaces and biholomorphic maps on the other side.

*Remark* 2.43. As a consequence of Lemma 2.41, in the case of 2d surfaces, the moduli space of conformal structures (Definition 2.35) and the moduli space of complex structures (183) are the same.

**Definition 2.44.** A smooth manifold $\Sigma$ of dimension 2 equipped with a complex structure is called a Riemann surface. Equivalently, a Riemann surface is a smooth 2-manifold equipped with orientation and conformal structure.[43]

**Definition 2.45.** We will call a Riemann surface *stable* if it does not admit nonzero conformal vector fields. In the case of a Riemann surface with marked points $p_1,\ldots,p_n$, we call it stable if there are no nonzero conformal vector fields which vanish at the point $p_i$.

Can move this definition to some later point, when stability becomes an issue (Teichmüller theory, uniformization,...)

2.8.3. *Deformations of a complex structure. Parametrization of deformations by Beltrami differentials.* Let $(M,J)$ be a complex manifold. A deformation of a complex structure in the class of almost complex structures can be described as a change of the Dolbeault operator $\bar\partial$:

$$\bar\partial\to\underbrace{\bar\partial-\mu}_{\bar\partial_\mu} \tag{174}$$

where the parameter of the deformation

$$\mu\in\Omega^{0,1}(M,T^{1,0}M) \tag{175}$$

is called the *Beltrami differential*; $\bar\mu\in\Omega^{1,0}(M,T^{0,1}M)$ is the complex conjugate object. In local complex coordinates, $\mu$ has the form

$$\mu=\mu^j_{\bar i}(z,\bar z)d\bar z^{\bar i}\frac{\partial}{\partial z^j} \tag{176}$$

using $z,\bar z$ notation here for a smooth function of $z$ – uniformize with Sec. 4.3?

where the coefficient functions $\mu^j_{\bar i}(z,\bar z)$ are arbitrary smooth complex-valued functions on $M$. In (174), we understand $\mu$ as a first-order differential operator $\Omega^{p,q}\to\Omega^{p,q+1}$. The deformed Dolbeault operator written locally thus has the form

$$\bar\partial_\mu=d\bar z^{\bar i}\left(\frac{\partial}{\partial\bar z^{\bar i}}+\mu^j_{\bar i}(z,\bar z)\frac{\partial}{\partial z^j}\right) \tag{177}$$

The deformation (174) is accompanied by the deformation of the holomorphic Dolbeault operator

$$\partial\to\partial-\bar\mu \tag{178}$$

[43] Note that a Riemann surface is not a Riemannian manifold: it does not come with a preferred metric.

where $\bar{\mu}$ is the complex conjugate of the Beltrami differential $\mu$.

Expressed as a deformation of $J$, (174) corresponds to the change

$$J_x \to J_x + 2i(\mu_x - \bar{\mu}_x) \tag{179}$$

for any $x \in M$ (in the first order in $\mu, \bar{\mu}$).

In order for the deformation (174) to be a complex structure (rather than almost complex), it must satisfy the integrability condition

$$(\bar{\partial}_\mu)^2 = 0 \quad \Leftrightarrow \quad \bar{\partial}\mu - \frac{1}{2}[\mu, \mu] = 0 \tag{180}$$

The equation on the right is called the Kodaira-Spencer equation.

*Remark* 2.46. In other words, deformations of a complex structure on a given complex manifold are governed by Maurer-Cartan elements of the differential graded Lie algebra

$$\Omega^{0,*}(M, T^{1,0}M), \;\; \bar{\partial}, \;\; [,] \tag{181}$$

of $(0, q)$-forms with coefficients in the holomorphic tangent bundle, with differential $\bar{\partial}$ and Lie bracket $[,]$ coming as the wedge product of forms tensored with the Lie bracket of $(1, 0)$-vector fields.[44]

We emphasize that the formula (174), with $\mu$ satisfying the Kodaira-Spencer equation (180), describes *finite* deformations of a complex structure, not just infinitesimal (first-order) deformations.

We also remark that if $\dim M = 2$, then the Kodaira-Spencer equation (180) holds trivially (as there are no $(0, 2)$-forms on $M$), cf. Lemma 2.40.

Lecture 10, 9/14/2022

Tangent space to the space of complex structures. The discussion above implies that the tangent space to the space of complex on a manifold $M$ at a complex structure $J$ is the space of $\bar{\partial}$-closed Beltrami differentials (with $\bar{\partial}$-closed condition being the first-order approximation of the Kodaira-Spencer equation (180)):

$$T_J\Big(\text{space of complex structures on } M\Big) \simeq \Omega^{0,1}_{\bar{\partial}-\text{closed}}(M, T^{1,0}M) \tag{182}$$

For the moduli space of of complex structures,[45]

$$\mathcal{M}_M = \{\text{complex structures on } M\}/\mathrm{Diff}(M), \tag{183}$$

the tangent space at the class of $J$ is given by the quotient of (182) modulo the action of (infinitesimal) diffeomorphisms on Beltrami differentials,

$$\mu \sim \mu + \bar{\partial} v^{1,0} \tag{184}$$

with $v^{1,0}$ the projection to $T^{1,0}$ of any vector field on $M$. I.e., one has

$$T_J\mathcal{M}_M = H^{0,1}(M, T^{1,0}M) \tag{185}$$

– the cohomology of the complex (181) in degree one.

[44] We should mention that there is a natural and very deep generalization of deformations of complex structures due to Barannikov-Kontsevich [5]. Here one replaces the dg Lie algebra (181) by a bigger one: $\Omega^{0,p}(M, \wedge^q T^{1,0}M)$, with total grading by $p+q-1$, and considers Maurer-Cartan elements there.

[45] In this subsection we use $\mathcal{M}_M$ for the moduli space of complex (not conformal) structures on $M$. Later, when we specialize to surfaces, there will be no difference between moduli complex and conformal structures, due to Lemma 2.41.

Cotangent space to the space of complex structures (case of surfaces). In the case of a 2-dimensional surface, the $\bar{\partial}$-closed condition in (182) is automatic. In this case, one can describe the *cotangent* space to the space of complex structures as

$$T^*_J\Big(\text{space of complex structures on } \Sigma\Big) = \Omega^{1,0}(\Sigma, K) \simeq \Gamma(\Sigma, K^{\otimes 2}) \tag{186}$$

where $K = (T^{1,0})^*\Sigma$ is the holomorphic cotangent bundle. Elements of (186) are quadratic differentials $\tau$ on $\Sigma$ – tensors written in a local complex coordinate chart as $\tau = f(z,\bar{z})(dz)^2$. The pairing between an element $\mu$ of (182) (a Beltrami differential) and an element $\tau$ of (186) is

$$\int_\Sigma \langle \mu, \tau\rangle \tag{187}$$

where $\langle,\rangle$ is a pairing between vectors $T^{1,0}_x\Sigma$ and covectors $(T^{1,0}_x\Sigma)^*$; thus, $\langle \mu,\tau\rangle$ is a $(1,1)$-form on $\Sigma$, i.e., a 2-form, which can be integrated.

For the cotangent space of the moduli space of complex structures $\mathcal{M}_\Sigma$, (186) implies

$$T^*_J\mathcal{M}_\Sigma \simeq \Omega^{1,0}_{\bar{\partial}-\mathrm{closed}}(\Sigma, K) = \{\text{holomorphic quadratic differentials on } \Sigma\} \tag{188}$$

– the space of *holomorphic* quadratic differentials, locally of the form $\tau = f(z)(dz)^2$ with a *holomorphic* coefficient function.

The holomorphicity condition in (188) arises because we are looking for the elements of (186) annihilating all vectors of the form

$$\bar{\partial}v^{1,0} \in T_J(\text{space of complex structures}),$$

cf. (184).

*Remark* 2.47. In 2d conformal field theory, the stress-energy tensor $T$ is a holomorphic quadratic differential, so it can be seen via (188) as a cotangent vector to the moduli space of complex structures.

2.8.4. *Uniformization theorem.* The following statement is a key result on Riemann surfaces, known as the Uniformization Theorem.

**Theorem 2.48** (Klein-Koebe-Poincaré)**.** *Any simply-connected Riemann surface $(\Sigma,\xi)$ is conformally equivalent to exactly one the following three model surfaces:*

*(i)* $\mathbb{CP}^1$,
*(ii)* $\mathbb{C}$,
*(iii) Open disk $D = \{z \in \mathbb{C} \mid |z| < 1\}$ ("Poincaré disk") or, equivalently (a conformally equivalent model), upper half-plane $\Pi_+ = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$.*

*Remark* 2.49. For each of the model surfaces from Theorem 2.48, there is a metric of constant scalar curvature $R = +1, 0, -1$ representing its conformal class:

(i) $\mathbb{CP}^1$ has a unique metric in its conformal class of scalar curvature $R = +1$ – the Fubini-Study metric $g = \frac{4dzd\bar{z}}{(1+z\bar{z})^2}$.
(ii) $\mathbb{C}$ has a unique up to scaling flat (i.e. $R = 0$) metric in its conformal class, $g = Cdzd\bar{z}$, for any $C > 0$.
(iii) $D$ has a unique metric of scalar curvature $R = -1$ in its conformal class, $g = \frac{4dzd\bar{z}}{(1-z\bar{z})^2}$. Equivalently, $\Pi_+$ has a unique $R = -1$ metric $g = \frac{dzd\bar{z}}{(\mathrm{Im}(z))^2}$.

We also remark that for these distinguished metrics, in cases (i) and (iii) the groups of isometries and all conformal automorphisms coincide (put another way, each conformal automorphism is an isometry).

For a general Riemann surface $\Sigma$ (not necessarily simply-connected), its universal cover $\widetilde{\Sigma}$ inherits a conformal structure from $\Sigma$, is simply-connected and corresponds to one of the model surfaces from Theorem 2.48. The group of covering transformations acts on $\widetilde{\Sigma}$ by conformal automorphisms. Thus, any Riemann surface $\Sigma$ is conformally equivalent to a surface of the form

$$\Sigma^{\mathrm{model}}/\Gamma \tag{189}$$

where $\Gamma$ is the image of a group homomorphism

$$\rho\colon \pi_1(\Sigma) \to \mathrm{Conf}(\Sigma^{\mathrm{model}}) \tag{190}$$

In particular, we need $\Gamma$ to be a discrete subgroup of $\mathrm{Conf}(\Sigma^{\mathrm{model}})$, acting freely on $\Sigma^{\mathrm{model}}$ (so that the quotient (189) is a smooth manifold).

*Remark* 2.50. If we change in (189) the subgroup $\Gamma$ to a conjugate subgroup $\chi\Gamma\chi^{-1}$ with $\chi \in \mathrm{Conf}(\Sigma^{\mathrm{model}})$ a fixed element (or, put another way, we change the homomorphism (190) to a conjugate one, $\rho \mapsto \chi\rho\chi^{-1}$), then the quotient (189) changes to a conformally equivalent surface.

This leads to the following classification of connected Riemann surfaces:

(i) $\mathbb{CP}^1$
(ii) (a) $\mathbb{C}$
  (b) $\mathbb{C}\backslash\{0\}$ or, equivalently, infinite cylinder $\mathbb{C}/\mathbb{Z}$.
  (c) 2-torus $\mathbb{C}/\Lambda$ where $\Lambda = u\mathbb{Z} \oplus v\mathbb{Z} \in \mathbb{C}$ is a lattice spanned by vectors $u, v \in \mathbb{C}$ with $u/v \notin \mathbb{R}$. Using rotation and scaling,[46] one can convert the pair $(u, v)$ to $(1, \tau)$ with $\tau \in \Pi_+$.
(iii) $\Pi_+/\Gamma$ for some $\Gamma \subset PSL_2(\mathbb{R})$ a "Fuchsian group" – a discrete subgroup of $PSL_2(\mathbb{R})$ isomorphic to $\pi_1(\Sigma)$. This case includes all surfaces of genus $g \geq 0$ with $n \geq 0$ boundary circles (the surfaces are considered as open – the boundary circles are not a part of $\Sigma$), with $\chi(\Sigma) = 2 - 2g - n < 0$, and also includes annulus (or finite cylinder) and punctured disk (or semi-infinite cylinder).

Surfaces of types (i), (ii), (iii) above are called, respectively, elliptic, parabolic and hyperbolic. Elliptic surfaces admit (in their conformal class) a unique metric of scalar curvature $+1$, parabolic surfaces – a unique-up-to-scaling flat metric, hyperbolic surfaces – a unique metric of scalar curvature $-1$.

**Example 2.51.** A closed Riemann surface of genus $g \geq 2$ falls into the type (iii) (hyperbolic). Using the standard presentation of the fundamental group of a surface as

$$\pi_1(\Sigma) = \langle \alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g \mid \prod_{i=1}^{g} \alpha_i\beta_i\alpha_i^{-1}\beta_i^{-1} = 1 \rangle$$

we see that its image in $PSL_2(\mathbb{R})$ under $\rho$ is a $2g$-tuple of elements

$$a_1, \ldots, a_g, b_1, \ldots, b_g \quad \in PSL_2(\mathbb{R})$$

[46] In this example, $\rho$ maps $\pi_1(S^1 \times S^1)$ to a lattice $\Lambda$ seen as a subgroup of $\{\text{translations}\} \subset \mathrm{Conf}(\mathbb{C})$. The change of the generators of $\Lambda$ by translation and scaling corresponds to the conjugation of $\rho$, as in Remark 2.50, by rotation and scaling.

subject to a relation

$$\prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} = 1.$$

Moreover, by Remark 2.50, two $2g$-tuples should be considered equivalent if they are related by conjugation by an element $h \in PSL_2(\mathbb{R})$:

$$(a_1, \ldots, a_g, b_1, \ldots, b_g) \sim (ha_1h^{-1}, \ldots, ha_gh^{-1}, hb_1h^{-1}, \ldots, hb_gh^{-1}). \tag{191}$$

2.8.5. *Moduli space $\mathcal{M}_{g,n}$ of complex structures on a surface with $n$ marked points.*

**Definition 2.52.** Fix a smooth closed oriented surface $\Sigma$ of genus $g$. Let $p_1, \ldots, p_n \in \Sigma$ be a collection of pairwise distinct points on $\Sigma$. The moduli space of complex structures on $\Sigma$ with $n$ marked points is the quotient space[47]

$$\mathcal{M}_{g,n} \colon = \{\text{complex structures on } \Sigma\}/\mathrm{Diff}_+(\Sigma, \{p_i\}), \tag{192}$$

where $\mathrm{Diff}_+(\Sigma, \{p_i\})$ stands for the orientation-preserving diffeomorphisms of $\Sigma$ that do not move each of the marked points $p_i$.[48]

There is another version of the moduli space where we quotient by orientation-preserving diffeomorphisms which are allowed to move a marked point to another marked point:

$$\mathrm{Diff}_+^{\text{unordered}}(\Sigma, \{p_i\}) \colon = \{\phi \in \mathrm{Diff}_+(\Sigma) \mid \phi(p_i) = p_{\sigma(i)} \text{ for some } \sigma \in S_n\}.$$

We denote the quotient of the space of complex structures on $\Sigma$ by such diffeomorphisms $\mathcal{M}_{g,n}^{\text{unordered}}$ (unordered marked points), whereas (192) is the moduli space of complex structures with $n$ *ordered* marked points, $\mathcal{M}_{g,n} =: \mathcal{M}_{g,n}^{\text{ordered}}$.

**Definition 2.53.** We call the *universal family* (or Riemann surfaces) the fiber bundle $\mathcal{E}_{g,n}$ over $\mathcal{M}_{g,n}$ where the fiber over the point corresponding to a Riemann surface $\Sigma$ with marked points $\{p_i\}$ is that same surface with same marked points.

is this def/terminology ok?

The idea of Teichmüller theory is to do the quotient (192) in two steps:

1. Take the quotient

$$\{\text{complex structures on } \Sigma\}/\mathrm{Diff}_0(\Sigma, \{p_0\}) =: \mathcal{T}_{g,n} \tag{193}$$

   with respect to the *connected component of identity* in the group of diffeomorphisms preserving the marked points, $\mathrm{Diff}_0 \subset \mathrm{Diff}_+$. The quotient (193) is called the Teichmüller space $\mathcal{T}_{g,n}$.[49] In the case $\chi = 2-2g-n < 0$ (the "stable" case), the Teichmüller space is diffeomorphic to $\mathbb{R}^{6g-6+2n}$. It carries a natural complex structure and several natural metrics.

[47]Again, there are different ways to understand the quotient here: as a topological space with quotient topology ("coarse" moduli space), as an orbifold, as a stack.

[48]Other names used for $\mathcal{M}_{g,n}$ include: "moduli space of conformal structures" (since in 2d, conformal and complex structures correspond to one another), "moduli space of Riemann surfaces" and (in the context of algebraic geometry) "moduli space of (algebraic) curves."

[49]The points of $\mathcal{T}_{g,n}$ correspond to equivalence classes of complex structures on $\Sigma$ (modulo diffeomorphisms fixing the marked points), equipped with a "marking" – a diffeomorphism $\phi \colon \Sigma_{g,n}^{\text{stand}} \to \Sigma$ from a "standard" surface to $\Sigma$ (taking marked points to marked points), where $\phi$ is considered up to isotopy.

2. Take the quotient of (193) by the discrete group of connected components of the diffeomorphism group appearing in (192),

$$\pi_0 \mathrm{Diff}(\Sigma, \{p_i\}) =: \mathrm{pMCG}_{g,n}. \tag{194}$$

This group is known as the "pure mapping class group" of a surface of genus $g$ with $n$ marked points. One has a natural action of $\mathrm{pMCG}_{g,n}$ on the Teichmüller space inherited from the action of diffeomorphisms on complex structures. Thus, we consider the quotient

$$\mathcal{M}_{g,n} = \mathcal{T}_{g,n}/\mathrm{pMCG}_{g,n}. \tag{195}$$

*Remark* 2.54. If one wants to construct the moduli space with unordered punctures, one extra step is needed: a quotient by the symmetric group $S_n$ (which acts by permuting the marked points):

$$\mathcal{M}^{\mathrm{unordered}}_{g,n} = \mathcal{M}_{g,n}/S_n. \tag{196}$$

Another way to write it is directly as a quotient of the Teichmüller space

$$\mathcal{M}^{\mathrm{unordered}}_{g,n} = \mathcal{T}_{g,n}/\mathrm{MCG}_{g,n} \tag{197}$$

by the full (not "pure") mapping class group

$$\mathrm{MCG}_{g,n} :\ = \pi_0 \mathrm{Diff}^{\mathrm{unordered}}_{+}(\Sigma, \{p_i\}). \tag{198}$$

factcheck

*Remark* 2.55. The action of the mapping class group on the Teichmüller space $\mathcal{T}_{g,n}$ is free almost everywhere, except for a discrete set of points where it has a discrete (in fact, finite, for $g, n$ sufficiently large) stabilizer. These points correspond to orbifold singularities of the quotient $\mathcal{M}_{g,n}$.

*Remark* 2.56. The following remark is from [32]. Given a closed surface $\Sigma$ of genus $g \geq 2$, by the Uniformization Theorem (see (189) and Remark 2.50) one has a map

$$\{\text{conformal structures on } \Sigma\} \to \{\text{subgroups } \Gamma \subset PSL_2(\mathbb{R}) \text{ s.t. } \Gamma \simeq \pi_1(\Sigma)\}/PSL_2(\mathbb{R}) \tag{199}$$

More specifically, one has a map

$$\mathcal{T}_{g,0} \xrightarrow{p} \mathrm{Hom}(\pi_1(\Sigma), PSL_2(\mathbb{R}))/PSL_2(\mathbb{R}) \tag{200}$$

In fact, $p$ is injective and its image is

$$\mathrm{im}(p) = \mathrm{Hom}^{df}(\pi_1(\Sigma), PSL_2(\mathbb{R}))/PSL_2(\mathbb{R}) \tag{201}$$

where superscript $d$ stands for "discrete" (so that 1 is not an accumulation point of the image of $\pi_1$), $f$ is for "faithful" (injective). One can also allow marked points – then one gets bijection

$$\mathcal{T}_{g,n} \xrightarrow{\sim} \mathrm{Hom}^{dfp}(\pi_1(\Sigma_{g,n}), PSL_2(\mathbb{R}))/PSL_2(\mathbb{R}) \tag{202}$$

where superscripts $d, f$ are as above and $p$ means "periferal cycles map to parabolic elements of $PSL_2(\mathbb{R})$" (i.e. elements with trace $\pm 2$). On the right hand side, $\Sigma_{g,n}$ is understood as a surface of genus $g$ with $n$ points removed. Thus, one has an identification of the Teichmüller space with a (part of) the moduli space of $PSL_2(\mathbb{R})$-local systems on $\Sigma$. For instance, the formula for the dimension of the Teichmüller space

$$\dim \mathcal{T}_{g,n} = 6g - 6 + 2n \tag{203}$$

follows from (202) immediately.

2.8.6. *Aside: cross-ratio.*

**Definition 2.57.** Given four pairwise distinct points $z_1, z_2, z_3, z_4$ in $\mathbb{CP}^1$, their *cross-ratio* is the number

$$[z_1, z_2 : z_3, z_4] := \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)} = \frac{z_1 - z_3}{z_1 - z_4} : \frac{z_2 - z_3}{z_2 - z_4} \quad \in \mathbb{C}\backslash\{0, 1\}. \tag{204}$$

**Lemma 2.58.** *The cross-ratio is invariant under Möbius transformations:*

$$[Az_1, Az_2 : Az_3, Az_4] = [z_1, z_2 : z_3 : z_4] \tag{205}$$

*for any $A \in PSL_2(\mathbb{C})$. Put another way, the cross-ratio is a function on the open configuration space $C_4(\mathbb{CP}^1)$ of 4 points on $\mathbb{CP}^1$ invariant under the diagonal action of $PSL_2(\mathbb{C})$.*

*Proof.* The Möbius group is generated by translations $z \to z + a$ with $a \in \mathbb{C}$, rotations plus dilations $z \to \lambda z$ with $\lambda \in \mathbb{C}^*$, and the transformation $z \to 1/z$. The expression (204) depends only on differences of $z$'s, so it is invariant under translations. It is a rational function of total homogeneity degree 0, so it is invariant under $z \to \lambda z$. The only thing left to check is that the cross-ratio is invariant under $z \to 1/z$. We have

$$[z_1^{-1}, z_2^{-1} : z_3^{-1}, z_4^{-1}] = \frac{(z_1^{-1} - z_3^{-1})(z_2^{-1} - z_4^{-1})}{(z_1^{-1} - z_4^{-1})(z_2^{-1} - z_3^{-1})} = \frac{(z_3 - z_1)(z_4 - z_2)}{(z_4 - z_1)(z_3 - z_2)} = [z_1, z_2 : z_3, z_4].$$

□

**Definition 2.59.** A an action of a group on a manifold $\rho: G \to \mathrm{Diff}(M)$ is said to be $k$-transitive, for some $k \geq 1$, if any $k$-tuple of distinct points in $M$ can be mapped to any other $k$-tuple of distinct points by acting with some element $g \in G$. Put another way, the action $\rho$ is $k$-transitive if the corresponding diagonal action on the open configuration space of $k$ points, $\rho: G \to \mathrm{Diff}(C_k(M))$ is transitive.

**Lemma 2.60.** *(a) The action of $PSL_2(\mathbb{C})$ on $\mathbb{CP}^1$ by Möbius transformations is 3-transitive.*

*(b) The Möbius transformation sending any one given triple of distinct points in $\mathbb{CP}^1$ to any other triple is unique.*

*Proof.* For (a), it suffices to check that for any triple of distinct points $(z_1, z_2, z_3)$ in $\mathbb{CP}^1$ there exists a Möbius transformation that moves it to the triple $(\infty, 0, 1)$. We can find it as the following composition of simple Möbius transformation:

$$\begin{aligned} (z_1, z_2, z_3) &\xrightarrow{z \to z^{-1}} (z_1^{-1}, z_2^{-1}, z_3^{-1}) \xrightarrow{z \to z - z_1^{-1}} (0, z_2^{-1} - z_1^{-1}, z_3^{-1} - z_1^{-1}) \to \\ &\xrightarrow{z \to z^{-1}} (\infty, \frac{z_1 z_2}{z_{12}}, \frac{z_1 z_3}{z_{13}}) \xrightarrow{z \to z - \frac{z_1 z_2}{z_{12}}} (\infty, 0, \frac{z_1^2 z_{32}}{z_{12} z_{13}}) \xrightarrow{z \to z \cdot \frac{z_{12} z_{13}}{z_1^2 z_{32}}} (\infty, 0, 1). \end{aligned} \tag{206}$$

Here we used a shorthand notation $z_{ij} := z_i - z_j$.

For, (b) it suffices to show that the only Möbius transformation mapping $(0, \infty, 1)$ to $(0, \infty, 1)$ is the identity map $z \to z$. Indeed, for a general Möbius transformation (92), we have

$$(0, \infty, 1) \mapsto (\frac{b}{d}, \frac{a}{c}, \frac{a + b}{c + d}).$$

For the right hand side to be $(0, \infty, 1)$, one needs $b = c = 0$ and $a = d$, thus the transformation (92) is the identity. □

**Lemma 2.61.** *The cross-ratio (204) has the following meaning: start with a quadruple of distinct points $(z_1, z_2, z_3, z_4)$ in $\mathbb{CP}^1$. Find the (unique) Möbius transformation that transforms the quadruple to one of the form $(\varkappa, 1, 0, \infty)$ with some $\varkappa \in \mathbb{CP}^1 \backslash \{0, 1, \infty\}$. Then one has*

$$[z_1, z_2 : z_3, z_4] = \varkappa. \tag{207}$$

*Proof.* By 3-transitivity of the Möbius transformations, it suffices to check that the cross-ratio $[\varkappa, 1 : 0, \infty]$ is $\varkappa$, and this is obvious from the definition (204). □

*Remark* 2.62. The group $S_4$ of permutations of $z_1, z_2, z_3, z_4$ acts on the cross-ratio. Its orbits consists of sextuples of the form

$$\varkappa \sim \frac{1}{\varkappa} \sim 1 - \varkappa \sim \frac{\varkappa}{\varkappa - 1} \sim \frac{1}{1 - \varkappa} \sim \frac{\varkappa - 1}{\varkappa}. \tag{208}$$

More precisely, one has a short exact sequence of groups

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \to S_4 \to S_3,$$

where $\mathbb{Z}_2 \times \mathbb{Z}_2$ (the "Klein four-group") is the symmetries of the cross-ratio – permutations of the four points that don't change it. Explicitly, these symmetries are:

$$[z_1, z_2 : z_3, z_4] = [z_2, z_1 : z_4, z_3] = [z_3, z_4 : z_1, z_2] = [z_4, z_3 : z_2, z_1].$$

Lecture 11, 9/16/2022

2.8.7. *Moduli space $\mathcal{M}_{0,n}$.* A sphere $\Sigma = S^2$ equipped with some conformal structure and $n$ (distinct) marked points is conformally equivalent to the standard $\mathbb{CP}^1$, by the Uniformization Theorem. Under this conformal equivalence, the points are mapped to the $n$-tuple of distinct points $z_1, \ldots, z_n \in \mathbb{CP}^1$. Note that the surfaces $(\mathbb{CP}^1, \{z_i\})$ and $(\mathbb{CP}^1, \{z'_i\})$ are conformally equivalent if and only if one can find a conformal automorphism $\alpha \in \mathrm{Conf}(\mathbb{CP}^1) = PSL_2(\mathbb{C})$ such that $z'_i = \alpha(z_i)$, for $i = 1, \ldots, n$.

Thus, we have the following:

- For $n = 3$, any three points can be mapped to $0, 1, \infty \in \mathbb{CP}^1$ by a Möbius transformation (in a unique way). Thus, all surfaces $(\mathbb{CP}^1, \{z_1, z_2, z_3\})$ are conformally equivalent to the standard one $(\mathbb{CP}^1, \{0, 1, \infty\})$. Hence the moduli space $\mathcal{M}_{0,3}$ is a single point.
- For $n = 4$, a quadruple of points can be mapped by unique Möbius transformation to the quadruple of the form $(\varkappa, 1, 0, \infty)$ where $\varkappa = [z_1, z_2 : z_3, z_4]$ – the cross-ratio. Thus, the surface $(\mathbb{CP}^1, \{z_1, z_2, z_3, z_4\})$ is conformally equivalent the surface of the form $(\mathbb{CP}^1, \{\varkappa, 1, 0, \infty\})$. So, genus 0 Riemann surfaces with 4 marked points up to conformal equivalence are parametrized by a single complex parameter $\varkappa \in \mathbb{CP}^1 \backslash \{0, 1, \infty\}$. Hence, we have
$$\mathcal{M}_{0,4} \simeq \mathbb{CP}^1 \backslash \{0, 1, \infty\} \tag{209}$$
and the coordinate on the moduli space is provided by the cross-ratio of the four marked points on $\Sigma = \mathbb{CP}^1$.
- For $n = 5$, one can map the last 3 out of 5 marked points to $1, 0, \infty$ by a unique Möbius transformation; this transformation moves the first two points to some $\varkappa_1 \neq \varkappa_2 \in \mathbb{CP}^1 \backslash \{0, 1, \infty\}$, with
$$\varkappa_{1,2} = [z_{1,2}, z_3 : z_4, z_5]$$

the cross-ratios. Thus, one has

$$\mathcal{M}_{0,5} \simeq C_2(\mathbb{CP}^1\backslash\{0,1,\infty\}) \tag{210}$$

– the open configuration space of two distinct points $\varkappa_1, \varkappa_2$ in $\mathbb{CP}^1\backslash\{0,1,\infty\}$.

- Similarly, for any $n \geq 3$, one has

$$\mathcal{M}_{0,n} \simeq C_{n-3}(\mathbb{CP}^1\backslash\{0,1,\infty\}) \tag{211}$$

where the surface $(\mathbb{CP}^1, \{z_1, \ldots, z_n\})$ corresponds to the point $(\varkappa_i = [z_i z_{n-2} : z_{n-1}, z_n])_{i=1}^{n-3}$ in the configuration space in the r.h.s. of (211).

- ("Unstable case.") For $n < 3$, one can fix $n$ marked points to standard positions, but by a non-unique Möbius transformation. So, the corresponding moduli can be thought of as a the quotient of a point (the standard $\mathbb{CP}^1$ with $n$ marked points in standard positions) by the subgroup $G_n \subset PSL_2(\mathbb{C})$ fixing the marked points:

$$\mathcal{M}_{0,n} \simeq \mathrm{pt}/G_n \tag{212}$$

– thought of as category with a single object and $G_n$ worth of morphisms, or as a stack. Explicitly, the groups $G_n$ are:

| $n$ | $G_n$ |
|---|---|
| 0 | $PSL_2(\mathbb{C})$ |
| 1 | $\mathrm{Stab}_\infty(PSL_2(\mathbb{C}) \circlearrowleft \mathbb{CP}^1) = \{\text{dilations}\} \oplus \{\text{rotations}\} \oplus \{\text{translations}\} \simeq \mathbb{C}^* \ltimes \mathbb{C}$ |
| 2 | $\mathrm{Stab}_\infty \cap \mathrm{Stab}_0(PSL_2(\mathbb{C}) \circlearrowleft \mathbb{CP}^1) = \{\text{dilations}\} \oplus \{\text{rotations}\} \simeq \mathbb{C}^*$ |

Deligne-Mumford compactification. The moduli space $\mathcal{M}_{0,n}$ with $n \geq 3$ is a smooth noncompact manifold. It admits the so-called Deligne-Mumford compactification $\overline{\mathcal{M}}_{0,n}$ – a stratified complex manifold. The main stratum (of codimension 0) is $\mathcal{M}_{0,n}$. A stratum $D_{S_1,S_2}$ of complex codimension 1 corresponds to a partitioning of the set of marked points $z_1, \ldots, z_n$ into two subsets $S_1, S_2$, each containing $\geq 2$ points; the stratum $D_{S_1,S_2}$ corresponds to "nodal curves/surfaces"[50]

$$(\mathbb{CP}^1, \{S_1, p\}) \cup_p (\mathbb{CP}^1, \{S_2, p\})$$

with "neck" at a point $p$. The moduli space of such nodal surfaces is

$$D_{S_1,S_2} \simeq \mathcal{M}_{0,|S_1|+1} \times \mathcal{M}_{0,|S_2|+1} \tag{213}$$

One adds higher-codimension strata by induction, compactifying the r.h.s. of (213).

We refer to all the strata of $\overline{\mathcal{M}}_{0,n}$ except for the main one ($\mathcal{M}_{0,n}$) as *compactification strata.*

straighten up terminology: nodal curves vs nodal surfaces

**Example 2.63** ($\overline{\mathcal{M}}_{0,4}$)**.** The Deligne-Mumford compactification of the moduli space $\mathcal{M}_{0,4}$ (209) glues back in the points $\varkappa = 0, 1, \infty$ (as compactificatation strata of complex codimension 1), thus

$$\overline{\mathcal{M}}_{0,4} = \underbrace{\mathbb{CP}^1\backslash\{0,1,\infty\}}_{\mathcal{M}_{0,4}} \cup \{0,1,\infty\} = \mathbb{CP}^1. \tag{214}$$

[50]There are competing terminologies for complex manifolds of complex dimension 1 – "curves" (mainly, in algebraic geometry literature) and "surfaces" (differential geometry literature). We will try to be consistent, sticking with "surfaces." In particular, instead of "nodal curve" (a standard term in algebraic geometry), we say "nodal surface."

E.g., the point $\varkappa = 0$ corresponds to the asymptotic situation for a surface $\mathbb{CP}^1, \{z_1, z_2, z_3, z_4\}$ where $z_1$ approaches $z_3$. Note that such configuration can be mapped by a Möbius transformation to one where $z_1, z_3$ stay at finite distance from each other but $z_2$ and $z_4$ approach one another. The limiting configuration is described by a nodal surface – two $\mathbb{CP}^1$'s, one containing $z_1, z_3$ and $p$ (the "neck") and the other containing $z_2, z_4$ and $p$. This singular surface is acted on by $PSL_2(\mathbb{C}) \times PSL_2(\mathbb{C})$ – independent Möbius transformations of both $\mathbb{CP}^1$'s. Thus, on both components of the singular surface, there are no moduli (3 marked points can be brought into standard position), so the stratum is $\mathcal{M}_{0,3} \times \mathcal{M}_{0,3} = \mathrm{pt}$.

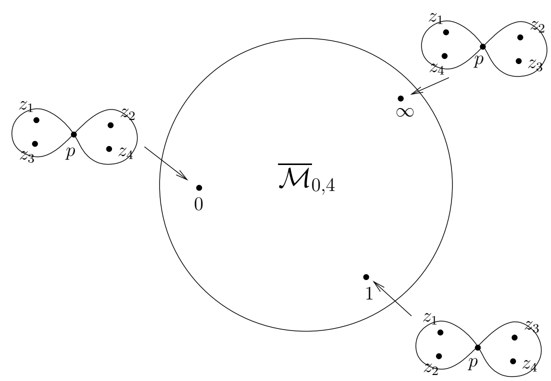



FIGURE 15. Deligne-Mumford compactification of $\mathcal{M}_{0,4}$. We are drawing the nodal surfaces corresponding to the compactification strata $\varkappa = 0$, $\varkappa = 1$,$\varkappa = \infty$. Put another way, the universal family (Definition 2.53) degenerates at these three points and we draw the degenerate fibers over them.

**Example 2.64** (Higher-codimension strata)**.** In the Deligne-Mumford compactification of $\mathcal{M}_{0,5}$, one can consider the $\mathrm{codim}_\mathbb{C} = 1$ compactification stratum of the form

$$\mathcal{M}_{0,3} \times \mathcal{M}_{0,4}, \tag{215}$$

corresponding to partitioning the marked points as $\{z_1, z_2\} \cup \{z_3, z_4, z_5\}$, i.e., nodal surfaces of the form

$$(\mathbb{CP}^1, \{z_1, z_2, p\}) \cup_p (\mathbb{CP}^1, \{p, z_3, z_4, z_5\}) \tag{216}$$

(corresponding to either $z_1$ approaching $z_2$ or, as alternative viewpoint, corresponding to $z_3, z_4, z_5$ colliding together). The right factor in (215) also should be further compactified, by adjoining to the product, e.g., the stratum $\mathcal{M}_{0,3} \times \mathcal{M}_{0,3} \times \mathcal{M}_{0,3}$ corresponding to surfaces with two necks, of the form

$$(\mathbb{CP}^1, \{z_1, z_2, p\}) \cup_p (\mathbb{CP}^1, \{p, z_3, q\}) \cup_q (\mathbb{CP}^1, \{q, z_4, z_5\}) \tag{217}$$

of complex codimension 2 (as a stratum in $\overline{\mathcal{M}}_{0,5}$; it corresponds to a stratum of complex codimension 1 in the right factor of (215)).

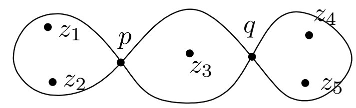



FIGURE 16. Nodal surface with two "necks," corresponding to a stratum in $\overline{\mathcal{M}}_{0,5}$ of complex codimension two.

*Remark* 2.65. The construction of Deligne-Mumford compactification extends to $\mathcal{M}_{g,n}$ with nonvanishing genus $g$. Then one has compactification strata (of complex codimension 1) of two types:

1. Strata isomorphic to $\mathcal{M}_{g_1,n_1+1} \times \mathcal{M}_{g_2,n_2+1}$ with $g_1+g_2=g$, $n_1+n_2=n$ – this is essentially the same construction as above, where not only marked points but also genus is distributed between the two components of the nodal surface.
2. Strata isomorphic to $\mathcal{M}_{g-1,n+2}$ – this corresponds to introducing a neck on handle, thus trading one handle for two extra marked points.

2.8.8. *Moduli space* $\mathcal{M}_{1,0}$. A 2-torus with conformal structure, is, by Uniformization Theorem, conformally equivalent to

$$\Sigma_\Lambda := \mathbb{C}/\Lambda \tag{218}$$

with

$$\Lambda = \mathrm{span}_{\mathbb{Z}}(u,v) \tag{219}$$

a lattice in $\mathbb{C}$ spanned by two non-collinear[51] vectors $u,v\in\mathbb{C}$. Since the order of $(u,v)$ does not matter, we may assume that $\mathrm{Im}(v/u)>0$. Surfaces (218) are conformally equivalent for lattices $\Lambda$, $\Lambda'$ if and only if the lattices are related by rotation and scaling. There is a unique rotation+scaling that transforms $v$ to 1. Thus, the surface (218) is equivalent to a surface of the form

$$T_\tau := \mathbb{C}/\Lambda_\tau \tag{220}$$

where $\Lambda_\tau := \mathrm{span}_{\mathbb{Z}}(\tau,1)$ with $\tau=\frac{u}{v}\in\Pi_+$.

Choosing a different basis in $\Lambda$,

$$(u,v)\mapsto(u'=au+bv, v'=cu+dv) \quad \text{with } \begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL_2(\mathbb{Z}),$$

one obtains that tori $T_\tau$ and $T_{\tau'}$ are equivalent if and only if

$$\tau'=\frac{a\tau+b}{c\tau+d} \quad \text{with } \begin{pmatrix} a & b \\ c & d \end{pmatrix}\in PSL_2(\mathbb{Z}). \tag{221}$$

Thus, we have the following.

**Theorem 2.66.** *The moduli space of complex structures on a 2-torus with no marked points is*

$$\mathcal{M}_{1,0}=\Pi_+/PSL_2(\mathbb{Z}). \tag{222}$$

*I.e. any complex torus is conformally equivalent to a torus of the form* $T_\tau=\mathbb{C}/(\mathbb{Z}\oplus\tau\mathbb{Z})$ *where the* modular parameter $\tau\in\Pi_+/PSL_2(\mathbb{C})$ *provides a complex coordinate on* $\mathcal{M}_{1,0}$.

[51]Otherwise, the quotient is not diffeomorphic to the 2-torus.

*Remark* 2.67. The standard way to choose a *fundamental domain*[52] $\mathcal{D} \subset \Pi_+$ for the action of $PSL_2(\mathbb{Z})$ on $\Pi_+$ is the following:

$$\mathcal{D} = \{z \in \mathbb{C} \mid \mathrm{Re}(z) \in [-\frac{1}{2}, \frac{1}{2}],\ |z| \geq 1\} \tag{223}$$

The action of $PSL_2(\mathbb{Z})$ identifies points on the boundary of $\mathcal{D}$ as follows:

$$\mathcal{M}_{1,0} \simeq \frac{\mathcal{D}}{-\frac{1}{2} + iy \sim \frac{1}{2} + iy \ \text{ for } y \geq \frac{\sqrt{3}}{2}, \quad e^{i\theta} \sim e^{i(\pi-\theta)} \ \text{ for } \theta \in [\frac{\pi}{3}, \frac{2\pi}{3}]} \tag{224}$$

Points $\tau = i$ and $\tau = e^{\pi i/3} \sim e^{2\pi i/3}$ in $\mathcal{D}$ have nontrivial stabilizers ($\mathbb{Z}_2$ and $\mathbb{Z}_3$, respectively) under the action of $PSL_2(\mathbb{Z})$ and correspond to orbifold singularities in $\mathcal{M}_{1,0}$.

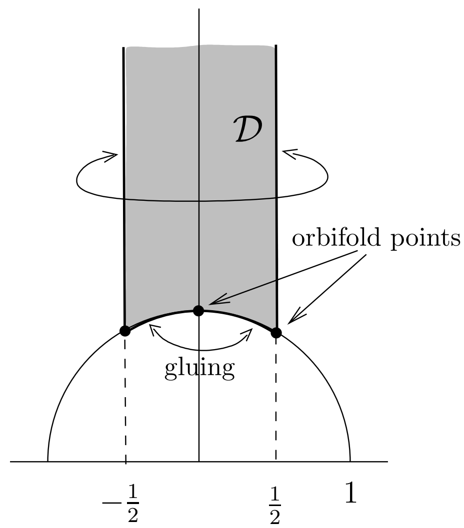



FIGURE 17. $\mathcal{M}_{1,0}$

*Remark* 2.68. Each complex torus $T_\tau$ has a nontrivial group of conformal automorphisms – translations by vectors in $T_\tau$,

$$\mathrm{Conf}(T_\tau) = T_\tau. \tag{225}$$

*Remark* 2.69. The moduli space $\mathcal{M}_{1,1}$ of complex tori with a single marked point can be identified with $\mathcal{M}_{1,0}$: one can convert the underlying complex torus to a standard one $T_\tau$ and then move the marked point to the standard position (say, $z = 0$) by a translation from (225).

2.8.9. *The mapping class group of a surface.* We refer to [11] as an excellent detailed introduction to the subject of mapping class groups of surfaces. Here we just want to give some simple examples.

[52] I.e. a subset of $\Pi_+$ such that each $PSL_2(\mathbb{Z})$-orbit intersects $\mathcal{D}$ and if two points in $\mathcal{D}$ are in the same orbit, then they are boundary points of $\mathcal{D}$.

**Example 2.70.** The mapping class group of a 2-torus (seen as a smooth manifold $\mathbb{R}^2/\mathbb{Z}^2$ with no marked points) is

$$\mathrm{MCG}_{1,0} = SL_2(\mathbb{Z}) \tag{226}$$

– elements of the mapping class group can be represented by linear sutomorphisms $\mathbb{R}^2 \to \mathbb{R}^2$ preserving the lattice $\mathbb{Z}^2$.

**Example 2.71.** The mapping class group of the sphere $S^2$ with $n$ marked points is the "spherical braid group on $n$ strands," i.e.,

$$\mathrm{MCG}_{0,n} = \pi_1 C_n^{\mathrm{non-ordered}}(S^2) \tag{227}$$

– the fundamental group of the open configuration space of $n$ non-ordered points on $S^2$.

The version for the pure mapping class group (respectively, pure spherical braid group on $n$ strands) is:

$$\mathrm{pMCG}_{0,n} = \pi_1 C_n^{\mathrm{ordered}}(S^2). \tag{228}$$

**Example 2.72.** The mapping class group of the annulus relative to the boundary (i.e. $\pi_0$ of diffeomorphisms of the annulus not moving the boundary points) is

$$\mathrm{MCG}(\mathrm{Ann}, \partial\mathrm{Ann}) \simeq \mathbb{Z} \tag{229}$$

This group is generated by the *Dehn twist*. Thinking of Ann as the domain $\{z \in \mathbb{C} \mid r \leq |z| \leq R\}$, the Dehn twist can be represented a diffeomorphism[53]

$$\begin{array}{ccc} \mathrm{Ann} & \to & \mathrm{Ann} \\ z & \mapsto & e^{2\pi i \frac{|z|-r}{R-r}} \cdot z \end{array} \tag{230}$$

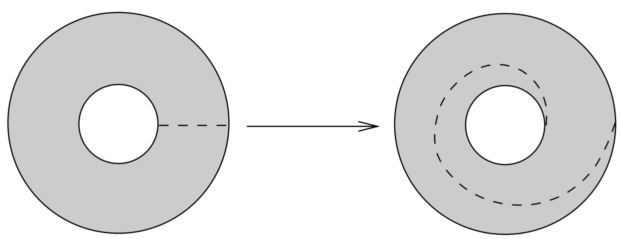


FIGURE 18. Dehn twist (illustrated by the image of the dashed curve).

For general genus $g$ and number $n$ of marked points, one can write a presentation of the mapping class group $\mathrm{MCG}_{g,n}$ with two types of generators:

- Dehn twists along a finite collection of nonseparating closed simple curves on the surface.[54]

[53]Equivalently, thinking of Ann as a cylinder $[0,1] \times S^1$, one can represent the Dehn twist by the diffeomorphism $(t, \theta) \mapsto (t, \theta + 2\pi t)$.

[54]The Dehn twist along a closed simple curve $\gamma$ on a surface $\Sigma$ is the diffeomorphism that is identity everywhere except in in a small tubular neighborhood $U_\gamma \subset \Sigma$ of $\gamma$; in $U_\gamma$ (which is diffeomorphic to an annulus or, equivalently, a cylinder), one performs the standard Dehn twist (Figure 18).

- "Dehn half-twists" which permute pairs of marked points.

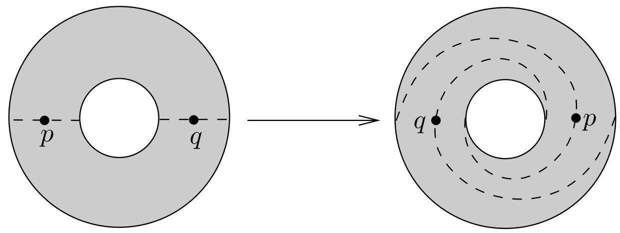



FIGURE 19. Dehn half-twist permuting the marked points $p$ and $q$.

These generators are subject to a set of relations. We refer to [11] for the details.

For the *pure* mapping class group $\mathrm{pMCG}_{g,n}$, one can make do with just Dehn twists (without half-twists). E.g., $\mathrm{pMCG}_{0,n}$ can be generated by Dehn twists along curves encircling pairs of marked points.

Let us also mention the following result, helpful in computing mapping class groups. Let $\mathrm{MCG}^{\pm}(\Sigma) = \pi_0 \mathrm{Diff}(\Sigma)$ – the group of isotopy classes of diffeomorphisms that either preserve or reverse the orientation of $\Sigma$. Note that there is a natural action of $\mathrm{MCG}^{\pm}$ on the fundamental group $\pi_1(\Sigma)$ (by pushing loops along the diffeomorphism).

**Theorem 2.73** (Dehn-Nielsen-Baer)**.**

$$\mathrm{MCG}^{\pm}(\Sigma) \simeq \mathrm{Out}(\pi_1(\Sigma)) \tag{231}$$

*where for $G$ a group,* $\mathrm{Out}(G)\colon = \mathrm{Aut}(G)/\mathrm{Inn}(G)$ *is the group of "outer automorphism" – the quotient of all automorphisms by inner ones.*

Then, the usual mapping class group (classes of orientation preserving diffeomorphisms) is an index two subgroup of (231).

Lecture 12, 9/19/2022

## 3. Symmetries in classical field theory, stress-energy tensor

3.1. **Classical field theory, Euler-Lagrange equations.** Recall (cf. Section 1.5) that a classical field theory on a (pseudo-)Riemannian $n$-manifold $M$ is determined by the following data:

(a) The space of fields
$$\mathrm{Fields}_M = \Gamma(M,E) \quad ni\phi$$
with $E \to M$ the field bundle.

(b) The action functional
$$S_{M,g}(\phi) = \int_M L(\phi, \partial\phi, \ldots; g) \tag{232}$$

The Lagrangian density $L$ in (232) is an $n$-form on $M$ depending on fields and on the metric on $M$. Recall that one can introduce the "variational bicomplex" (see [1] for details)

$$\Omega^{p,q}_{\mathrm{loc}}(M \times \mathrm{Fields}_M) \tag{233}$$

of "local" $q$-forms on $\mathrm{Fields}_M$ valued in $p$-forms on $M$; locality means that for $\omega \in \Omega^{p,q}_{\mathrm{loc}}$, its value at $x \in M$ depends on $x$ and the jet of fields at $x$ (but not on the values of the fields away from $x$). The bicomplex $\Omega^{\bullet,\bullet}_{\mathrm{loc}}(M, \mathrm{Fields}_M)$ comes with

- the "vertical" differential $\delta\colon \Omega^{p,q}_{\mathrm{loc}} \to \Omega^{p,q+1}_{\mathrm{loc}}$ – the de Rham operator on the space of fields.[55]
- the "horizontal" differential $d\colon \Omega^{p,q}_{\mathrm{loc}} \to \Omega^{p+1,q}_{\mathrm{loc}}$ – the de Rham operator on $M$,[56]

The two differentials $d, \delta$ both square to zero and anticommute with each other.[57]

In this language, the Lagrangian density in (232) is an element

$$L \in \Omega^{n,0}_{\mathrm{loc}}(M \times \mathrm{Fields}_M). \tag{234}$$

Covariance. We are assuming "covariance" of the given classical field theory (as assignment of action functionals to (pseudo-)Riemannian $n$-manifolds): for a diffeomorphism $m\colon M \xrightarrow{\sim} M'$ of smooth manifolds, one has

$$S_{M,g}(\phi) = S_{M',(m^{-1})^* g}((m^{-1})^*\phi) \tag{235}$$

3.1.1. *Euler-Lagrange equations.* The Euler-Lagrange equation (or "equation of motion") is the condition on a field $\phi \in \mathrm{Fields}_M$ that for any curve $\phi_t$ in $\mathrm{Fields}_M$ such that $\phi_0 = \phi$ and such that $\phi_t$ coincides with $\phi$ in a neighborhood of the boundary $\partial M \subset M$, one has

$$\left.\frac{d}{dt}\right|_{t=0} S(\phi_t) = 0. \tag{236}$$

(Put another way, Fréchet derivatives of $S$ at $\phi$ in the directions given by fluctuations of $\phi$ supported away from $\partial M$ vanish.) This condition is a equivalent to a PDE on $\phi$. We will sometimes shorten "Euler-Lagrange equation" to just "EL."

More explicitly, applying the de Rham derivative in fields to $S$ (or in other words, considering the variation of $S$ with respect to the variation of the field), one can write the result in the form

$$\delta S = \int_M \mathcal{EL}(\phi)_a \delta\phi^a + \int_{\partial M} \underline{\alpha} \tag{237}$$

where $\mathcal{EL}(\phi)_a\delta\phi^a \in \Omega^{n,1}_{\mathrm{loc}}(M \times \mathrm{Fields}_M)$ is an expression containing variations of the field not hit by derivatives along $M$ (labels $a$ refer to the local trivialization of the field bundle $E$). To obtain an answer in this form, one has to integrate by parts (to move the geometric derivatives from $\delta\phi$), which results in the appearance of the boundary term in (237). The integrand $\underline{\alpha}$ in the boundary term of (237) is an element of $\Omega^{n-1,1}_{\mathrm{loc}}(M \times \mathrm{Fields}_M)$; the whole boundary term $\int_{\partial M} \underline{\alpha}$ is a 1-form

---

[55] Locally, in a local trivialization of $E$, one has $\delta = \sum_{r\geq 0} \delta\phi^a_{i_1\cdots i_r} \frac{\partial}{\partial \phi^a_{i_1\cdots i_r}}$ where $a$ labels the field components (coordinates in the fiber of the field bundle $E$); $\phi^a_{i_1\cdots i_r} = \partial_{i_1}\cdots\partial_{i_r}\phi^a$ are components of the $r$-th jet of the field.

[56] It is understood that $d$ also acts on fields. Locally, one has $d = dx^i \left( \frac{\partial}{\partial x^i} + \sum_{r\geq 0} \phi^a_{ii_1\cdots i_r} \frac{\partial}{\partial \phi^a_{i_1\cdots i_r}} + \sum_{r\geq 0} (\delta\phi^a_{ii_1\cdots i_r}) \frac{\partial}{\partial(\delta\phi^a_{i_1\cdots i_r})} \right)$.

[57] Another viewpoint on the bicomplex $\Omega^{\bullet,\bullet}_{\mathrm{loc}}$ is as follows. Consider the composition of maps $M \times \underbrace{\Gamma(M,E)}_{\mathrm{Fields}_M} \xrightarrow{\mathrm{id}\times j_\infty} M \times \Gamma(M, \mathrm{Jet}_\infty E) \xrightarrow{\mathrm{ev}} \mathrm{Jet}_\infty E$ where $j_\infty$ takes the jet of a section of $E$ at each point of $M$; ev is the evaluation of a section at a point of $M$. Consider the complex of forms $\Omega(\mathrm{Jet}_\infty E)$ on the total space of the jet bundle. Then $\Omega_{\mathrm{loc}}(M \times F_M)$ is the image of $\Omega(\mathrm{Jet}_\infty E)$ under the pullback $(\mathrm{ev} \circ (\mathrm{id} \times j_\infty))^*$.

on $\mathrm{Fields}_M$, known as the Noether 1-form (thus, $\underline{\alpha}$ is the density of the Noether 1-form).

Expressed in terms of densities, the equation (237) is:

$$\delta L = (-1)^n \Big(\mathcal{EL}_a(\phi)\delta\phi^a + d\underline{\alpha}\Big) \tag{238}$$

The Euler-Lagrange equation then is:

$$\mathcal{EL}_a(\phi) = 0$$

with $\mathcal{EL}_a(\phi)$ an expression in the jet of the field appearing in (237).

<u>Aside on source forms.</u> In the variational bicomplex one can consider the subspace of "source forms"

$$\Omega^{n,1\ \mathrm{source}}_{\mathrm{loc}} \subset \Omega^{n,1}_{\mathrm{loc}} \tag{239}$$

– the set of elements of the form $\omega = \omega_a(x,\phi,\partial\phi,\ldots)\delta\phi^a$, i.e., not depending on variations of derivatives of fields $\delta\phi^a_{i_1\cdots i_r}$ for $r \geq 1$. Then one has a straightforward lemma.

**Lemma 3.1.**

$$\Omega^{n,1}_{\mathrm{loc}} = \Omega^{n,1\ \mathrm{source}}_{\mathrm{loc}} \oplus d\left(\Omega^{n-1,1}_{\mathrm{loc}}\right) \tag{240}$$

*I.e., any $(n,1)$-form $\beta$ can be written in a unique way as $\beta = \omega + d\eta$ with $\omega$ a source form.*

*Proof.* It is proven straightforwardly, by moving the derivatives from $\delta\phi$ to its prefactor in $\beta$, at the cost of adding a $d$-exact term:

$$\begin{aligned} &\beta_a^{i_1\cdots i_r}(x,\phi,\partial\phi,\cdots)\delta\phi^a_{i_1\cdots i_r} = \\ &\qquad = -(\partial_{i_r}\beta_a^{i_1\cdots i_r})\delta\phi^a_{i_1\cdots i_{r-1}} + \underbrace{\partial_{i_r}(\beta_a^{i_1\cdots i_r}\delta\phi^a_{i_1\cdots i_{r-1}})}_{d\iota_{\partial_{i_r}}(\beta_a^{i_1\cdots i_r}\delta\phi^a_{i_1\cdots i_{r-1}})} \\ &= \cdots = (-1)^r(\partial_{i_1}\cdots\partial_{i_r}\beta_a^{i_1\cdots i_r})\delta\phi^a + d\left(\sum_{k=0}^{r-1}(-1)^k\iota_{\partial_{i_{r-k}}}((\partial_{i_r}\cdots\partial_{i_{r-k+1}}\beta_a^{i_1\cdots i_r})\delta\phi^a_{i_1\cdots i_{r-k-1}})\right)\end{aligned} \tag{241}$$

One extends this computation by $\mathbb{R}$-linearity to general $\beta$'s. This gives a splitting $\beta = \omega + d\eta$, with $\omega$ a source form. The fact that the splitting is unique follows from the observation that $d$ of a field-dependent $(*,1)$-form will necessarily contain a term depending on $\delta\phi^a_{i_1\cdots i_r}$ with $r \geq 1$. Thus, $\Omega^{n,1\ \mathrm{source}}_{\mathrm{loc}} \cap d(\Omega^{n-1,1}_{\mathrm{loc}}) = 0$. □

Equation (238) is an application of this lemma to $\beta = \delta L \in \Omega^{n,1}$; Euler-Lagrange equation says that $[\delta L]_{\mathrm{source}} = 0$ where $[\cdots]_{\mathrm{source}}$ is the projection onto the source forms.

Add example: classical mechanics

**Example 3.2** (Free massive scalar field)**.** Let $(M,g)$ be a (pseudo-)Riemannian $n$-manifold. The fields are smooth functions on $M$, $\phi \in C^\infty(M)$ (i.e., the field bundle is $E = M \times \mathbb{R} \to M$ – the trivial rank one bundle over $M$). The action is

$$S(\phi) = \int_M \underbrace{\frac{1}{2}d\phi \wedge *d\phi + \frac{m^2}{2}\phi^2 d\mathrm{vol}_g}_{L} \tag{242}$$

with $*$ the Hodge star associated with the metric $g$ and $d\mathrm{vol}_g = *1$ the metric volume element; $m \geq 0$ is a fixed number ("mass"). It is obvious that the assignment (242) satisfies the covariance property (235) – essentially because the action is written in terms of natural geometric operations on forms.

We have

$$\delta S = \int_M (-1)^{n+1} \underbrace{d\delta\phi \wedge *d\phi}_{d(\delta\phi\wedge *d\phi)+\delta\phi\wedge d*d\phi} + (-1)^n m^2 \delta\phi\, \phi\, d\mathrm{vol}_g = \tag{243}$$

$$= \int_M \underbrace{d\mathrm{vol}_g (\Delta + m^2)\phi \wedge \delta\phi}_{\mathcal{EL}(\phi)\delta\phi} + d(\underbrace{*d\phi \wedge \delta\phi}_{\underline{\alpha}}) = \int_M d\mathrm{vol}_g(\Delta+m^2)\phi\wedge\delta\phi + \underbrace{\int_{\partial M} *d\phi \wedge \delta\phi}_{\text{Noether 1-form}}.$$

Thus, the Euler-Lagrange equation is the linear PDE

$$(\Delta + m^2)\phi = 0 \tag{244}$$

with $\Delta = - * d * d$ the Laplace-Beltrami operator.

Note that if we start instead with the action

$$S(\phi) = \int_M \frac{1}{2} d\phi \wedge *d\phi + V(\phi) d\mathrm{vol}_g$$

with $V$ some smooth function on $\mathbb{R}$ (the "potential"), then by repeating the computation we see that the Noether 1-form $\alpha$ (and its density $\underline{\alpha}$) does not change from (243) but the Euler-Lagrange equation becomes a nonlinear PDE

$$\Delta\phi + V'(\phi) = 0. \tag{245}$$

**Example 3.3** (Yang-Mills theory)**.** Fix a Lie group $G$ with a nondegenerate ad-invariant quadratic form $\langle , \rangle$ on its Lie algebra $\mathfrak{g}$. Let $(M, g)$ be a (pseudo-)Riemannian $n$-manifold. The fields of the theory are pairs $(\mathcal{P}, A)$ consisting of a principal $G$-bundle $\mathcal{P}$ over $M$ and a connection $A$ in $\mathcal{P}$. The action of Yang-Mills theory is

$$S(A) = \int_M \frac{1}{2} \langle F_A \overset{\wedge}{,} *F_A \rangle \tag{246}$$

where $F_A \in \Omega^2(M, \mathrm{ad}(\mathcal{P}))$ is the curvature 2-form of the connection $A$; $*$ is again the Hodge star. In a local trivialization of $\mathcal{P}$, $A$ is represented by a $\mathfrak{g}$-valued 1-form on $M$ (or rather on the trivializing neighborhood $U \subset M$) and $F_A$ is represented by the $\mathfrak{g}$-valued 2-form $dA + \frac{1}{2}[A, A]$.

The corresponding Euler-Lagrange equation is:

$$d_A * F_A = 0 \tag{247}$$

with $d_A \colon \Omega^\bullet(M, \mathrm{ad}(\mathcal{P})) \to \Omega^{\bullet+1}(M, \mathrm{ad}(\mathcal{P}))$ the covariant derivative operator associated with $A$. The equation (247) is a nonlinear PDE (for nonabelian $G$) – the "Yang-Mills equation."

In the special case $G = \mathbb{R}$, the Yang-Mills theory drastically simplifies (in this case, it is called electrodynamics or Maxwell theory): fields are global 1-forms $A \in \Omega^1(M)$, the action is $S(A) = \frac{1}{2}\int_M dA \wedge *dA$ and the Euler-Lagrange equation becomes the Maxwell equation

$$d * dA = 0$$

– a linear PDE.

**Example 3.4** (Chern-Simons theory)**.** Fix a simply connected Lie group $G$ with a nondegenerate ad-invariant bilinear form $\langle , \rangle$ on its Lie algebra $\mathfrak{g}$. Let $M$ be an oriented 3-manifold. The fields of the theory are connections $A$ in the trivial principal bundle $\mathcal{P} = M \times G$ over $M$.[58] Since $\mathcal{P}$ is trivial, connections can be identified with $\mathfrak{g}$-valued 1-forms, $A \in \Omega^1(M, \mathfrak{g})$. The action is defined as

$$S(A) = \int_M \frac{1}{2}\langle A \overset{\wedge}{,} dA\rangle + \frac{1}{6}\langle A \overset{\wedge}{,} [A, A]\rangle \tag{248}$$

We have

$$\delta S = \int_M -\frac{1}{2}\langle \delta A, dA\rangle - \frac{1}{2}\underbrace{\langle A, d\delta A\rangle}_{-d\langle A,\delta A\rangle + \langle dA, \delta A\rangle} -\frac{1}{2}\langle \delta A, [A, A]\rangle = \tag{249}$$

$$= \int_M -\langle \delta A, dA + \frac{1}{2}[A, A]\rangle + \int_{\partial M} \frac{1}{2}\langle A, \delta A\rangle = \int_M -\langle \delta A, F_A\rangle + \int_{\partial M} \underbrace{\frac{1}{2}\langle A, \delta A\rangle}_{\alpha}$$

where $F_A = dA + \frac{1}{2}[A, A] \in \Omega^2(M, \mathfrak{g})$ is the curvature 2-form. Thus, the Euler-Lagrange equation is

$$F_A = 0 \tag{250}$$

– zero-curvature (or "flatness") condition for the connection field $A$.

Note that the action (248) does not depend on a metric on $M$ (it only depends on the orientation, no other geometric structure is used). – It is an example of a topological field theory (in the sense of Schwarz).

**Example 3.5** (Nonlinear sigma model)**.** Fix a (pseudo-)Riemannian manifold $(X, h)$ (the target) and let $(M, g)$ be a (pseudo-)Riemannian $n$-manifold (the source). The fields are smooth maps $\Phi\colon M \to X$ (one can think of them as section of the field bundle $E = M \times X \to X$ – the trivial fiber bundle with fiber $X$) and the action is

$$S(\Phi) = \int_M \frac{1}{2}\langle d\Phi \overset{\wedge}{,} *d\Phi\rangle_{\Phi^* h} \tag{251}$$

where $* = *_g\colon \Omega^\bullet(M) \to \Omega^{n-\bullet}$ is the Hodge star associated with the source metric $g$; $d\Phi \in \Omega^1(M, \Phi^* TX)$ is the differential of the map $\Phi$, $\langle , \rangle_{\Phi^* h}$ is the fiberwise metric on the vector bundle $\Phi^* TX \to M$ coming from the pullback of the target metric. Using local coordinates $u^a$ on the target and local coordinates $x^i$ on the source, the action (251) can be written as

(252)
$$S(\Phi) = \int_M \frac{1}{2} h_{ab}(\Phi)\, d\Phi^a \wedge *d\Phi^b = \int_M \frac{1}{2} g^{ij}(x)\, h_{ab}(\Phi(x))\, \partial_i \Phi^a(x)\, \partial_j \Phi^b(x)\, d\mathrm{vol}_g.$$

The variation of the action is:

(253)
$$\delta S = \int_M (-1)^n \Big(\frac{1}{2}\partial_c h_{ab}(\Phi) d\Phi^a \wedge * d\Phi^b - \underbrace{h_{ab}(\Phi) d\delta\Phi^a \wedge *d\Phi^b}_{d(h_{ab}(\Phi)\delta\Phi^a \wedge *d\Phi^b) + (-1)^n d(h_{ab}(\Phi) * d\Phi^b)\wedge \delta\Phi^a}\Big)$$

$$= \int_M \Big(\frac{1}{2}\partial_a h_{bc}(\Phi) d\Phi^b \wedge * d\Phi^c - d\Big(h_{ab}(\Phi) * d\Phi^b\Big)\Big) \wedge \delta\Phi^a + \int_{\partial M} \Big(h_{ab}(\Phi) * d\Phi^b\Big) \wedge \delta\Phi^a$$

[58] In fact, one should allow connections in all principal $G$-bundles over $M$. However, for $G$ simply connected and $M$ 3-dimensional there are no nontrivial $G$-bundles over $M$ (since $BG$ is 3-connected and hence there is a unique homotopy class of classifying maps $M \to BG$). This is why we asked $G$ to be simply connected – to have this simplification. Case of non-simply connected $G$ can be treated but requires more care.

$$= \int_M \left(\frac{1}{2}\partial_a h_{bc}(\Phi) d\Phi^b \wedge * d\Phi^c - \partial_b h_{ac}(\Phi) d\Phi^b \wedge * d\Phi^c - h_{ab}(\Phi) d * d\Phi^b\right) \wedge \delta\Phi^a + \int_{\partial M} \left(h_{ab}(\Phi) * d\Phi^b\right) \wedge \delta\Phi^a$$
$$= \int_M \left(h_{ab}(\Phi) * \Delta\Phi^b - \Gamma_{abc}(\Phi) d\Phi^b \wedge * d\Phi^c\right) \wedge \delta\Phi^a + \int_{\partial M} \left(h_{ab}(\Phi) * d\Phi^b\right) \wedge \delta\Phi^a$$
$$= \int_M \mathrm{dvol}_g h_{ab}(\Phi)(\Delta\Phi^b - \Gamma^b_{cd}(\Phi)\langle d\Phi^c, d\Phi^d\rangle_{g^{-1}}) \wedge \delta\Phi^a + \int_{\partial M} \underbrace{\left(h_{ab}(\Phi) * d\Phi^b\right) \wedge \delta\Phi^a}_{\underline{\alpha}}.$$

Here $\Gamma^{\bullet}_{\bullet\bullet}$ are the Christoffel symbols of the target metric; $\Delta = - * d * d$ is the Laplacian on $(M, g)$. Thus, the Euler-Lagrange equation is

$$\Delta\Phi^a - \Gamma^a_{bc}(\Phi)\langle d\Phi^b, d\Phi^c\rangle_{g^{-1}} = 0. \tag{254}$$

Note that in the special case $\dim M = 1$, (254) becomes the equation of geodesic motion on $X$.

In the other extreme case, $X = \mathbb{R}$ the model becomes the massless free scalar field.

One can also consider a modification of the sigma model action functional (251) by a potential:

$$S(\Phi) = \int_M \frac{1}{2}\langle d\Phi \overset{\wedge}{,} * d\Phi\rangle_{\Phi^* h} - V(\Phi)\, \mathrm{dvol}_g \tag{255}$$

with $V \in C^\infty(X)$ a fixed function on the target manifold (the "potential"). This modification does not change the density

$$\underline{\alpha} = \langle * d\Phi \overset{\wedge}{,} \delta\Phi\rangle_{h^*\Phi} \tag{256}$$

of the Noether 1-form, as one can see by repeating the computation (253)). However it changes the Euler-Lagrange equation (254) to

$$\Delta\Phi^a - \Gamma^a_{bc}(\Phi)\langle d\Phi^b, d\Phi^c\rangle_{g^{-1}} - h^{ab}(\Phi)\partial_b V(\Phi) = 0. \tag{257}$$

3.2. **Symmetries and Noether currents.** Consider an infinitesimal transformation of fields of the field theory given as

$$\phi(x) \mapsto \phi(x) + \epsilon v(\phi(x), \partial\phi(x), \ldots) \tag{258}$$

given by a local vector field

$$v \in \mathfrak{X}_{\mathrm{loc}}(\mathrm{Fields}_M) \tag{259}$$

where loc means that the infinitesimal change of the field given by $v$ at the point $x \in M$ depends only on the jet of the field at $x$.

**Definition 3.6.** We say that $v$ is an *infinitesimal symmetry* of the given classical field theory if one has

$$\mathcal{L}_v L = d\Lambda \tag{260}$$

for some element $\Lambda \in \Omega^{n-1,0}_{\mathrm{loc}}(M, \mathrm{Fields}_M)$; here $\mathcal{L}_v$ stands for the Lie derivative in the direction of $v$.[59]

[59] The vector field (259) naturally induces an "evolutionary" (i.e. commuting with derivatives along $M$) vertical vector field $v^{\mathrm{evo}}$ on the jet bundle $\mathrm{Jet}_\infty E \to M$ (see [1]). It is that latter vector field that we act with in (260); by an abuse of notation, we still denote it $v$. Cf. Example 3.13 and footnote 60 below.

An equivalent formulation of (260) is: one has

$$\mathcal{L}_v S_N = \int_{\partial N} \Lambda. \tag{261}$$

for any $n$-dimensional submanifold with boundary $N \subset M$.

Strengthen to if and only if? Also: talk about families of finite symmetries?

**Lemma 3.7.** *If $v$ is a symmetry in the sense of (260), then the corresponding infinitesimal transformation of fields (260) takes solutions of Euler-Lagrange equation to solutions of Euler-Lagrange equations.*

Make it more clear? Write a second proof, within var bicomplex?

*Proof.* Consider a path $\phi_t$ in $\mathrm{Fields}_M$ with $\frac{d}{dt}\phi_t$ supported away from $\partial M$, as in the beginning of Section 3.1.1 and assume that $\phi_0 = \phi$ is a solution of the Euler-Lagrange equation (236). Then

$$\frac{d}{dt}\Big|_{t=0} S(\phi_t + \epsilon v(\phi_t)) = \epsilon \frac{d}{dt}\Big|_{t=0} \mathcal{L}_v S(\phi_t) = \epsilon \frac{d}{dt}\Big|_{t=0} \int_{\partial M} \Lambda(\phi_t) = 0 \quad \mod \epsilon^2.$$

In the last step we used that $\dot{\phi}_t$ (and its jet) vanishes on the boundary. Thus, any fluctuation away from the boundary of the transformed $\phi$ (the r.h.s. of (258)) preserves the value of the action in the first order in fluctuation (i.e., first order in $t$). □

**Definition 3.8.** Given an infinitesimal symmetry $v \in \mathfrak{X}_{\mathrm{loc}}(\mathrm{Fields}_M)$ of a given classical field theory, defines an element $J \in \Omega^{n-1,0}_{\mathrm{loc}}(M, \mathrm{Fields}_M)$ by the formula

$$J \colon = (-1)^n \iota_v \underline{\alpha} + \Lambda. \tag{262}$$

$J$ is called the "Noether current" associated with the symmetry $v$.

**Theorem 3.9** (Noether theorem)**.** *The field-dependent $(n-1)$-form $J$ defined by (262) is closed on $M$ when restricted to the solutions of the Euler-Lagrange equation.*

**Notation.** For two expressions $A, B$ depending on the field we will write

$$A = B \bmod \mathrm{EL} \qquad \text{or} \qquad A \underset{\mathrm{EL}}{\sim} B$$

to indicate that an equality holds "modulo the Euler-Lagrange equation," i.e., when both sides are evaluated on a field configuration $\phi$ satisfying the Euler-Lagrange equation.

Thus, Noether theorem reads

$$dJ = 0 \bmod \mathrm{EL}. \tag{263}$$

*Proof of Theorem 3.9.* Applying $d$ to the definition (262), we have

$$dJ = (-1)^{n+1}\iota_v \underbrace{d\underline{\alpha}}_{\underset{(238)}{=}(-1)^n \delta L - \mathcal{E}\mathcal{L}_a \delta\phi^a} + \underbrace{d\Lambda}_{\underset{(260)}{=}\mathcal{L}_v L} = \tag{264}$$

$$= -\mathcal{L}_v L + \underbrace{(-1)^n \mathcal{E}\mathcal{L}_a v^a}_{=0 \bmod \mathrm{EL}} + \mathcal{L}_v L = 0 \bmod \mathrm{EL}.$$

□

**Corollary 3.10.** *Assume we have a symmetry $v$ and $J$ is the associated Noether current. Then for $\gamma_1, \gamma_2$ two* cobordant *submanifolds of $M$ of codimension* 1*, one has*

$$\int_{\gamma_1} J = \int_{\gamma_2} J \quad \mod \mathrm{EL}. \tag{265}$$

*Proof.* Let $N \subset M$ be the cobordism between $\gamma_1$ and $\gamma_2$, i.e., $\partial N = \gamma_2 - \gamma_1$. Then we have

$$\int_{\gamma_2} J - \int_{\gamma_1} J \underset{\text{Stokes}'}{=} \int_N dJ \underset{\text{Noether thm}}{=} 0 \mod \text{EL}. \tag{266}$$

□

The expression $\int_\gamma J$, with $J$ a Noether current associated with a symmetry and with $\gamma \subset M$ a codimension 1 submanifold, is called a "Noether charge."

Equation (265) expresses the conservation property of the Noether charge (for a fixed field configuration satisfying EL), as one slides $\gamma$ along $M$.

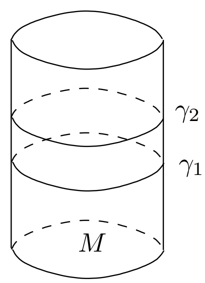



FIGURE 20. Noether charge is conserved (modulo EL) when changing the hypersurface in its cobordism class $\gamma_1 \to \gamma_2$.

**Definition 3.11.** One calls an element of $J \in \Omega^{n-1,0}_{\text{loc}}(M \times \text{Fields}_M)$ a "conserved current" if it satisfies

$$dJ \underset{EL}{\sim} 0 \tag{267}$$

A conserved current gives rise to a charge $\int_\gamma J$, with $\gamma \subset M$ a hypersurface, which is "conserved" – independent under deformations of $\gamma$, cf. (265).

Noether theorem gives a mechanism producing conserved currents out of infinitesimal symmetries.

**Definition 3.12.** We call two conserved currents $J, J'$ "equivalent" if one has

$$J' \underset{EL}{\sim} J + dK \tag{268}$$

for some element $K \in \Omega^{n-2,0}_{\text{loc}}(M \times \text{Fields}_M)$.

In particular, if $J$ is conserved ($dJ \underset{EL}{\sim} 0$), then the equivalent current $J'$ is automatically conserved. Also, if $J$ and $J'$ are equivalent, they yield the same (modulo EL) charges,

$$\int_\gamma J \underset{EL}{\sim} \int_\gamma J',$$

for $\gamma$ closed.

**Example 3.13.** Consider the classical mechanics of a particle moving on a target manifold $\mathbb{R}$. The spacetime (source cobordism) is $M = [t_0, t_1]$, fields are maps $x\colon [t_0, t_1] \to \mathbb{R}$ and the action is

$$S[x(\tau)] = \int_{t_0}^{t_1} \underbrace{d\tau \left( m\frac{\dot{x}(\tau)^2}{2} - U(x(\tau)) \right)}_{L} \tag{269}$$

The derivative in fields yields

$$\delta S = \int_{t_0}^{t_1} d\tau (m\dot{x}\delta\dot{x} - U'(x)\delta x) = \int_{t_0}^{t_1} \underbrace{d\tau(-m\ddot{x} - U'(x))\delta x}_{\mathcal{EL}\delta x} + \Big| \underbrace{m\dot{x}\delta x}_{\underline{\alpha}} \Big|_{t_0}^{t_1} \tag{270}$$

The Euler-Lagrange equation is:

$$m\ddot{x} + U'(x) = 0. \tag{271}$$

Consider the infinitesimal transformation of fields

$$x(\tau) \mapsto x(\tau) - \epsilon\dot{x}(\tau)$$

The corresponding vector field is $v = -\int_{t_0}^{t_1} \dot{x}\frac{\delta}{\delta x}$.[60] Acting with it on $L$ yields

$$\mathcal{L}_v L = d\tau(-m\dot{x}\ddot{x} + U'(x)\dot{x}) = d\Big( \underbrace{-m\frac{\dot{x}^2}{2} + U(x)}_{\Lambda} \Big) \tag{272}$$

where $d = d\tau\frac{d}{d\tau}$ the "horizontal" differential. Thus, the Noether current is

$$J = -\iota_v\underline{\alpha} + \Lambda = m\dot{x}^2 - m\frac{\dot{x}^2}{2} + U(x) = m\frac{\dot{x}(\tau)^2}{2} + U(x(\tau)) \quad \in \Omega^{0,0}([t_0, t_1] \times \mathrm{Fields}_{[t_0,t_1]}) \tag{273}$$

– this is the "energy" of the particle.

The conservation law (265) says that if $\gamma_1 = \{\tau_1\}$, $\gamma_2 = \{\tau_2\}$ are two points on the time interval $M = [t_0, t_1]$, then, assuming the trajectory $x(\tau)$ satisfies the Euler-Lagrange equation, the expression (273) yields the same number at time $\tau_1$ and at time $\tau_2$. In other words, we get that the energy (273) is constant in $\tau \in [t_0, t_1]$, if $[x(\tau)]$ is a solution of EL. Of course, we can verify this statement directly: applying $\frac{d}{d\tau}$ to the r.h.s. of (273), we obtain minus the l.h.s. of (271).

Lecture 13, 9/21/22

*Remark* 3.14 (Noether current as a vector field). In Definition 3.8 we introduced the Noether current as a field-dependent $(n-1)$-form on $M$, $J \in \Omega^{n-1}(M)$, closed modulo $M$. One can consider the associated field-dependent vector field $J^\# \in \mathfrak{X}(M)$ uniquely determined by $\iota_{J^\#} d\mathrm{vol}_g = J$. Then:

- The conservation property $dJ \underset{EL}{\sim} 0$ corresponds in terms of the vector-field Noether current to the property

$$\mathrm{div}_{d\mathrm{vol}_g} J^\# \underset{EL}{\sim} 0 \tag{274}$$

[60] When acting on jets of fields at $\tau$, $v$ acts as $v^{\mathrm{evo}} = -(\dot{x}\frac{\partial}{\partial x} + \ddot{x}\frac{\partial}{\partial \dot{x}} + \dddot{x}\frac{\partial}{\partial \ddot{x}} + \cdots)$ where the superscript "evo" stands for "evolutionary" (i.e. commuting with $d_\tau$) prolongation of $v$.

– the divergence of the vector field $J^\#$ w.r.t. the metric volume form on $M$ vanishes.[61] Equivalently, in the index notation,

$$\nabla_i (J^\#)^i \underset{EL}{\sim} 0. \tag{275}$$

- The Noether charge $\int_\gamma J$ is the flux of the vector field $J^\#$ through the hypersurface $\gamma$.

**Example 3.15.** Consider the free massless scalar on a Riemannian $n$-manifold $(M, g)$, defined by the action

$$S(\phi) = \int_M \underbrace{\frac{1}{2} d\phi \wedge *d\phi}_{L} \tag{276}$$

Consider the infinitesimal transformation of fields

$$\phi \to \phi + \epsilon \tag{277}$$

– a shift of the value of the field $\phi$ by a constant function. The corresponding vector field on the space of fields is

$$v = \int_M \frac{\delta}{\delta\phi(x)} \in \mathfrak{X}(\mathrm{Fields}_M) \tag{278}$$

The transformation (277) clearly does not change the action $S$ and the Lagrangian $L$ and clearly takes a solution of the Euler-Lagrange equation

$$\Delta\phi = 0 \tag{279}$$

to another solution. In particular, (277) is a symmetry with $\Lambda = 0$ (cf. Definition 3.6). Thus, the Noether current (262) corresponding to (277) is

(280)

$$J = (-1)^n \iota_v \underline{\alpha} = (-1)^n \iota_v \left( (-1)^{n+1} \delta\phi \wedge *d\phi \right) = \boxed{- * d\phi} \quad \in \Omega^{n-1,0}_{\mathrm{loc}}(M \times \mathrm{Fields}_M)$$

where we used $\underline{\alpha}$ we obtained before, in the computation (243). Noether theorem the tells that $J = - * d\phi$ is conserved (closed) modulo EL. One can check it independently:

$$dJ = -d * d\phi = *(\Delta\phi) \underset{EL}{\sim} 0, \tag{281}$$

cf. the Euler-Lagrange equation (279).

3.2.1. *Canonical stress-energy tensor.*

**Example 3.16** (Canonical stress-energy tensor for the free massive scalar field)**.** Consider the free massive scalar field on $M = \mathbb{R}^n$ (equipped with standard Euclidean metric), with the action

$$S(\phi) = \int_{\mathbb{R}^n} \underbrace{\frac{1}{2} d\phi \wedge *d\phi + \frac{m^2}{2} \phi^2 \mathrm{dvol}}_{L} \tag{282}$$

[61] Recall that to define the divergence of a vector field $u$ on a manifold $M$, one needs to specify a volume form $\mu$ on $M$. Then the divergence is defined via $\int_M \mu u(f) = -\int_M \mu f \mathrm{div}_\mu(u)$ for any compactly supported test function $f$. Equivalent definition: $\mathrm{div}_\mu(u) = \frac{\mathcal{L}_u \mu}{\mu}$.

Consider the symmetry given by a translation on $\mathbb{R}^n$:

$$\begin{array}{rccc} R_\epsilon\colon & \mathbb{R}^n & \to & \mathbb{R}^n \\ & \vec{x} & \mapsto & \vec{x}' = \vec{x} + \epsilon\vec{a} \end{array} \tag{283}$$

with $\vec{a} \in \mathbb{R}^n$ a fixed vector. This symmetry acts on fields as

$$\phi \to (R_\epsilon^{-1})^*\phi = \phi - \epsilon a^i \partial_i \phi + O(\epsilon^2) \tag{284}$$

This transformation is described by the vector field

$$v = -\int_{\mathbb{R}^n} a^i \partial_i \phi \frac{\partial}{\partial \phi}. \tag{285}$$

We have

$$\mathcal{L}_v L = -a^i \frac{\partial}{\partial x^i} L = -\mathcal{L}_{\vec{a}} L = d(\underbrace{-\iota_{\vec{a}} L}_{\Lambda}). \tag{286}$$

Here the derivatives $\frac{\partial}{\partial x^i}$ act on fields, $\vec{a}$ is understood as a constant vector field on $\mathbb{R}^n$. Computation (286) in particular shows that (284) is indeed a symmetry, in the sense of Definition 3.6. The corresponding Noether current is

$$J_{\vec{a}} = (-1)^n \iota_v \underline{\alpha} + \Lambda = *d\phi \langle \vec{a}, d\phi \rangle - \iota_{\vec{a}} \underbrace{\left( \frac{1}{2} d\phi \wedge *d\phi + \frac{m^2}{2} \phi^2 d\mathrm{vol} \right)}_{L} \tag{287}$$

So, one gets a family of conserved charges parametrized by $\vec{a} \in \mathbb{R}^n$. This family is linear in $\vec{a}$, so it can be written as

$$J_{\vec{a}} = \langle \vec{a}, T_{\mathrm{can}} \rangle \tag{288}$$

where the generating object of the family

$$T_{\mathrm{can}} \in \Omega^{n-1}(M) \otimes_{C^\infty(M)} \Omega^1(M) = \Gamma(M, \wedge^{n-1} T^*M \otimes T^*M) \tag{289}$$

(depending on a field) is called the "canonical stress-energy tensor." In (288), the second factor in (289) (covectors) is contracted with the constant vector field $\vec{a}$.

By Noether theorem, one has

$$(d \otimes \mathrm{id}) T_{\mathrm{can}} \underset{EL}{\sim} 0. \tag{290}$$

If we switch in (289) from $(n-1)$-forms on $M$ to vector fields on $M$ by contacting with metric volume form, we obtain the tensor

$$(T_{\mathrm{can}})^\bullet{}_\bullet = \left( \partial^i \phi\, \partial_j \phi - \delta^i_j \left( \frac{1}{2} \partial_k \phi\, \partial^k \phi + \frac{m^2}{2} \phi^2 \right) \right) \partial_i \otimes dx^j \quad \in \Gamma(M, TM \otimes T^*M). \tag{291}$$

Here the bullets $(\cdots)^\bullet{}_\bullet$ indicate the location of indices – the type of tensor – once covariant and once contravariant.

*Remark* 3.17. More generally, one can repeat the computation of Example 3.16 for any classical field theory on $M = \mathbb{R}^{p,q}$ (or any full-dimensional submanifold $M$ of $\mathbb{R}^{p,q}$), defined by some Lagrangian density

$$L = d^n x\, \mathsf{L}(\phi, \partial\phi) \tag{292}$$

Then one obtains as the generating object for Noether currents associated with translations (283) on $\mathbb{R}^{p,q}$, the "canonical stress-energy tensor"

$$(293)\qquad \begin{gathered}(T_{\mathrm{can}})^{\bullet}{}_{\bullet} = T^i{}_j \partial_i \otimes dx^j \quad \in \Gamma(M, TM \otimes T^*M) \\ \text{with} \quad T^i{}_j = \frac{\partial \mathsf{L}(\phi, \partial\phi)}{\partial(\partial_i \phi^A)} \partial_j \phi^A - \delta^i_j \mathsf{L}(\phi, \partial\phi)\end{gathered}$$

By Noether theorem, it satisfies the conservation property

$$(294)\qquad (\mathrm{div} \otimes \mathrm{id})(T_{\mathrm{can}})^{\bullet}{}_{\bullet} \underset{EL}{\sim} 0 \quad \text{or} \quad \partial_i T^i{}_j \underset{EL}{\sim} 0.$$

*Remark* 3.18. One can trade tangent and cotangent coefficient bundles in (293) (i.e. raise/lower indices), using the standard metric on $M = \mathbb{R}^{p,q}$. In particular, one has the versions

$$(295)\qquad (T_{\mathrm{can}})^{\bullet\bullet} = (T_{\mathrm{can}})^{ij}\, \partial_i \otimes \partial_j \quad \in \quad \Gamma(M, TM \otimes TM),$$

$$(296)\qquad (T_{\mathrm{can}})_{\bullet\bullet} = (T_{\mathrm{can}})_{ij}\, dx^i \otimes dx^j \quad \in \quad \Gamma(M, T^*M \otimes T^*M)$$

In the example of the free massive scalar field (Example 3.16), these two versions of the canonical stress-energy tensor happen to be symmetric. However, in a general (not necessarily scalar) field theory on $\mathbb{R}^{p,q}$ this fails: the canonical stress energy tensor is generally not symmetric.

3.3. **Hilbert stress-energy tensor.** The notion of canonical stress-energy tensor above has deficiencies. Most importantly, it is only defined on a flat manifold. There is different version of the stress-energy tensor, which is defined on any manifold, is always symmetric and conserved.

**Definition 3.19.** Given a covariant (see (235)) classical field theory, one defines the *Hilbert stress-energy tensor* as a field-dependent tensor

$$(297)\qquad T = T^{ij} \partial_i \cdot \partial_j \quad \in \Gamma(M, \mathrm{Sym}^2 TM)$$

characterized by

$$(298)\qquad \delta_g S_{M,g}(\phi) = -\frac{1}{2} \int_M \mathrm{dvol}_g T^{ij} \delta g_{ij}$$

where $\delta_g S$ means "variation of $S$ w.r.t. the variation of the metric $g \to g + \delta g$." In other words, (298) means

$$(299)\qquad \frac{d}{d\epsilon}\Big|_{\epsilon=0} S_{M,g+\epsilon h} = -\frac{1}{2} \int_M \mathrm{dvol}_g T^{ij} h_{ij}$$

for any fluctuation of the metric $h \in \Gamma(M, \mathrm{Sym}^2 TM)$.

Equivalently, one defines $T$ as the variational derivative of $S_{M,g}$ w.r.t. the metric at a given point:

$$(300)\qquad T^{ij}(x) \colon = -\frac{2}{\sqrt{\det(g)}} \frac{\delta S_{M,g}}{\delta g_{ij}(x)}.$$

From now on when we say "stress-energy tensor" we will mean "Hilbert stress-energy tensor," unless stated otherwise.

Hilbert stress-energy tensor satisfies the following properties.

**Lemma 3.20.** *(i) $T$ is a symmetric tensor: $T^{ij} = T^{ji}$.*

*(ii) $T$ is conserved, in the sense that*

$$\nabla_i T^{ij} \underset{EL}{\sim} 0 \tag{301}$$

*(or, in coordinate-free language,* $(\mathrm{div}_{\mathrm{dvol}} \otimes \mathrm{id}) T \underset{EL}{\sim} 0$*.)*

*Proof.* (i) is obvious by construction.

Proof of (ii): Let $r \in \mathfrak{X}(M)$ be aby vector field vanishing in a neighborhood of the boundary; let $R_\epsilon \in \mathrm{Diff}(M)$ be the flow along $r$ in time $\epsilon$. Covariance (235) implies

$$S_{M,g}(\phi) = S_{M,(R_\epsilon^{-1})^* g}((R_\epsilon^{-1})^* \phi) \tag{302}$$

Taking the derivative of both sides in $\epsilon$ at $\epsilon = 0$, we get

$$\begin{aligned} 0 = \frac{d}{d\epsilon}\Big|_{\epsilon=0} S_{M,(R_\epsilon^{-1})^* g}(\phi) + \frac{d}{d\epsilon}\Big|_{\epsilon=0} S_{M,g}((R_\epsilon^{-1})^* \phi) = \\ = -\frac{1}{2}\int_M \mathrm{dvol}_g T^{ij}(\nabla_i r_j + \nabla_j r_i) + \underbrace{(\cdots)}_{\underset{EL}{\sim} 0} \underset{EL}{\sim} 2\int_M \mathrm{dvol}_g (\nabla_i T^{ij}) r_j. \end{aligned} \tag{303}$$

In the last step we integrated by parts and used that $r$ vanished near $\partial M$. Since the computation (302) holds for any $r$ supported away from $\partial M$, we get $\nabla_i T^{ij} \underset{EL}{\sim} 0$. □

**Definition 3.21.** Given a covariant classical field theory on a Riemannian manifold $(M, g)$, we say that a vector field $r \in \mathfrak{X}(M)$ is a "source symmetry" (or "spacetime symmetry," or "horizontal symmetry") if for any $n$-dimensional submanifold $N \subset M$, possibly with boundary, one has

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} S_{R_\epsilon(N),g}((R_\epsilon^{-1})^* \phi) = 0, \tag{304}$$

where $R_\epsilon$ is the flow of $r$ in time $\epsilon$, in a neighborhood of $N$.[62]

Equivalently, $r$ is a source symmetry if $v = \mathcal{L}_r \in \mathfrak{X}_{\mathrm{loc}}(\mathrm{Fields})$ is a symmetry in the sense of Definition 3.6, with $\Lambda = \iota_r L$.

For instance, in Example 3.16, *constant* vector fields on $\mathbb{R}^n$ are source symmetries for the massive scalar field theory.

**Definition 3.22.** We call an infinitesimal symmetry (258) a "target symmetry" (or "vertical symmetry") if it has the form

$$\phi(x) \mapsto \phi(x) + \epsilon v(\phi(x)), \tag{305}$$

with $v$ not depending on derivatives of the field, and if it is a symmetry in the sense of (260).

For instance, constant shift of the field in Example 3.15 is a target symmetry. More generally, in the sigma model (251), a Killing vector field on the target (infinitesimal isometry) gives rise to a target symmetry (305).

[62] Note that in (304) the metric is not pushed forward by the flow in the r.h.s. If it were, the property would hold automatically for any vector field $r$ by covariance (235).

**Lemma 3.23.** *Let $r$ be a vector field on $M$. Then $r$ source symmetry of the theory if and only if the expression*

$$J_r^i := T^{ij} r_j \tag{306}$$

*is a conserved current, i.e.,*

$$\nabla_i J_r^i \underset{EL}{\sim} 0 \tag{307}$$

*(or, in coordinate-free language,* $\mathrm{div}_{\mathrm{dvol}} J_r \underset{EL}{\sim} 0$*).*

*Proof.* Assume that $r$ is a source symmetry. Applying covariance relation (235) with $m = R_\epsilon^{-1} \colon R_\epsilon(N) \to N$ to (304), we have

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} S_{N,R_\epsilon^* g}(\phi) = 0. \tag{308}$$

By (298), this means

$$-\int_N \mathrm{dvol}_g \quad \underbrace{T^{ij}\nabla_i r_j}_{\nabla_i(T^{ij}r_j)-(\nabla_i T^{ij})r_j} \quad = 0 \tag{309}$$

Since $\nabla_i T^{ij} \underset{EL}{\sim} 0$ and since (309) holds in particular for any small disk $N$ in $M$, we infer that

$$\nabla_i (T^{ij} r_j) \underset{EL}{\sim} 0 \tag{310}$$

everywhere on $M$.

The converse is proven by reversing the argument: conservation of $J_r^i$ implies (308), which implies (by covariance) the source symmetry property of $r$. $\square$

In particular, (307) can be interpreted as follows: $T$ converts source symmetries of the theory into conserved currents:

$$r \to J_r = \langle T, r \rangle. \tag{311}$$

We remark that the conserved current $\langle T, r\rangle$ does not generally coincide with the conserved current associated with the source symmetry $r$ by the Noether theorem, see Example 3.28 below.

I think they are always equivalent though?

Lecture 14, 9/23/2022

**Example 3.24** ($T$ for the free massive scalar field)**.** Consider the free massive scalar field (Example 3.2). The variation of the action w.r.t. metric is

$$\begin{aligned}
\delta_g S_{M,g}(\phi) &= S_{M,g+\delta g}(\phi) - S_{M,g}(\phi) \mod (\delta g)^2 = \\
&= \delta_g \int_M \underbrace{\Big(\frac{1}{2}(g^{-1})^{ij}\partial_i\phi\,\partial_j\phi + \frac{m^2}{2}\phi^2\Big)}_{\mathsf{L}} \underbrace{\sqrt{\det(g)}d^n x}_{\mathrm{dvol}_g} \\
&= \int_M \frac{1}{2}\Big(-(g^{-1})^{ik}\delta g_{kl}(g^{-1})^{lj}\Big)\partial_i\phi\,\partial_j\phi\sqrt{\det(g)}d^n x + \\
&\quad + \mathsf{L}\frac{1}{2}(g^{-1})^{kl}\delta g_{kl}\sqrt{\det(g)}d^n x \\
&= -\frac{1}{2}\int_M \sqrt{\det(g)}d^n x\,\delta g_{ij} \underbrace{\Big(\partial^i\phi\,\partial^j\phi - (g^{-1})^{ij}\mathsf{L}\Big)}_{T^{ij}}.
\end{aligned} \tag{312}$$

Thus, the Hilbert stress-energy tensor is:

$$T = T^{ij}\,\partial_i \cdot \partial_j = \left(\partial^i\phi\,\partial^j\phi - (g^{-1})^{ij}\left(\frac{1}{2}\partial_k\phi\,\partial^k\phi + \frac{m^2}{2}\phi^2\right)\right)\,\partial_i\cdot\partial_j, \tag{313}$$

where the indices are raised using the metric, e.g., $\partial^i\phi\colon=(g^{-1})^{il}\partial_l$. In a coordinate-free language, one has

$$T = (d\phi)^\# \cdot (d\phi)^\# - g^{-1}\left(\frac{1}{2}\langle d\phi, d\phi\rangle_{g^{-1}} + \frac{m^2}{2}\phi^2\right), \tag{314}$$

where $(\cdots)^\#$ is the bundle map $T^*M \to TM$ provided by the metric $g$ ("index-raising").

We remark that the Hilbert stress-energy tensor we computed coincides (for $M=\mathbb{R}^n$) with the canonical stress-energy tensor we found in Example 3.16: (313) coincides with (293) (upon raising an index). However, in more general classical field theories it does not happen.

**Example 3.25.** For the sigma model with target potential (Example 3.5, (255)), the Hilbert stress-energy tensor is

$$T = T^{ij}\partial_i\cdot\partial_j = \left(h_{ab}(\Phi)\partial^i\Phi^a\partial^j\Phi^b - (g^{-1})^{ij}\left(\frac{1}{2}h_{ab}(\Phi)\langle d\Phi^a, d\Phi^b\rangle_{g^{-1}} - V(\Phi)\right)\right)\partial_i\cdot\partial_j, \tag{315}$$

by a computation similar to (312).

**Example 3.26** ($T$ in the Yang-Mills theory)**.** Consider the Yang-Mills theory (Example 3.3). The variation of the action

$$S_{M,g}(A) = \frac{1}{2}\int_M \langle F_A \overset{\wedge}{,} *F_A\rangle = \frac{1}{4}\int_M \sqrt{\det(g)}d^nx\; (g^{-1})^{ik}(g^{-1})^{jl}\langle (F_A)_{ij}, (F_A)_{kl}\rangle \tag{316}$$

with respect to the metric. The computation is similar to (312) and yields

$$T = T^{ij}\partial_i\cdot\partial_j = \left(\langle F^{ik}, F^j{}_k\rangle - \frac{1}{4}(g^{-1})^{ij}\langle F_{kl}, F^{kl}\rangle\right)\partial_i\cdot\partial_j \tag{317}$$

where $F\colon=F_A$ the curvature of the connection.

**Example 3.27** ($T$ in Chern-Simons theory)**.** Consider Chern-Simons theory (Example 3.4). Since the action (248) does not depend on the metric, Hilbert stress-energy tensor (300) automatically vanishes:

$$T = 0. \tag{318}$$

improve the result

We remark that the canonical (rather than Hilbert) stress-energy tensor (293) for Chern-Simons theory on $\mathbb{R}^3$, is nonzero. Seen as an element of $\Omega^2(M)\otimes_{C^\infty(M)}\Omega^1(M)$ and then projected to 3-forms (i.e. skew-symmetrized), it is

$$[T_{\mathrm{can}}]_{\Omega^3} = -\langle A, F_A\rangle. \tag{319}$$

This expression is not zero on the nose, but vanishes modulo EL.

**Example 3.28** (Noether current for source symmetries in Chern-Simons theory)**.** In Chern-Simons theory (and generally, in any metric-independent, i.e. topological field theory) any vector field $r\in\mathfrak{X}(M)$ is a source symmetry. The corresponding Noether current is

$$J_r^{\text{Noether}} = -\iota_{\mathcal{L}_r}\underline{\alpha} + \underbrace{\Lambda}_{\iota_r L} = \tag{320}$$
$$= -\iota_{\mathcal{L}_r}(\frac{1}{2}\langle \delta A, A\rangle) + \iota_r\left(\frac{1}{2}\langle A, dA\rangle + \frac{1}{6}\langle A,[A,A]\rangle\right)$$
$$= -\frac{1}{2}\langle \mathcal{L}_r A, A\rangle + \iota_r\left(\frac{1}{2}\langle A, dA\rangle + \frac{1}{6}\langle A,[A,A]\rangle\right)$$
$$= -\frac{1}{2}\langle d\iota_r A + \cancel{\iota_r dA}, A\rangle + \underbrace{\frac{1}{2}\langle \iota_r A, dA\rangle}_{-\frac{1}{2}\langle \iota_r A, dA\rangle + \langle \iota_r A, dA\rangle} \cancel{-\frac{1}{2}\langle A, \iota_r dA\rangle} + \frac{1}{2}\langle \iota_r A, [A,A]\rangle$$
$$= d(-\frac{1}{2}\langle \iota_r A, A\rangle) + \langle \iota_r A, \underbrace{dA + \frac{1}{2}[A,A]}_{\underset{EL}{\sim} 0}\rangle$$

So, it is a $d$-exact term plus a term vanishing modulo EL. On the other hand, the conserved current associated to $r$ by Lemma 3.23 is identically zero, since the stress-energy tensor vanishes:

$$J_r = 0.$$

Note that although the currents $J_r^{\text{Noether}}$ and $J_r$ are different on the nose, they are equivalent in the sense of Definition 3.12.

### 3.4. Conformally invariant classical field theories.

**Definition 3.29.** We say that a classical field theory is "conformally invariant" (or just "conformal") if its action is invariant under Weyl transformations of the metric:

$$S_{M,g}(\phi) = S_{M,\Omega\cdot g}(\phi) \tag{321}$$

for any positive function $\Omega \in C^\infty_{>0}(M)$.

**Theorem 3.30.** *A classical field theory is conformally invariant if and only if its Hilbert stress-energy tensor is traceless*

$$\operatorname{tr} T = 0. \tag{322}$$

*Here the trace of the stress-energy tensor is understood as*

$$\operatorname{tr} T = \operatorname{tr} T^{\bullet}{}_{\bullet} = T^i{}_i = g_{ij}T^{ij} \quad \in \Omega^{0,0}_{\text{loc}}(M \times \text{Fields}_M) \tag{323}$$

*Proof.* Given a Weyl-invariant classical field theory, we have

$$0 = \frac{d}{d\epsilon}\Big|_{\epsilon=0} S_{M,(1+\epsilon\omega)g}(\phi) = -\frac{1}{2}\int_M d\text{vol}_g \underbrace{T^{ij}(x)g_{ij}(x)}_{\operatorname{tr} T(x)}\omega(x) \tag{324}$$

for any function $\omega \in C^\infty(M)$. Hence, $\operatorname{tr} T = 0$.

For the "only if" part: given that $\operatorname{tr} T = 0$ we have by the same computation (read right-to-left) that $S$ is invariant under infinitesimal Weyl transformations. Since Weyl orbits are path connected, this implies the full Weyl-invariance property (321). □

Weyl-invariance (321) implies (via covariance) that every *conformal* vector field $r \in \text{conf}(M)$ is a source symmetry. In particular, by Lemma 3.23,

$$J_r := \langle T, r\rangle \tag{325}$$

is a conserved current for any conformal vector field $M$.

Given a conformally invariant classical field theory, the stress-energy tensor depends on the metric (in addition to its dependence on fields) and is generally not Weyl-invariant. However, it is Weyl-equivariant. More precisely, we have

$$(T_{\Omega g})^{\bullet\bullet} = \Omega^{-1-\frac{n}{2}}(T_g)^{\bullet\bullet}, \tag{326}$$
$$(T_{\Omega g})_{\bullet\bullet} = \Omega^{1-\frac{n}{2}}(T_g)_{\bullet\bullet} \tag{327}$$

where we are indicating the background metric as a subscript; $n$ is the dimension of the spacetime manifold $M$.

Indeed, to see (326), we compute

$$\begin{array}{ccc} S_{\Omega(g+\delta g)} & = & -\frac{1}{2}\int \underbrace{\sqrt{\det(\Omega\, g)}}_{\Omega^{\frac{n}{2}}\sqrt{\det(g)}}\, d^n x\, T^{ij}_{\Omega g}\Omega\delta g_{ij} \\ \| & & \\ S_{g+\delta g} & = & -\frac{1}{2}\int \sqrt{\det(g)}\, d^n x\, T^{ij}_g \delta g_{ij} \end{array} \tag{328}$$

which immediately implies (326). We get (327) by contracting (326) with two copies of $\Omega g$.

In particular, (327) implies that for a *2-dimensional* conformally invariant classical field theory the stress-energy tensor $T_{\bullet\bullet}$ is Weyl-invariant – depends only on the conformal class of the metric $g$.

**Example 3.31.** Consider again the massive scalar field (Examples 3.2, 3.24), $S = \int \frac{1}{2}d\phi\wedge * d\phi + \frac{m^2}{2}\phi^2 \mathrm{dvol}_g$. Using (313), the trace of the stress-energy tensor is

$$\mathrm{tr}\, T = \frac{2-n}{2}\partial^i\phi\,\partial_i\phi - n\frac{m^2}{2}\phi^2. \tag{329}$$

The only way this expression can be identically zero is if $n = 2$ and $m = 0$. I.e., *only the 2d massless scalar field theory is conformally invariant* (among all scalar field theories in different dimensions and with different masses).

Another way to see this is to look directly at the action where one performs a Weyl transformation with the metric:

$$S_{M,\Omega g}(\phi) = \int_M \Omega^{\frac{n}{2}-1}\frac{1}{2}d\phi\wedge *_g d\phi + \Omega^{\frac{n}{2}}\frac{m^2}{2}\mathrm{dvol}_g \tag{330}$$

It is independent of $\Omega$ (and coincides with $S_{M,g}(\phi)$) if and only if $n = 2$ (which makes the first term $\Omega$-independent) and $m = 0$ (which makes the second term $\Omega$-independent). Here we made use of the fact that the Hodge star behaves under Weyl transformations as

$$*_{\Omega g}\,\alpha = \Omega^{\frac{n}{2}-p} *_g \alpha, \tag{331}$$

where $\alpha\in\Omega^p(M)$ is any $p$-form.

**Example 3.32.** Similarly to the previous example, the sigma model (Example 3.5, 3.25) is conformally invariant if and only if $n = 2$ and the potential $V(\Phi)$ is zero.

**Example 3.33.** Consider again the Yang-Mills theory (Examples 3.3, 3.26). The trace of the stress-energy tensor (317) is

$$\mathrm{tr}\, T = \frac{n-4}{4}\langle F_{ij}, F^{ij}\rangle \tag{332}$$

Thus, $n$-dimensional Yang-Mills theory is conformally invariant if and only if $n = 4$.

Another way to see this is via a computation similar to (330):

$$S_{M,\Omega g}(A) = \frac{1}{2}\int_M \Omega^{\frac{n}{2}-2}\langle F_A \overset{\wedge}{,} *_g F_A\rangle. \tag{333}$$

This expression is independent of $\Omega$ (and coincides with $S_{M,g}(A)$) if and only if the power of $\Omega$ in the integrand vanishes, i.e., if $n = 4$.

**Example 3.34.** 3d Chern-Simons theory (Example 3.4) is conformally invariant a fortiori: stress-energy tensor vanishes identically, in particular its trace vanishes. Put another way, the model does not depend on metric, hence it is invariant under Weyl transformations of metric.

3.5. **2d classical conformal field theory.** Consider a conformal classical field theory on a Riemann surface $\Sigma$.

3.5.1. *Stress-energy tensor in local complex coordinates.* Let us use local coordinates $x^1 = x, x^2 = y$ in which the conformal structure is represented by the metric $(dx)^2 + (dy)^2$. Symmetry and tracelessness of the stress-energy tensor implies the ansatz

$$T_{ij} = \begin{pmatrix} T_{11} & T_{12} \\ T_{12} & -T_{11} \end{pmatrix} \tag{334}$$

Thus, there are two independent components $T_{11}$, $T_{12}$ The conservation property $\partial^i T_{ij} \underset{EL}{\sim} 0$ is tantamount to

$$\partial_1 T_{11} + \partial_2 T_{12} \underset{EL}{\sim} 0, \tag{335}$$

$$\partial_1 T_{12} - \partial_2 T_{11} \underset{EL}{\sim} 0. \tag{336}$$

Switching to local complex coordinates $z, \bar{z}$, we have

$$\begin{aligned} T_{\bullet\bullet} = T_{ij}dx^i dx^j = T_{11}\underbrace{(dx^2 - dy^2)}_{\frac{1}{2}(dz^2 + d\bar{z}^2)} + T_{12}\underbrace{2dx\, dy}_{\frac{1}{2i}(dz^2 - d\bar{z}^2)} = \\ = \underbrace{\frac{T_{11} - iT_{12}}{2}}_{T_{zz}}(dz)^2 + \underbrace{\frac{T_{11} + iT_{12}}{2}}_{T_{\bar{z}\bar{z}}}(d\bar{z})^2 = T_{zz}(dz)^2 + T_{\bar{z}\bar{z}}(d\bar{z})^2. \end{aligned} \tag{337}$$

Thus, $T$ is a sum of a quadratic differential and its complex conjugate. Note that the the mixed term $T_{z\bar{z}}dzd\bar{z}$ does not appear. In fact, this property is equivalent to conformal invariance, since[63]

$$T_{z\bar{z}} = \frac{1}{4}\mathrm{tr}\, T \underset{\text{conformal invariance}}{=} 0. \tag{338}$$

[63] Since the metric is $g = dz \cdot d\bar{z}$, the inverse metric is $g^{-1} = 4\partial_z \cdot \partial_{\bar{z}}$; the matrix of the latter in $z, \bar{z}$-coordinates is $(g^{-1})^{ij} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$. Hence, $\mathrm{tr}\, T = (g^{-1})^{ij}T_{ij} = (g^{-1})^{z\bar{z}}T_{z\bar{z}} + (g^{-1})^{\bar{z}z}T_{\bar{z}z} = 2T_{z\bar{z}} + 2T_{\bar{z}z} = 4T_{z\bar{z}}$.

Conservation property (335), (336) in the complex coordinates reads

$$\partial_{\bar{z}} T_{zz} \underset{EL}{\sim} 0, \quad \partial_z T_{\bar{z}\bar{z}} \underset{EL}{\sim} 0. \tag{339}$$

So, modulo EL, $T_{zz}$ is a holomorphic function and $T_{\bar{z}\bar{z}}$ is antiholomorphic. Thus, modulo EL, the stress-energy tensor (337) is a sum of a holomorphic quadratic differential

$$T_{zz}(z)(dz)^2$$

and its complex conjugate.

*Remark* 3.35. Note that holomorphic quadratic differentials arise

- as a parametrization of the cotangent space to the moduli space of complex structures on a surface (cf. (188)),
- as a component of the stress-energy in a 2d conformal classical field theory.

These two occurrences are related: the variation $\delta_g S_{\Sigma,g}(\phi)$ is (for a fixed field $\phi$) a cotangent vector to the space of metrics, but due to Weyl-invariance it descends to a cotangent vector to the space of conformal (or complex) structures on the surface.

Next, if the field $\phi$ satisfies the Euler-Lagrange equation, then for $\psi_t \in \mathrm{Diff}(\Sigma)$ the flow of some (not necessarily conformal) vector field $u$ on $\Sigma$, one has

$$\frac{d}{dt}\Big|_{t=0} S_{\Sigma,\psi_t^* g}(\phi) \underset{\text{covariance}}{=} \frac{d}{dt}\Big|_{t=0} S_{\Sigma,g}((\psi_t^{-1})^*\phi) \underset{EL}{=} 0$$

Thus, for $\phi$ satisfying EL, $\delta_g S_{\Sigma,g}(\phi)$ actually gives a cotangent vector to the Teichmüller space

$$\mathcal{T}_\Sigma = \{\text{conformal structures}\}/\{\text{action by vector fields}\}$$

Lecture 15, 9/26/2022

and hence to the moduli space of complex structures $\mathcal{M}_\Sigma$.

3.5.2. *Conserved currents and charges associated to conformal symmetry.* Given a conformal vector field on $\Sigma$

$$r = u(z)\partial_z + \bar{u}(\bar{z})\partial_{\bar{z}} \in \mathrm{conf}(\Sigma)$$

(which is automatically a source symmetry for a conformal field theory), the associated conserved current (325) is

$$J_r = \langle T, r\rangle = uT_{zz}dz + \bar{u}T_{\bar{z}\bar{z}}d\bar{z} \tag{340}$$

– a field-dependent 1-form on $\Sigma$, which is closed modulo EL. Indeed, we see from (339) that

$$d(uT_{zz}dz) = \bar{\partial}(uT_{zz}dz) = \partial_{\bar{z}}(u(z)T_{zz})d\bar{z}\wedge dz = u(z)(\partial_{\bar{z}}T_{zz})d\bar{z}\wedge dz \underset{EL}{\sim} 0 \tag{341}$$

and similarly for the second term in (340).

Given a closed loop $\gamma \subset \Sigma$, one has the corresponding conserved charge is

$$C_{r,\gamma} := \oint_\gamma J_r \tag{342}$$

and it is invariant under deformations of $\gamma$ modulo EL – by Stokes' theorem (as an integral of a closed 1-form), or equivalently by Cauchy theorem (as an integral of a holomorphic 1-form, plus complex conjugate).

3.5.3. *Example: massless scalar field on a Riemann surface.* Fields are smooth real-valued functions $\phi \in C^\infty(M)$. The action written in real local coordinates on $\Sigma$ reads

$$S(\phi) = \int_\Sigma \sqrt{\det(g)} dx \wedge dy \, \frac{1}{2} (g^{-1})^{ij} \partial_i \phi \, \partial_j \phi. \tag{343}$$

Here $g$ can be any metric withing the given conformal class (the combination $\sqrt{\det(g)}(g^{-1})^{ij}$ is Weyl-invariant).

Written in local complex coordinates $z, \bar{z}$ on $\Sigma$, the action reads

$$S(\phi) = \int_\Sigma \frac{i}{2} dz \wedge d\bar{z} \; 2\partial_z \phi \, \partial_{\bar{z}} \phi. \tag{344}$$

To see this, it is sufficient to consider the Lagrangian density in (343) in the standard real/complex coordinates on the standard $\mathbb{R}^2 \simeq \mathbb{C}$, since this locally describes the general surface. In the standard metric one has $dz \wedge d\bar{z} = (dx + idy) \wedge (dx - idy) = -2idx \wedge dy$, thus $dx \wedge dy = \frac{i}{2} dz \wedge d\bar{z}$. Also, one has $\frac{1}{2}((\partial_x \phi)^2 + (\partial_y \phi)^2) = 2\partial_z \phi \partial_{\bar{z}} \phi$. This proves (344).

The Euler-Lagrange equation reads

$$\Delta \phi = 0 \tag{345}$$

or equivalently, in complex coordinates,

$$\partial_z \partial_{\bar{z}} \phi = 0. \tag{346}$$

I.e., $\phi$ satisfies EL if and only if it is a harmonic function on $\Sigma$. We remark that although the Laplacian

$$\Delta = \frac{1}{\sqrt{\det(g)}} \partial_i \sqrt{\det(g)} (g^{-1})^{ij} \partial_j$$

itself is not a Weyl-invariant operator on a surface (it changes under Weyl transformations as $\Delta_{\Omega g} = \Omega^{-1} \Delta_g$ on a 2d manifold), the equation (345) is Weyl-invariant.

The components of the stress-energy tensor in complex coordinates read

$$T_{zz} = (\partial_z \phi)^2, \qquad T_{\bar{z}\bar{z}} = (\partial_{\bar{z}} \phi)^2. \tag{347}$$

## 4. 2D QUANTUM FREE MASSLESS SCALAR FIELD

To edit

In this section our goal is to study the 2d massless scalar field as a quantum conformal field theory (in Euclidean signature): the space of states for the circle $\mathcal{H}$, correlation functions on the plane, partition function of a torus, Virasoro algebra action on $\mathcal{H}$ and the operator product expansions.

The logic of the approach is as follows:

(i) We start by constructing the quantum theory on a Minkowski cylinder (via canonical quantization of the classical theory) – along the way we identify the space of states for the circle. To start ourselves on this way, we start with the quantization of a simple 1d system – the harmonic oscillator; as we will see, the free scalar field on a cylinder can be represented (via Fourier transform on $S^1$) as a tensor product of a family of harmonic oscillators.

(ii) We switch from Minkowski to Euclidean metric on the cylinder by Wick's rotation. Then we identify – via the exponential map – the Euclidean cylinder with the punctured complex plane $\mathbb{C}^*$. At this point we are ready to calculate correlation functions of several point observables on $\mathbb{C}$.

## 4.1. **A warm-up: harmonic oscillator.**

4.1.1. *Harmonic oscillator as a classical mechanical system.* In classical mechanics, in Hamiltonian formalism, the harmonic oscillator is a system with the phase space

$$\Phi = T^*\mathbb{R} \tag{348}$$

seen as a symplectic vector space, with symplectic form

$$\omega_{\mathrm{symp}} = dp \wedge dx \tag{349}$$

where $x$ is the coordinate on $\mathbb{R}$ and $p$ – the coordinate on the cotangent fiber. The symplectic form equips the algebra of smooth functions $C^\infty(\Phi)$ with the Poisson bracket

$$\{-,-\}\colon \Phi \times \Phi \to \Phi \tag{350}$$

– a skew-symmetric bilinear (over $\mathbb{R}$) operation which is a derivation in either slot and satisfies the generating relation

$$\{p, x\} = 1. \tag{351}$$

A more geometric definition of the Poisson bracket (valid for any symplectic manifold $(\Phi, \omega_{\mathrm{symp}})$) is:

- For each function $f \in C^\infty(\Phi)$, there is the corresponding Hamiltonian vector field $X_f \in \mathfrak{X}(\Phi)$ uniquely characterized by the property

$$\iota_{X_f}\omega_{\mathrm{symp}} = -df. \tag{352}$$

- The Poisson bracket is defined by

$$\{f, g\}\colon = X_f(g) \tag{353}$$

  for any $f, g \in C^\infty(\Phi)$.

Back to the harmonic oscillator: the phase space $\Phi$ is equipped with the function

$$H = \frac{p^2}{2} + \omega^2 \frac{x^2}{2} \quad \in C^\infty(\Phi) \tag{354}$$

– the classical Hamiltonian; here $\omega > 0$ is a parameter of the system ("frequency"). The function $H$ generates the Hamiltonian vector field

$$X_H = \{H, -\} = p\frac{\partial}{\partial x} - \omega^2 x \frac{\partial}{\partial p}. \tag{355}$$

Hamilton's equations of motion of the system is the equation of an integral curve of the vector field $X_H$ on $\Phi$. In the case of the oscillator, they are:

$$\dot{x} = \{H, x\} \quad = \quad p, \tag{356}$$

$$\dot{p} = \{H, p\} \quad = \quad -\omega^2 x. \tag{357}$$

Solving this system is straightforward: one combines this system to the single equation on $x$

$$\ddot{x} + \omega^2 x = 0 \tag{358}$$

which has general solution $x(t) = A\cos(\omega t) + B\sin(\omega t)$ – oscillatory motion with frequency $\omega$ and $A, B$ arbitrary parameters. Then one uses (356) to find $p(t)$.

In Lagrangian mechanics, the same system is described by space of fields

$$\mathrm{Fields}_{[t_0,t_1]} = \mathrm{Map}([t_0, t_1], \mathbb{R}) \tag{359}$$

– maps from the source (or "worldline") interval $[t_0, t_1]$ to the target $\mathbb{R}$ (the base of the cotangent bundle (348)). The action for a function $x(\tau)$ is

$$S[x(\tau)] = \int_{t_0}^{t_1} d\tau \left( \frac{\dot{x}^2}{2} - \frac{\omega^2}{2} x^2 \right) \tag{360}$$

The corresponding Euler-Lagrange equation is exactly the equation (358). Thus, indeed, the Euler-Lagrange equations for the action (360) are equivalent to the Hamilton's equations corresponding to the Hamiltonian (354).

4.1.2. *Correspondence between Lagrangian and Hamiltonian descriptions of classical mechanics.* Stepping aside from the harmonic oscillator for the moment, consider the general classical mechanical system in Lagrangian formalism, with fields

$$\mathrm{Fields}_{[t_0,t_1]} = \mathrm{Map}([t_0, t_1], X), \tag{361}$$

with $X$ some target manifold, and with action functional

$$S[x(\tau)] = \int_{t_0}^{t_1} d\tau \; \mathsf{L}(x(\tau), \dot{x}(\tau)), \tag{362}$$

where

$$\mathsf{L}(x, v) \in C^\infty(TX) \tag{363}$$

is some function on the tangent bundle of the target $X$; here $v \in T_x X$ is a tangent vector. Then the Euler-Lagrange equation is

$$\frac{\partial \mathsf{L}(x, \dot{x})}{\partial x^i} - \frac{d}{dt} \left( \frac{\partial \mathsf{L}(x, v)}{\partial v^i} \Big|_{v = \dot{x}} \right) = 0 \tag{364}$$

– an ODE on the map $x \colon [t_0, t_1] \to X$; here we use local coordinates $x^i$ on $X$.

The same system can be described as a Hamiltonian system with the phase space

$$\Phi = T^* X \tag{365}$$

– the cotangent bundle of the target $X$ equipped with the canonical symplectic form of the cotangent bundle, $\omega_{\mathrm{symp}} = dp_i \wedge dx^i$. The Hamiltonian function $H \in C^\infty(\Phi)$ is obtained as the *Legendre transform* of the Lagrangian $\mathsf{L}$, trading velocity $v$ for momentum $p$:

$$H(x, p) \colon = v^i p_i - \mathsf{L}(x, v), \tag{366}$$

where $v = v(x, p)$ determined implicitly by the equation

$$p_i = \frac{\partial \mathsf{L}(x, v)}{\partial v^i}. \tag{367}$$

For the Legendre transform to exist and be invertible, one needs $\mathsf{L}(x, v)$ to be a convex function in $v$ (for any $x$).

The key observation is that the Hamiltonian equations generated by $H$ and Euler-Lagrange equations determined by the action (362) are equivalent, provided that the Lagrnagian $\mathsf{L}$ and the Hamiltonian $H$ are linked by the Legendre transform (366), (367). Indeed, the Hamiltonian equations read

$$\dot{x}^i = \frac{\partial H}{\partial p_i} \underset{(366)}{=} v^i + \cancel{p_j \frac{\partial v^j}{\partial p_i}} - \cancel{\frac{\partial v^j}{\partial p_i} \underbrace{\frac{\partial \mathsf{L}}{\partial v^j}}_{p_j}} = v^i, \tag{368}$$

$$\dot{p}_i = -\frac{\partial H}{\partial x^i} = -p_j \frac{\partial v^j}{\partial x^i} + \frac{\partial \mathsf{L}}{\partial x^i}\Big|_{p=\text{const}} = \cancel{-p_j \frac{\partial v^j}{\partial x^i}} + \Big( \frac{\partial \mathsf{L}}{\partial x^i}\Big|_{v=\text{const}} + \cancel{\underbrace{\frac{\partial \mathsf{L}}{\partial v^j}}_{p_j} \frac{\partial v^j}{\partial x^i}} \Big) = \frac{\partial \mathsf{L}}{\partial x^i}.$$

Substituting (367) in the second equation above, we get the Euler-Lagrange equation (364).

*Remark* 4.1. Legendre transform admits the following geometric description. If $l(v)$ is convex[64] smooth function on a vector space $V \ni v$ then its Legendre transform $h(p)$ is a smooth convex function on $V^* \ni p$ with the property that the Lagrangian submanifold $V \oplus V^*$ that is the graph of $dl$ (here we think of $V \oplus V^*$ as the cotangent bundle $T^*V$ with the standard symplectic form $dp_i \wedge dv^i$) is also described as the graph of $dh$ (where we think of $V \oplus V^*$ as $T^*(V^*)$ with symplectic structure $dv^i \wedge dp_i$):

(369)
$$\text{graph}(dl) = \left\{ (v,p) \mid p_i = \frac{\partial l}{\partial v^i} \right\} \quad = \quad \text{graph}(dh) = \left\{ (v,p) \mid v_i = \frac{\partial h}{\partial p_i} \right\} \qquad \subset V \oplus V^*$$

Put another way, the Lagrangian submanifold (369) has $l$ as its generating function on $V$ and $h$ as its generating function on $V^*$. If $l$ is given, the property (369) determines $h$ uniquely up to a possible shift by a constant function. FINISH

In (366), (367), the Legendre transform is done pointwise on $X$, with $V = T_x X$, $V^* = T_x^* X$, $l(v) = \mathsf{L}(x,v)$ and $h(p) = H(x,p)$ for any point $x \in X$.

4.1.3. *Preparing for canonical quantization: Weyl algebra and Heisenberg Lie algebra.*

**Definition 4.2.** Let $(V, \omega_{\text{symp}})$ be a (real) symplectic vector space and let $V_\mathbb{C} = \mathbb{C} \otimes V$ be its complexification. One defines the Heisenberg Lie algebra associated to $(V, \omega_{\text{symp}})$ as the Lie $*$-algebra

$$\text{Heis}(V, \omega_{\text{symp}}) = V_\mathbb{C} \oplus \mathbb{C} \cdot \mathbb{K} \tag{370}$$

where $\mathbb{K}$ is a central element and one has the commutators

$$[\widehat{u}, \widehat{v}] = -i\omega_{\text{symp}}(u,v) \cdot \mathbb{K} \tag{371}$$

for $u, v \in V$. We put a hat on an element of $V$ when we think of it as an element of Heis. Elements $\widehat{v}$ and $\mathbb{K}$ are understood as self-adjoint.

Thus, Heisenberg Lie algebra is a central extension of $V_\mathbb{C}$ seen as an abelian Lie algebra,

$$\mathbb{C} \to \text{Heis}(V, \omega_{\text{symp}}) \to V_\mathbb{C}, \tag{372}$$

with the Lie 2-cocycle of $V$ defining the central extension being $\omega_{\text{symp}}$.

**Theorem 4.3** (Stone-von Neumann)**.** *Assume that $V$ is finite-dimensional. Then there exists a unique (up to isomorphism) irreducible unitary representation of* $\text{Heis}(V, \omega_{\text{symp}})$.

"Unitary" here means that the representation is on a Hilbert space $\mathcal{H}$ and for each $v \in V$, $\widehat{v}$ is represented by a hermitian operator.

[64]Convexity implies that the Lagrangian submanifold (369) is projectable onto both $V$ and $V^*$.

**Definition 4.4.** Weyl algebra of the symplectic vector space $(V, \omega)$ is defined as the following associative $*$-algebra over the ring of formal power series $\mathbb{C}[[\hbar]]$:

$$\mathrm{Weyl}(V, \omega_{\mathrm{symp}}) \colon = \mathbb{C}[[\hbar]] \otimes U\mathrm{Heis}(V, \omega_{\mathrm{symp}})/(\mathbb{K} = \hbar) \tag{373}$$

– the universal enveloping of the Heiseberg Lie algebra (with scalars extended to formal power series), with the central element $\mathbb{K}$ identified with the scalar $\hbar$. The involution (hermitian conjugation) maps the Heisenberg generators $\widehat{v}$ to themselves.

Here we think of the Planck constant $\hbar$ as an infinitesimal formal parameter.

**Example 4.5** (Main example)**.** Consider $V = T^*\mathbb{R}^n$ with coordinates $x_1, \ldots, x_n$ on the base $\mathbb{R}^n$ and dual fiber coordinates $p^1, \ldots, p^n$, with standard symplectic form

$$\omega_{\mathrm{symp}} = \sum_i dp_i \wedge dx^i. \tag{374}$$

The corresponding Weyl algebra is generated by elements $\widehat{x}^i, \widehat{p}_i$, $i = 1, \ldots, n$, subject to relations

$$[\widehat{x}^i, \widehat{x}^j] = 0, \quad [\widehat{p}_i, \widehat{p}_j] = 0, \quad [\widehat{p}_i, \widehat{x}^j] = -i\hbar\,\delta_i^j, \qquad \forall\ 0 \leq i, j \leq n \tag{375}$$

– the "canonical commutation relations."

The standard representation of this algebra – the Schrödinger representation – is on the Hilbert space $\mathcal{H} = L^2_{\mathbb{C}}(\mathbb{R}^n)$ of complex-valued square-integrable function on $\mathbb{R}^n$, with hermitian structure

$$\langle \psi_1, \psi_2 \rangle \colon = \int_{\mathbb{R}^n} d^n x\, \overline{\psi_1(x)} \psi_2(x)$$

for $\psi_1, \psi_2$ two square-integrable functions on $\mathbb{R}^n$. The generators $\widehat{x}^i$, $\widehat{p}_i$ of the Weyl algebra act on $\mathcal{H}$ as the following hermitian operators:

$$\widehat{x}^i \colon \psi(x) \mapsto x^i \psi(x), \quad \widehat{p}_j \colon \psi(x) \mapsto -i\hbar \frac{\partial}{\partial x^i} \psi(x) \tag{376}$$

I.e. $\widehat{x}^i$ acts as a multiplication operator (by a coordinate function) and $\widehat{p}^i$ acts as a derivation.[65]

In particular, using this representation, one can identify the Weyl algebra of $T^*\mathbb{R}^n$ with the algebra of polynomial differential operators in $n$ variables.

4.1.4. *Canonical quantization of the harmonic oscillator.* The idea of canonical quantization is to start with a classical system in Hamiltonian formalism with phase space $\Phi = T^*\mathbb{R}^n$ and lift (or "quantize") the Hamiltonian function $H(x, p)$ to an element $\widehat{H} = H(\widehat{x}, \widehat{p})$ of the corresponding Weyl algebra – the quantum Hamiltonian.

By quantizing/lifting a polynomial function $f$ on $\Phi$ we mean choosing a preimage of $f$ under the "dequantization map"

$$\pi \colon \mathrm{Weyl}(\Phi) \xrightarrow{\mathrm{mod}\ \hbar} C^\infty_{\mathrm{poly}}(\Phi). \tag{377}$$

where $C^\infty_{\mathrm{poly}}(\Phi) = \mathrm{Sym}^\bullet \Phi^*$ is the algebra of polynomial functions on $\Phi$. Put another way, we take a polynomial function $f(x, p) \in C^\infty_{\mathrm{poly}}(\Phi)$ and replace $x^i$, $p_j$ with corresponding generators of the Weyl algebra $\widehat{x}^i$, $\widehat{p}_j$, where we are allowed to add any terms proportional to $\hbar^k$ for $k > 0$. The possibility to add such terms reflects the ordering ambiguity. E.g., $xp = px$ as functions on $\Phi = T^*\mathbb{R}$, but $\widehat{x}\widehat{p} = \widehat{p}\widehat{x} + i\hbar$ in the

[65] These operators are unbounded on $L^2_{\mathbb{C}}(\mathbb{R}^n)$.

Weyl algebra; so both $\widehat{x}\widehat{p}$ and $\widehat{p}\widehat{x}$ should be considered as legitimate quantizations of the monomial $xp$, and these quantizations are different.

A systematic approach to lifting is to choose a "quantization map" (or "operator ordering").

**Definition 4.6.** We call a "quantization map" a $\mathbb{C}$-linear map

$$q\colon C^\infty_{\mathrm{poly}}(\Phi) \to \mathrm{Weyl}(\Phi) \tag{378}$$

which satisfies $\pi \circ q = \mathrm{id}$, where $\pi$ is the map (377).

Note that $q$ is not required to be an algebra morphism; in fact, it cannot be one.

**Example 4.7** (Weyl quantization map)**.** Consider the map $q\colon C^\infty_{\mathrm{poly}}(\Phi) \to \mathrm{Weyl}(\Phi)$ which sends a monomial in $x^i, p_j$ to the corresponding monomial in $\widehat{x}^i$, $\widehat{p}_j$, where one averages over all possible orderings of the factors (i.e. for a monomial of degree $d$, one averages over the symmetric group $S_d$). Then one extends $q$ to general polynomials by linearity. One calls this map $q$ the Weyl (or "symmetric") quantization map.

In the case of harmonic oscillator, we lift the coordinate function $x, p$ on the phase space $\Phi = T^*\mathbb{R}$ to the generators of the Weyl algebra $\widehat{x}, \widehat{p}$ satisfying the relation

$$[\widehat{p}, \widehat{x}] = -i\hbar. \tag{379}$$

We lift the Hamiltonian function to the element

$$\widehat{H} = \frac{\widehat{p}^2}{2} + \omega^2 \frac{\widehat{x}^2}{2} \tag{380}$$

of the Weyl algebra.

**Disclaimer.** In the discussion below, we will be thinking of $\hbar$ as a small positive real number (rather than a formal parameter), and formulae involving $\hbar$ should be thought of as a family over $\hbar \in \mathbb{R}_{>0}$.

In the Schrödinger representation, the Weyl algebra is acting on the Hilbert space

$$\mathcal{H} = L^2_{\mathbb{C}}(\mathbb{R}), \tag{381}$$

with

$$\widehat{x} = x\cdot, \quad \widehat{p} = -i\hbar \frac{\partial}{\partial x}. \tag{382}$$

The quantum Hamiltonian (380) is then represented as the differential operator

$$\widehat{H} = -\frac{\hbar^2}{2}\frac{\partial^2}{\partial x^2} + \frac{\omega^2}{2} x^2. \tag{383}$$

To construct the evolution operator of the quantum system

$$U(t) = e^{-\frac{it\widehat{H}}{\hbar}} \quad \in U(\mathcal{H}), \tag{384}$$

where $U(\mathcal{H})$ is the unitary group, one needs to find the eigenvalues and eigenvectors (as square-integrable functions) of $\widehat{H}$. I.e., one is looking for all pairs $\psi \neq 0 \in L^2_{\mathbb{C}}(\mathbb{R})$, $E \in \mathbb{R}$ such that

$$\left(-\frac{\hbar^2}{2}\frac{\partial^2}{\partial x^2} + \frac{\omega^2}{2}x^2\right)\psi(x) = E\psi(x). \tag{385}$$

This is a well-known instance of a singular Sturm-Liouville problem. The answer is:

**Theorem 4.8.** *The operator (383) admits a complete orthonormal system of eigenvectors $\{\psi_n\}_{n\geq 0}$ in $L^2(\mathcal{H})$ of the form*

$$\psi_n = C_n e^{-\frac{\omega x^2}{2\hbar}} H_n\left(\sqrt{\frac{\omega}{\hbar}}\, x\right) \tag{386}$$

*where*

$$H_n(x) := (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \tag{387}$$

*are Hermite polynomials; $C_n = \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{4}} (2^n n!)^{-\frac{1}{2}}$ is a normalization constant. The eigenvalue of $\widehat{H}$ corresponding to $\psi_n$ is*

$$E_n = \hbar\omega(n + \frac{1}{2}). \tag{388}$$

The first few Hermite polynomials are:

| $n$ | $H_n(x)$ |
|---|---|
| 0 | 1 |
| 1 | $2x$ |
| 2 | $4x^2 - 2$ |
| 3 | $8x^3 - 12x$ |
| 4 | $16x^4 - 48x^2 + 12$ |
| $\vdots$ | $\vdots$ |

The evolution operator (384) is then

$$\begin{array}{rccc} U(t): & \mathcal{H} & \to & \mathcal{H} \\ & \psi & \mapsto & \sum_{n\geq 0} e^{-\frac{iE_n t}{\hbar}} \langle \psi_n, \psi\rangle\, \psi_n = \sum_{n\geq 0} e^{-i(n+\frac{1}{2})\omega t} \langle\psi_n,\psi\rangle\,\psi_n \end{array} \tag{389}$$

4.1.5. *Creation/annihilation operators.* Instead of directly looking for eigenvectors and eigenvalues of the operator (383), one can obtain the result of Theorem 4.8 by exploiting the hidden algebraic structure of the operator $\widehat{H}$ (specific to the harmonic oscillator case).

Let us introduce two new elements of the Weyl algebra[66] – special complex linear combinations of $\widehat{x}$, $\widehat{p}$:

$$\widehat{a} = \sqrt{\frac{\omega}{2\hbar}}(\widehat{x} + \frac{i}{\omega}\,\widehat{p}), \tag{390}$$

$$\widehat{a}^+ = \sqrt{\frac{\omega}{2\hbar}}(\widehat{x} - \frac{i}{\omega}\,\widehat{p}). \tag{391}$$

Operators $\widehat{a}, \widehat{a}^+$ are called the "annihilation operator" and "creation operator," respectively. The are hermitian conjugates of one another and satisfy the commutation relation

$$[\widehat{a}, \widehat{a}^+] = 1 \tag{392}$$

[66]More pedantically, here we extend the ring of scalars in the Weyl algebra by tensoring it with $\mathbb{C}[\hbar^{1/2}, \hbar^{-1/2}]$.

as a consequence of the canonical commutation relation (379). The inverse formulae to (390), (391) are:

$$\widehat{x} = \sqrt{\frac{\hbar}{2\omega}}(\widehat{a}^+ + \widehat{a}), \tag{393}$$

$$\widehat{p} = i\sqrt{\frac{\hbar\omega}{2}}(\widehat{a}^+ - \widehat{a}). \tag{394}$$

The quantum Hamiltonian (383) expressed in terms of $\widehat{a}, \widehat{a}^+$ is

$$\widehat{H} = \hbar\omega\frac{1}{2}(\widehat{a}^+\widehat{a} + \widehat{a}\widehat{a}^+) = \hbar\omega\left(\widehat{a}^+\widehat{a} + \frac{1}{2}\right) \tag{395}$$

The relation (392) implies the commutation relations between $\widehat{H}$ and $\widehat{a}$, $\widehat{a}^+$:

$$[\widehat{H}, \widehat{a}] = -\hbar\omega\widehat{a} \tag{396}$$

$$[\widehat{H}, \widehat{a}^+] = \hbar\omega\widehat{a}^+ \tag{397}$$

This implies that if in a representation of the Weyl algebra on a Hilbert space $\mathcal{H}$, a vector $\psi \in \mathcal{H}$ is an eigenvector of $\widehat{H}$ with eigenvalue $E$, then

$$\widehat{H}(\widehat{a}\psi) = (E - \hbar\omega)(\widehat{a}\psi), \tag{398}$$

$$\widehat{H}(\widehat{a}^+\psi) = (E + \hbar\omega)(\widehat{a}^+\psi). \tag{399}$$

Thus, $\widehat{a}$, $\widehat{a}^+$ take eigenvectors of $\widehat{H}$ to eigenvectors; applying $\widehat{a}^+$ *raises* the eigenvalue of by $\hbar\omega$, while $\widehat{a}$ *lowers* the eigenvalue by $\hbar\omega$.

Lecture 16, 9/28/2022

We can construct an irreducible unitary representation $\mathcal{H}^{\mathrm{osc}}$ of the Weyl algebra as follows: let $|0\rangle \in \mathcal{H}^{\mathrm{osc}}$ be the "vacuum vector" – a vector with the property

$$\widehat{a}|0\rangle = 0. \tag{400}$$

We will assume that $|0\rangle$ has norm 1 in $\mathcal{H}^{\mathrm{osc}}$. From (395) we infer that

$$\widehat{H}|0\rangle = \frac{\hbar\omega}{2}|0\rangle. \tag{401}$$

We then introduce the vectors $|n\rangle \in \mathcal{H}^{\mathrm{osc}}$ with $n = 1, 2, 3, \ldots$ as

$$|n\rangle := \alpha_n(\widehat{a}^+)^n|0\rangle \tag{402}$$

where $\alpha_n$ is a normalization factor, chosen in such a way that the vectors $|n\rangle$ are of norm 1. From (399) we infer that

$$\widehat{H}|n\rangle = \left(n + \frac{1}{2}\right)\hbar\omega|n\rangle \tag{403}$$

The representation space $\mathcal{H}^{\mathrm{osc}}$ of the Weyl algebra is then

$$\mathcal{H}^{\mathrm{osc}} = \Big\{ \sum_{n\geq 0} c_n|n\rangle \;\Big|\; c_n \in \mathbb{C}, \sum_{n\geq 0} |c_n|^2 < \infty \Big\}. \tag{404}$$

One can calculate the norms/inner products of vectors in $\mathcal{H}^{\mathrm{osc}}$ from the fact that $\widehat{a}$, $\widehat{a}^+$ are Hermitian conjugate, using the commutation relation (392). E.g., one has

$$\Big\langle \widehat{a}^+|0\rangle, \widehat{a}^+|0\rangle \Big\rangle = \Big\langle |0\rangle, \widehat{a}\widehat{a}^+|0\rangle \Big\rangle = \langle 0| \underbrace{\widehat{a}\widehat{a}^+}_{\widehat{a}^+\widehat{a}+1} |0\rangle = \langle 0|\widehat{a}^+ \underbrace{\widehat{a}|0\rangle}_{0} + \underbrace{\langle 0|0\rangle}_{||\,|0\rangle\,||^2=1} = 1 \tag{405}$$

Here we used the Dirac's notation: a covector in $(\mathcal{H}^{\mathrm{osc}})^*$ dual to the vector $|\psi\rangle \in \mathcal{H}^{\mathrm{osc}}$ is denoted $\langle\psi|$; the inner product $\Big\langle |\psi_1\rangle, |\psi_2\rangle \Big\rangle_{\mathcal{H}^{\mathrm{osc}}}$ of two vectors in $\mathcal{H}^{\mathrm{osc}}$ is also denoted $\langle\psi_1|\psi_2\rangle$.

More generally, using the same strategy – commuting $\widehat{a}$ to the right of the word of creation/annihilition operators – one can show the following.

**Lemma 4.9.** *For $n, m = 0, 1, 2, \ldots,$ one has*

$$\langle 0|\widehat{a}^m(\widehat{a}^+)^n|0\rangle = n!\, \delta_{nm}. \tag{406}$$

*Proof.* First note that we have the commutation relation

$$[\widehat{a}, (\widehat{a}^+)^n] = \sum_{k=1}^{n} (\widehat{a}^+)^{k-1} \underbrace{[\widehat{a}, \widehat{a}^+]}_{1} (\widehat{a}^+)^{n-k} = n(\widehat{a}^+)^{n-1}. \tag{407}$$

Using it, we find

$$\widehat{a}(\widehat{a}^+)^n|0\rangle = [\widehat{a}, (\widehat{a}^+)^n]|0\rangle + (\widehat{a}^+)^n \underbrace{\widehat{a}|0\rangle}_{0} = n(\widehat{a}^+)^{n-1}|0\rangle. \tag{408}$$

Thus, for $m \leq n$, we have

$$\begin{aligned}(\widehat{a})^m(\widehat{a}^+)^n|0\rangle &= (\widehat{a})^{m-1}\widehat{a}(\widehat{a}^+)^n|0\rangle = (\widehat{a})^{m-1}n(\widehat{a}^+)^{n-1}|0\rangle = \\ &= (\widehat{a})^{m-2}n\widehat{a}(\widehat{a}^+)^{n-1}|0\rangle = (\widehat{a})^{m-2}n(n-1)(\widehat{a}^+)^{n-2}|0\rangle \\ &= \cdots = n(n-1)\cdots(n-m+1)(\widehat{a}^+)^{n-m}|0\rangle\end{aligned} \tag{409}$$

In particular, for $m = n$ we have

$$(\widehat{a})^n(\widehat{a}^+)^n|0\rangle = n!|0\rangle, \tag{410}$$

which implies (406) for $m = n$.

If $m < n$, we have

$$\langle 0|\widehat{a}^m(\widehat{a}^+)^n|0\rangle = \frac{n!}{(n-m)!}\langle 0|(\widehat{a}^+)^{n-m}|0\rangle = 0, \tag{411}$$

where we use the fact that $\langle 0|\widehat{a}^+$ is the covector dual to $\widehat{a}|0\rangle$ and thus vanishes.

Likewise, if $m > n$, we have

$$\langle 0|\widehat{a}^m(\widehat{a}^+)^n|0\rangle = n!\langle 0| \underbrace{\widehat{a}^{m-n}|0\rangle}_{0} = 0. \tag{412}$$

□

In particular, vectors (402) with $n = 0, 1, 2, \ldots$ form an orthonormal basis for $\mathcal{H}$ if we set the normalization factors to be

$$\alpha_n = \frac{1}{\sqrt{n!}}. \tag{413}$$

In this basis, the operators $\widehat{a}, \widehat{a}^+$ act as

$$\widehat{a}|n\rangle = \frac{1}{\sqrt{n!}} \underbrace{[\widehat{a}, (\widehat{a}^+)^n]}_{n(\widehat{a}^+)^{n-1}} |0\rangle = \sqrt{n}\,|n-1\rangle \tag{414}$$

and

$$\widehat{a}^+|n\rangle = \underbrace{\frac{1}{\sqrt{n!}}}_{\frac{\sqrt{n+1}}{\sqrt{(n+1)!}}} (\widehat{a}^+)^{n+1}|0\rangle = \sqrt{n+1}\,|n+1\rangle. \tag{415}$$

By Stone-von Neumann theorem, there is an isomorphism of representations of the Weyl algebra

$$\mathcal{H}^{\mathrm{osc}} \simeq L^2_{\mathbb{C}}(\mathbb{R}) \tag{416}$$

– the "oscillator representation" and Schrödinger representation. Under this isomorphism vectors $|n\rangle \in \mathcal{H}^{\mathrm{osc}}$ correspond to vectors (386). In fact, one can obtain the formula (386) from (402). Indeed: in Schrödinger representation, the operators $\widehat{a}$, $\widehat{a}^+$ are

$$\widehat{a} = \frac{1}{\sqrt{2}}\left(y + \frac{\partial}{\partial y}\right) = \frac{1}{\sqrt{2}} e^{-\frac{y^2}{2}} \frac{\partial}{\partial y} e^{\frac{y^2}{2}}, \tag{417}$$

$$\widehat{a}^+ = \frac{1}{\sqrt{2}}\left(y - \frac{\partial}{\partial y}\right) = \frac{-1}{\sqrt{2}} e^{\frac{y^2}{2}} \frac{\partial}{\partial y} e^{-\frac{y^2}{2}}, \tag{418}$$

where we denoted $y = \sqrt{\frac{\omega}{\hbar}}\, x$. Thus, the vacuum vector $|0\rangle$ in Schrödinger representation is a function $\psi_0$ satisfying the first-order ODE

$$\widehat{a}\psi_0 = 0 \quad \Leftrightarrow \quad \frac{\partial}{\partial y}\left(e^{\frac{y^2}{2}}\psi_0(y)\right) = 0 \quad \Leftrightarrow \quad \psi_0(y) = C_0 e^{-\frac{y^2}{2}} \tag{419}$$

with $C_0$ a constant (which can be chosen to normalize $\psi_0$ to unit norm). Vectors $|n\rangle$ in Schrödinger representation are then

$$\begin{aligned}\psi_n(y) = \alpha_n (\widehat{a}^+)^n |0\rangle = \alpha_n (-1)^n 2^{-\frac{n}{2}} e^{\frac{y^2}{2}} \frac{\partial^n}{\partial y^n}\left(e^{-\frac{y^2}{2}}\psi_0(y)\right) = \\ = 2^{-\frac{n}{2}} C_0 \alpha_n e^{-\frac{y^2}{2}} \underbrace{\left((-1)^n e^{y^2} \frac{\partial^n}{\partial y^n} e^{-y^2}\right)}_{H_n(y)}.\end{aligned} \tag{420}$$

This is exactly the formula (386).

In terms of the basis $\{|n\rangle\}$ in the Hilbert space $\mathcal{H}^{\mathrm{osc}}$, the evolution operator (389) acts as

$$U(t) = e^{-\frac{i\widehat{H}t}{\hbar}} : \quad \sum_{n\geq 0} c_n |n\rangle \mapsto \sum_{n\geq 0} c_n e^{-i(n+\frac{1}{2})\omega t}|n\rangle. \tag{421}$$

*Remark* 4.10. The partition function of the harmonic oscillator on the circle of length $t$ (cf. Example 1.10) is

$$Z(S^1_t) = \mathrm{tr}_{\mathcal{H}^{\mathrm{osc}}} U(t) = \sum_{n\geq 0} e^{-i(n+\frac{1}{2})\omega t} = \frac{e^{-\frac{i\omega t}{2}}}{1-e^{-i\omega t}} = \frac{1}{e^{\frac{i\omega t}{2}} - e^{-\frac{i\omega t}{2}}} = \frac{1}{2i \sin\frac{\omega t}{2}}. \tag{422}$$

The Euclidean version of the partition function is obtained by the Wick rotation $t = -iT_{\mathrm{Eucl}}$ with $T_{\mathrm{Eucl}} > 0$. In this version, the sum over eigenvalues in (422) becomes absolutely convergent and one has

$$Z_{\mathrm{Eucl}}(S^1_{T_{\mathrm{Eucl}}}):=Z(S^1_{t=-iT_{\mathrm{Eucl}}})=\sum_{n\geq 0}e^{-(n+\frac{1}{2})\omega T_{\mathrm{Eucl}}}=\frac{1}{2\sinh\frac{\omega T_{\mathrm{Eucl}}}{2}}. \tag{423}$$

*Remark* 4.11. The algebra of creation/annihilation operators (392) admits another useful representation (unitarily isomorphic to $\mathcal{H}^{\mathrm{osc}}$ and to the Schrödinger representation), on the Segal-Bargmann space, constructed as follows. Consider the following hermitian inner product on the space $\mathrm{Hol}(\mathbb{C})$ of holomorphic functions on $\mathbb{C}$:

$$\langle f,g\rangle_{\mathrm{SB}}=\frac{1}{\pi}\int\frac{i}{2}dz\wedge d\bar{z}\; e^{-|z|^2}\overline{f(z)}g(z). \tag{424}$$

Then the Segal-Bargmann space is defined as

$$\mathcal{H}_{\mathrm{SB}}:=\{f\in\mathrm{Hol}(\mathbb{C})\mid\langle f,f\rangle_{\mathrm{SB}}<\infty\}. \tag{425}$$

In this representation, creation and annihilation operators act as

$$\widehat{a}=\frac{\partial}{\partial z},\quad \widehat{a}^+=z\cdot \tag{426}$$

– holomorphic derivative and multiplication operator by the holomorphic coordinate, respectively; these operators are hermitian conjugate of one another w.r.t. the inner product (424). The vacuum vector $|0\rangle$ can be identified with the function $1\in\mathcal{H}_{\mathrm{SB}}$; then the vectors $|n\rangle$ are identified with $\frac{1}{\sqrt{n!}}z^n\in\mathcal{H}_{\mathrm{SB}}$. The Hamiltonian $\widehat{H}=\hbar\omega(z\frac{\partial}{\partial z}+\frac{1}{2})$, up to normalization and a shift, is the Euler vector field and thus counts the monomial degree of a function in $z$.

Normal ordering. Normal ordering is an operation acting on linear combination words in the creation-annihilation operators $\widehat{a},\widehat{a}^+$ which reshuffles the letters in each word, putting annihilation operators $\widehat{a}$ to the right and creation operators $\widehat{a}^+$ to the left. Normal ordering applied to a word $W$ is denoted $:W:$. For instance, one has

$$:\widehat{a}\widehat{a}^+\widehat{a}\widehat{a}^+:=\widehat{a}^+\widehat{a}^+\widehat{a}\widehat{a}. \tag{427}$$

We stress that normal ordering is an operation on words – it does not descend to the Weyl algebra.

An important property of normal ordering is that if $O$ is a sum of words, each containing at least one creation or annihilation operator (i.e. no constant summand in $O$), then one has

$$\langle 0|:O:|0\rangle=0. \tag{428}$$

This property is obvious: for each normally ordered word, the expression (428) will contain a term $\widehat{a}|0\rangle$ and/or a term $\langle 0|\widehat{a}^+$, both of which vanish.

In particular, if we represent the Hamiltonian of the harmonic oscillator by the combination of words $\widehat{H}=\hbar\omega\frac{1}{2}(\widehat{a}\widehat{a}^++\widehat{a}^+\widehat{a})$, then we have

$$:\widehat{H}:=\hbar\omega\widehat{a}^+\widehat{a}. \tag{429}$$

– which differs from (395) by $\frac{\hbar\omega}{2}$. In particular, one has

$$:\widehat{H}:|n\rangle=\hbar\omega n|n\rangle. \tag{430}$$

In particular, the vacuum vector $|0\rangle$ is has zero eigenvalue w.r.t. the normally ordered Hamiltonian $:\widehat{H}:$,

$$:\widehat{H}:|0\rangle = 0. \tag{431}$$

4.2. **Free massless scalar field on Minkowski cylinder.**

4.2.1. *Lagrangian formalism.* Consider the massless scalar field on the cylinder $\Sigma = \mathbb{R}\times S^1$ with Minkowski metric $g = (dt)^2 - (d\sigma)^2$. Here $t$ (time) is the coordinate on $\mathbb{R}$ and $\sigma \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$ is the "spatial coordinate."

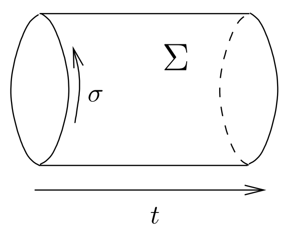



FIGURE 21. Cylinder.

Fields of the theory are smooth real functions $\phi(t,\sigma)$ on $\Sigma$ and the action functional is

$$S(\phi) = \frac{\kappa}{2}\int_\Sigma dt\, d\sigma(\dot\phi^2 - (\partial_\sigma\phi)^2) \tag{432}$$

where dot means the derivative in $t$. We put a normalization factor $\kappa$ in the definition of the action – we will fix it later.

The space of fields of the theory Fields $= \mathrm{Map}(\mathbb{R}\times S^1,\mathbb{R})$ can be thought of as $\mathrm{Map}(\mathbb{R},\mathrm{Map}(S^1,\mathbb{R}))$. Thus, one can think of the field theory on the cylinder $\Sigma$ as classical mechanics on the worldline $\mathbb{R}$ with target

$$X = \mathrm{Map}(S^1,\mathbb{R}) = C^\infty(S^1) \quad \ni \phi(\sigma) \tag{433}$$

and Lagrangian

$$\mathsf{L} = \frac{\kappa}{2}\oint_{S^1} d\sigma(\dot\phi^2 - (\partial_\sigma\phi)^2) \tag{434}$$

– a function on $TX$ (cf. (363)). We understand $\phi(\sigma)$[67] as a point in the base of $TX$ and $\dot\phi(\sigma)$ as a tangent vector to $X$ at $\phi(\sigma)$.

The Euler-Lagrange equation of the theory is the wave equation

$$\ddot\phi - \partial_\sigma^2\phi = 0. \tag{435}$$

Its solution can be though of as a path in $X$ parametrized by $t\in\mathbb{R}$.

Let us expand $\phi(\sigma)$ in the Fourier series

$$\phi(\sigma) = \sum_{n\in\mathbb{Z}}\phi_n e^{in\sigma}. \tag{436}$$

Since $\phi$ is real-valued, the Fourier coefficients (or "modes") $\phi_n\in\mathbb{C}$ must satisfy the reality condition $\phi_{-n} = \overline{\phi}_n$. A path in $X$ is then specified by a collection of Fourier modes $\phi_n(t)$ as functions of $t\in\mathbb{R}$.

[67] Here we mean the function on $S^1$, not its value at some particular $\sigma$.

In terms of Fourier modes, the Lagrangian (434) is

$$\mathsf{L} = \frac{\kappa}{2} 2\pi \sum_{n\in\mathbb{Z}} \left(\dot{\phi}_n \dot{\phi}_{-n} - n^2 \phi_n \phi_{-n}\right) \tag{437}$$

4.2.2. *Hamiltonian formalism.* In Hamiltonian formalism, the phase space of the system is

$$\Phi = T^* X, \tag{438}$$

with $X$ as in (433). Since $X$ is a linear space, we identify $T^*X$ with $X \times T^*X$ – pairs of a function $\phi(\sigma)$ on $S^1$ and a distribution $\pi(\sigma)$ on $S^1$ (the "momentum"). The canonical symplectic form on $\Phi$ is

$$\omega_{\text{symp}} = \oint_{S^1} dt\ \delta\pi(\sigma) \wedge \delta\phi(\sigma). \tag{439}$$

The corresponding Poisson brackets between $\phi(\sigma)$, $\pi(\sigma')$ (thought of as coordinate functions on $\Phi$) are

$$\{\phi(\sigma), \pi(\sigma')\} = -\delta_{\text{per}}(\sigma - \sigma'), \quad \{\phi(\sigma), \phi(\sigma')\} = 0, \quad \{\pi(\sigma), \pi(\sigma')\} = 0, \tag{440}$$

where $\delta_{\text{per}}$ is the periodic Dirac delta-distibution on $S^1$, $\delta_{\text{per}}(\sigma) = \sum_{n\in\mathbb{Z}} \delta(\sigma + 2\pi n)$ (where on the right $\delta$ are the usual Dirac delta-distributions on $\mathbb{R}$).

To find the Legendre transform of the Lagrangian (434), we first find the relation between momenta and velocities:

$$\pi(\sigma) = \frac{\delta \mathsf{L}}{\delta\dot{\phi}(\sigma)} = \kappa\dot{\phi}(\sigma), \tag{441}$$

cf. (367). Then we find the Hamiltonian (cf. (366)) as

$$H = \oint_{S^1} d\sigma \pi(\sigma)\dot{\phi}(\sigma) - \mathsf{L} = \oint_{S^1} d\sigma \left(\frac{\pi(\sigma)^2}{2\kappa} + \frac{\kappa}{2}(\partial_\sigma \phi)^2\right), \tag{442}$$

where in the second step we expresses velocities in terms of momenta using (441).

The Hamiltonian equations generated by the Hamiltonian $H$ are

$$\dot{\phi} = \frac{1}{\kappa}\pi, \quad \dot{\pi} = \kappa\partial_\sigma^2 \phi. \tag{443}$$

In particular, these equations imply the wave equation (435) for $\phi$.

*Remark* 4.12. The components of the stress-energy tensor of the theory are

$$T_{00} = T_{11} = \frac{\kappa}{2}(\dot{\phi}^2 + (\partial_\sigma \phi)^2), \tag{444}$$

$$T_{01} = T_{10} = \kappa\dot{\phi}\partial_\sigma\phi \tag{445}$$

We note that integrating $T_{00}$ over $\{t\} \times S^1$ one gets

$$H = \oint_{S^1} d\sigma\, T_{00} \tag{446}$$

– the Hamiltonian (or "total energy"). Integrating $T_{01}$ over $\{t\} \times S^1$ one gets

$$P \colon = \oint_{S^1} d\sigma\, T_{01} \tag{447}$$

– the "total momentum" of the system.

Modulo equations of motion, $H$ and $P$ do not depend on $t$ – the position of the spatial slice. One can infer this from Lemma 3.23: translations along $\mathbb{R}$ and rotations along $S^1$ are source symmetries and yield conserved currents, $T_{i0}$ and $T_{i1}$, hence the corresponding charges (fluxes through a spatial slice $\{t\}\times S^1$) are conserved – independent of $t$ modulo equations of motion.

Expanding the field $\phi(\sigma)$ and the momentum $\pi(\sigma)$ in Fourier modes on $S^1$, we have

$$\phi(\sigma)=\sum_{n\in\mathbb{Z}}\phi_n e^{in\sigma},\quad \pi(\sigma)=\frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\pi_n e^{in\sigma}, \tag{448}$$

with reality conditions $\phi_{-n}=\overline{\phi_n}$ and $\pi_{-n}=\overline{\pi_n}$. Poisson brackets (440) correspond to the following brackets between the modes:

$$\{\phi_n,\pi_m\}=-\delta_{n,-m},\quad \{\phi_n,\phi_m\}=0,\quad \{\pi_n,\pi_m\}=0. \tag{449}$$

The Hamiltonian (442) written in terms of the parametrization of the phase space by Fourier modes $\phi_n,\pi_n$ is:

$$H=\sum_{n\in\mathbb{Z}}\frac{1}{2}\frac{1}{2\pi\kappa}\pi_n\pi_{-n}+\frac{1}{2}2\pi\kappa n^2\phi_n\phi_{-n}. \tag{450}$$

At this point we want to fix the normalization factor $\kappa$ to the value

$$\kappa=\frac{1}{4\pi}. \tag{451}$$

Then we have

$$H=\sum_{n\in\mathbb{Z}}\pi_n\pi_{-n}+\frac{1}{4}n^2\phi_n\phi_{-n}=(\pi_0)^2+2\sum_{n>0}\left(|\pi_n|^2+\frac{1}{4}n^2|\phi_n|^2\right). \tag{452}$$

Lecture 17, 9/30/2022

Similarly, the total momentum (447) is:

$$P=\sum_{n\in\mathbb{Z}}in\pi_{-n}\phi_n. \tag{453}$$

The Hamiltonian equations (443) spelled in terms of coordinates $\phi_n,\pi_n$ on the phase space read

$$\dot{\phi}_n=2\pi_n,\quad \dot{\pi}_n=-\frac{n^2}{2}\phi_n. \tag{454}$$

As a consequence, $\phi_n$ satisfies the second-order ODE $\ddot{\phi}_n+n^2\phi_n=0$ (cf. (358)).

Thus the system is a superposition of a collection of non-interacting subsystems: variables $(\phi_0,\pi_0)$ describe a free particle of mass $\mu=\frac{1}{2}$ while variables $(\phi_n,\pi_n)$ for $n\neq 0$ describe a complex harmonic oscillator with frequency $\omega_n=|n|$.

Explain more/better? Also: $(\phi_n,\pi_n)$ or $(\phi_n,\pi_{-n})$?

Real oscillators. To get a better understanding of how the system breaks up into a collection of harmonic oscillators (plus a free particle), it useful to rewrite it in the real parametrization. Introduce the real variables $\phi_n^{(1,2)}$, $\pi_n^{(1,2)}$, with $n>0$, related to complex variables $\phi_n,\pi_n$ by

$$\phi_n=\phi_n^{(1)}+i\phi_n^{(2)},\quad \pi_n=\frac{1}{2}(\pi_n^{(1)}+i\pi_n^{(2)})\qquad \text{for } n>0. \tag{455}$$

I.e., $\phi_n^{(1,2)}$ are the real/imaginary parts of $\phi_n$, $n>0$, and similarly for $\pi_n$. The real variables satisfy the Poisson brackets

$$\{\phi_n^{(\alpha)},\pi_m^{(\beta)}\}=\delta_{nm}\delta_{\alpha\beta},\quad \{\phi_n^{(\alpha)},\phi_m^{(\beta)}\}=0,\quad \{\pi_n^{(\alpha)},\pi_m^{(\beta)}\}=0 \tag{456}$$

for $n>0$ and $\alpha,\beta\in\{1,2\}$. The Hamiltonian (452) in these variables reads

$$(457)\quad H = \pi_0^2 + \sum_{n\geq 1}\sum_{\alpha=1}^{2}\left(\frac{(\pi_n^{(\alpha)})^2}{2} + \frac{n^2}{2}(\phi_n^{(\alpha)})^2\right) = H_{\text{free particle},\,\mu=\frac{1}{2}} + \sum_{n\geq 1}\sum_{\alpha=1}^{2} H_{\text{harmonic oscillator},\,\omega_n=n}$$

The general solution of the Hamiltonian equations (443) is

$$(458)\quad \phi(t,\sigma) = \sum_{n\neq 0}\Big(\underbrace{A_n e^{in(t+\sigma)} + B_n e^{in(-t+\sigma)}}_{\phi_n(t)e^{in\sigma}}\Big) + \underbrace{Ct+D}_{\phi_0(t)},$$

$$(459)\quad \pi(t,\sigma) = \frac{1}{2\pi}\Big(\sum_{n\neq 0}\underbrace{\frac{in}{2}\Big(A_n e^{in(t+\sigma)} - B_n e^{in(-t+\sigma)}\Big)}_{\pi_n(t)e^{in\sigma}} + \underbrace{\frac{C}{2}}_{\pi_0(t)}\Big),$$

where $A_n, B_n, C, D$ are arbitrary constants subject to the reality constraints

$$(460)\quad A_{-n} = \overline{A_n},\ B_{-n} = \overline{B_n}\ \text{ for } n\neq 0, \qquad C, D \in \mathbb{R}.$$

*Remark* 4.13. For the *massive* scalar field (242) on the Minkowski cylinder we can repeat all the computations above, introducing the same parametrization of the phase space by modes $\phi_n, \pi_n$. The Hamiltonian instead of (452) will then be

$$(461)\quad H = \sum_{n\in\mathbb{Z}} \pi_n\pi_{-n} + \frac{1}{4}\omega_n^2\phi_n\phi_{-n}$$

with

$$(462)\quad \omega_n = \sqrt{n^2+m^2}$$

(with $m$ the mass of the scalar field). Thus, the system is a collection of non-interacting harmonic oscillators, one for each $n \in \mathbb{Z}$, with $n$-th oscillator having frequency (462).

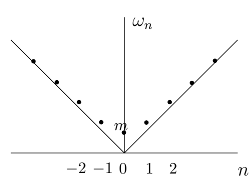



FIGURE 22. Frequencies $\omega_n$ of oscillators comprising the free massive scalar field.

In the massless limit $m \to 0$, the frequencies become $\omega_n \to |n|$. In particular, the $n = 0$ oscillator in the limit becomes a free particle.

4.2.3. *Aside: free particle.* Since the free massless scalar field on a cylinder splits into a family of harmonic oscillators and a single free particle (cf. (457)), we stop for a moment to discuss the free particle, as a classical and as a quantum mechanical system.

The free particle moving on $\mathbb{R}$ is the Lagrangian formalism is defined by the space of fields Fields $= \mathrm{Map}([t_0,t_1],\mathbb{R})$ with action functional

$$S[x(t)] = \int_{t_0}^{t_1} d\tau\, \underbrace{\frac{\mu\dot{x}^2}{2}}_{\mathsf{L}}, \tag{463}$$

where $\mu > 0$ is a parameter – "mass" of the particle.

In the Hamiltonian formalism, the system is described by the phase space $\Phi = T^*\mathbb{R}$ and the Hamiltonian

$$H = \frac{p^2}{2\mu} \tag{464}$$

(which is in particular the Legendre transform of the Lagrangian $\mathsf{L} = \frac{\mu v^2}{2}$).

In canonical quantization, we have the Weyl algebra generated by $\widehat{x},\widehat{p}$ subject to $[\widehat{p},\widehat{x}] = -i\hbar$, and the quantum Hamiltonian (using the symmetric Weyl quantization) is

$$\widehat{H} = \frac{\widehat{p}^2}{2\mu}. \tag{465}$$

In Schrödinger representation of the Weyl algebra, the Hamiltonian acts as the differential operator

$$\widehat{H} = -\frac{\hbar^2}{2\mu}\partial_x^2 \tag{466}$$

on the Hilbert space $\mathcal{H} = L^2_{\mathbb{C}}(\mathbb{R})$.

The eigenvectors of $\widehat{H}$ are the vectors

$$|p\rangle : = e^{\frac{i}{\hbar}px} \tag{467}$$

with $p \in \mathbb{R}$ a parameter (momentum). One then has

$$\widehat{H}|p\rangle = \frac{p^2}{2\mu}|p\rangle. \tag{468}$$

In particular, the operator $\widehat{H}$ has a continuum eigenvalue spectrum $[0,\infty)$, where the eigenvalue 0 is nondegenerate and all positive eigenvalues have multiplicity 2. We also note that eigenvectors (467) are not points of $L^2_{\mathbb{C}}(\mathbb{R})$ (not square-integrable), but rather are limit points of the space (which is the usual case for a continuum spectrum).

4.2.4. *Canonical quantization.* We now proceed to the canonical quantization of the free massless scalar field on the Minkowski cylinder.

We promote the modes $\phi_n,\pi_n$ to generators $\widehat{\phi}_n,\widehat{\pi}_n$ of the Weyl algebra, subject to the relations

$$[\widehat{\pi}_n,\widehat{\phi}_m] = -i\delta_{n,-m}, \quad [\widehat{\phi}_n,\widehat{\phi}_m] = 0, \quad [\widehat{\pi}_n,\widehat{\pi}_m] = 0. \tag{469}$$

For convenience we set $\hbar = 1$.

Next, we introduce a system of creation/annihilation operators $\widehat{a}_n,\widehat{\bar{a}}_n$, $n \neq 0$, subject to hermitian conjugation properties

$$(\widehat{a}_n)^+ = \widehat{a}_{-n}, \quad (\widehat{\bar{a}}_n)^+ = \widehat{\bar{a}}_{-n} \tag{470}$$

and related to the Weyl generators $\widehat{\phi}_n, \widehat{\pi}_n$, with $n \neq 0$ by[68]

$$\begin{aligned} \widehat{\phi}_n &= \frac{i}{n}(-\widehat{a}_{-n} + \widehat{\overline{a}}_n), \\ \widehat{\pi}_n &= \frac{\widehat{a}_{-n} + \widehat{\overline{a}}_n}{2}. \end{aligned} \tag{471}$$

The commutation relations corresponding to (469) are

$$[\widehat{a}_n, \widehat{a}_m] = n\delta_{n,-m}, \quad [\widehat{\overline{a}}_n, \widehat{\overline{a}}_m] = n\delta_{n,-m}, \quad [\widehat{a}_n, \widehat{\overline{a}}_m] = 0. \tag{472}$$

In terms of these creation/annihilation operators (and the zero-mode operators $\widehat{\phi}_0$, $\widehat{\pi}_0$ which need to be treated separately), the quantum Hamiltonian (obtained by symmetric Weyl quantization) is:

$$\widehat{H} = \sum_{n\neq 0} \frac{\widehat{a}_{-n}\widehat{a}_n + \widehat{\overline{a}}_{-n}\widehat{\overline{a}}_n}{2} + (\widehat{\pi}_0)^2 = \frac{1}{2}\sum_{n\in\mathbb{Z}} \left(\widehat{a}_{-n}\widehat{a}_n + \widehat{\overline{a}}_{-n}\widehat{\overline{a}}_n\right). \tag{473}$$

In the second equality we introduced the notation

$$\widehat{a}_0 = \widehat{\overline{a}}_0 := \widehat{\pi}_0. \tag{474}$$

The canonical quantization of the total momentum operator (453), written in terms of creation/annihilation operators, is

$$\widehat{P} = \frac{1}{2}\sum_{n\in\mathbb{Z}} \left(\widehat{a}_{-n}\widehat{a}_n - \widehat{\overline{a}}_{-n}\widehat{\overline{a}}_n\right). \tag{475}$$

*Remark* 4.14 (Heisenberg Lie algebra). One can consider the Lie $*$-algebra (the Heisenberg Lie algebra)

$$\mathrm{Heis} := \mathrm{Span}_{\mathbb{C}}(\{\widehat{a}_n\}_{n\in\mathbb{Z}}, \mathbb{K}) \tag{476}$$

where $\mathbb{K}$ is the central element and the commutation relations are

$$[\widehat{a}_n, \widehat{a}_m] = n\delta_{n,-m}\mathbb{K}. \tag{477}$$

with the involution (hermitian conjugation) acting as $\widehat{a}_n^+ = \widehat{a}_{-n}$, $\mathbb{K}^+ = \mathbb{K}$. It is the special case of the general Heisenberg Lie algebra (Definition 4.2), for the symplectic vector space $V$ of Laurent series on $\mathbb{C}^*$

$$V = \left\{ f(z) = \sum_{n\in\mathbb{Z}} f_n z^{-n} \right\} \tag{478}$$

with symplectic form

$$\omega_{\mathrm{symp}}(f,g) = i\,\mathrm{res}_{z=0}(fdg) \tag{479}$$

– the residue at $z = 0$ (i.e. the coefficient of $z^{-1}dz$) of the meromorphic 1-form $fdg$.[69] The basis vectors $z^{-n}$ in $V$ correspond to the generators $\widehat{a}_n$ of Heis.

[68]One can also express the operators $\widehat{a}_n, \widehat{\overline{a}}_n$ in terms of the standard creation/annihilation operators (390), (391) for the real oscillators, as in (455): for $n > 0$ one sets $\widehat{a}_n = \sqrt{\frac{n}{2}}(-i\widehat{a}_n^{(1)} - \widehat{a}_n^{(2)})$, $\widehat{a}_{-n} = \sqrt{\frac{n}{2}}(i\widehat{a}_n^{(1)+} - \widehat{a}_n^{(2)+})$, $\widehat{\overline{a}}_n = \sqrt{\frac{n}{2}}(-i\widehat{a}_n^{(1)} + \widehat{a}_n^{(2)})$, $\widehat{\overline{a}}_{-n} = \sqrt{\frac{n}{2}}(i\widehat{a}_n^{(1)+} + \widehat{a}_n^{(2)+})$.

[69]The normalization factor $i$ in (479) compensates the factor $-i$ in the general definition of Heisenberg Lie algebra (371), i.e., one has the commutation relation $[\widehat{f}, \widehat{g}] = \mathrm{res}_{z=0}(fdg)\,\mathbb{K}$.

The full Lie algebra of mode operators of the free massless scalar fields can then be described via two copies Heis, $\overline{\mathrm{Heis}}$ of the algebra above:

$$\mathrm{Span}_{\mathbb{C}}(\{\widehat{\phi}_n,\widehat{\pi}_n\}_{n\in\mathbb{Z}},\mathbb{K}) = \frac{\mathrm{Heis}\oplus\overline{\mathrm{Heis}}}{\widehat{a}_0=\widehat{\overline{a}}_0,\mathbb{K}=\overline{\mathbb{K}}}\oplus\mathbb{C}\cdot\widehat{\phi}_0, \tag{480}$$

where on the right the extra generator $\widehat{\phi}_0$ interacts with the Heisenberg Lie algebras via

$$[\widehat{a}_0,\widehat{\phi}_0] = -i\mathbb{K}. \tag{481}$$

From (473) and (472) one easily finds the commutators between the Hamiltonian $\widehat{H}$ and the operators $\widehat{a}_n$, $\widehat{\overline{a}}_n$:

$$[\widehat{H},\widehat{a}_n] = -n\widehat{a}_n,\quad [\widehat{H},\widehat{\overline{a}}_n] = -n\widehat{\overline{a}}_n,\qquad n\in\mathbb{Z}. \tag{482}$$

In particular, for $n>0$ applying $\widehat{a}_n$ or $\widehat{\overline{a}}_n$ to an eigenvector of $\widehat{H}$ *decreases* the eigenvalue (total energy of the state) by $n$, while applying $\widehat{a}_{-n}$ or $\widehat{\overline{a}}_{-n}$ *increases* the eigenvalue by $n$. Thus, it is natural to think of $\widehat{a}_n,\widehat{\overline{a}}_n$ as annihilation operators and of $\widehat{a}_{-n},\widehat{\overline{a}}_{-n}$ as creation operators.

Next, consider the commutators of $\widehat{a}_n$, $\widehat{\overline{a}}_n$ with the total momentum operator (475):

$$[\widehat{P},\widehat{a}_n] = -n\widehat{a}_n,\quad [\widehat{P},\widehat{\overline{a}}_n] = +n\widehat{\overline{a}}_n,\qquad n\in\mathbb{Z}. \tag{483}$$

Thus, for $n>0$, applying $\widehat{a}_{-n}$ to a joint eigenvector of $\widehat{H}$ and $\widehat{P}$ increases the energy and the total momentum of the system (creates – or adjoins to the system – a "left-mover" – a quantum with positive momentum), while applying $\widehat{\overline{a}}_{-n}$ increases the energy but decreases the total momentum (creates a "right-mover").

To summarize, we have the following table for each $n>0$.

| | annihilation operator | creation operator |
|---|---|---|
| left-mover | $\widehat{a}_n$ | $\widehat{a}_{-n}$ |
| right-mover | $\widehat{\overline{a}}_n$ | $\widehat{\overline{a}}_{-n}$ |

The space of states. The space of states of the full system (the massless free scalar theory) can be described as the tensor product of the spaces of states for the constituent subsystems:

$$\mathcal{H} = \mathcal{H}_{\text{free particle}}\otimes\bigotimes_{n\neq 0}\mathcal{H}_{\text{harmonic oscillator }\omega_n=|n|}. \tag{484}$$

One can choose to represent each factor in (484) by the Schrödinger representation, thereby obtaining a tensor product of countably many copies of $L^2(\mathbb{R})$.

A different (better) description of $\mathcal{H}$ is as a "Fock space" – in the vein of the description (404) of the space of states of harmonic oscillator as spanned by excitations of a vacuum state given by repeatedly applying creation operators (Verma module description). In the case of the free massless scalar field, we pick from the first factor of (484) any vector $|\pi_0\rangle$ (cf. (467)), with $\pi_0\in\mathbb{R}$ the zero-mode momentum, tensored with vacua $|0\rangle$ in each oscillator factor – we denote the result by abuse of notations again $|\pi_0\rangle$ (this vector is referred to as "psedovacuum").[70]

[70]Note that by construction we have $\widehat{a}_n|\pi_0\rangle = \widehat{\overline{a}}_n|\pi_0\rangle = 0$ for any $n>0$.

Then we act on $|\pi_0\rangle$ by the creation operators corresponding to different oscillators, creating an excited state; this gives a basis for $\mathcal{H}$:

$$\mathcal{H} = \bigoplus_{r\geq 0, s\geq 0} \mathrm{Span}_{\mathbb{C}} \Big\{ \prod_{i=1}^{r} \widehat{a}_{-n_i} \prod_{j=1}^{s} \widehat{\overline{a}}_{-\overline{n}_j} |\pi_0\rangle \;\Big|\; \begin{array}{c} 1\leq n_1 \leq n_2 \leq \cdots \leq n_r, \\ 1 \leq \overline{n}_1 \leq \overline{n}_2 \leq \cdots \leq \overline{n}_s, \\ \pi_0 \in \mathbb{R} \end{array} \Big\}. \tag{485}$$

Let us denote the basis vectors spanning $\mathcal{H}$ by

$$|\pi_0; \{n_i\}, \{\overline{n}_j\}\rangle := \prod_{i=1}^{r} \widehat{a}_{-n_i} \prod_{j=1}^{s} \widehat{\overline{a}}_{-\overline{n}_j} |\pi_0\rangle. \tag{486}$$

We think of the basis vector $|\pi_0; \{n_i\}, \{\overline{n}_j\}\rangle$ as a multiparticle state, consisting of

- $r$ left-moving quanta carrying energy-momentum 2-vectors $(n_i, n_i)$, $i = 1, \ldots, r$ and
- $s$ right-moving quanta carrying energy-momentum $(\bar{n}_j, -\bar{n}_j)$, $j = 1, \ldots, s$.

We motivate this interpretation more below, after (492).

This is a bit murky, I should explain better

*Remark* 4.15. Thinking of the system a string moving in the target $\mathbb{R}$ (for each time $t$, we have a map $\phi\colon \{t\} \times S^1 \to \mathbb{R}$), the zero-mode momentum $\pi_0$ can be understood as the (target) momentum of the center-of-mass of the string, and has nothing to do with the (source) total momentum $P$.

An equivalent description of $\mathcal{H}$ as a Fock space (a different way to enumerate the basis vectors) is as follows:

$$\mathcal{H} = \mathrm{Span}_{\mathbb{C}} \left\{ \prod_{n\geq 1} (\widehat{a}_{-n})^{k_n} \prod_{\overline{n}\geq 1} (\widehat{\overline{a}}_{-\overline{n}})^{\overline{k}_{\overline{n}}} |\pi_0\rangle \;\Big|\; \begin{array}{l} k_n \geq 0, \overline{k}_{\overline{n}} \geq 0, \\ \text{finitely many of } k_n, \overline{k}_{\overline{n}} \text{ are nonzero} \end{array} \right\}. \tag{487}$$

The numbers $k_n$, $\overline{k}_{\overline{n}}$ are the "occupation numbers" for the excitations with energy-momentum $(n, n)$ and $(\overline{n}, -\overline{n})$, respectively (i.e. $k_n$, $\overline{k}_{\overline{n}}$ are the numbers of quanta of these types).

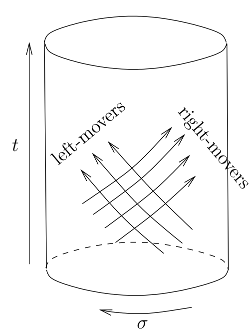



FIGURE 23. Left- and right-movers on a cylinder.

Lecture 18, 10/3/2022

Note that applying the Hamiltonian (473) to the pseudovacuum $|\pi_0\rangle$ we obtain

$$\widehat{H}|\pi_0\rangle = \widehat{\pi}_0^2|\pi_0\rangle + \frac{1}{2}\sum_{n>0} \Big( \widehat{a}_{-n} \underbrace{\widehat{a}_n|\pi_0\rangle}_{0} + \widehat{\overline{a}}_{-n} \underbrace{\widehat{\overline{a}}_n|\pi_0\rangle}_{0} \Big) + \tag{488}$$

$$+\frac{1}{2}\sum_{n<0}\Big(\underbrace{\widehat{a}_{-n}\widehat{a}_n}_{-n+\widehat{a}_n\widehat{a}_{-n}}|\pi_0\rangle+\underbrace{\widehat{\overline{a}}_{-n}\widehat{\overline{a}}_n}_{-n+\widehat{\overline{a}}_n\widehat{\overline{a}}_{-n}}|\pi_0\rangle\Big)=\Big(\pi_0^2+\underbrace{\sum_{n<0}(-n)}_{\text{divergence!}}\Big)|\pi_0\rangle$$

– $|\pi_0\rangle$ multiplied by a divergent factor. By a similar reason, each basis vector $|\pi_0;\{n_i\},\{\overline{n}_j\}\rangle$ is an eigenvector of $\widehat{H}$ with a divergent eigenvalue. To deal with this problem, one uses normal ordering.

Uniformize the conventions (Free-Ass to Weyl) with harmonic oscillator case.

Normal ordering. Normal ordering (in the context of the free massless scalar field) is defined as a $\mathbb{C}$-linear map $:\cdots:$ from the free associative algebra generated by the operators $\{\widehat{a}_n,\widehat{\overline{a}}_n\}_{n\in\mathbb{Z}}$ to the Weyl algebra (i.e., to the quotient of the free associative algebra by relations (472)). Acting on a word, it puts the annihilation operators $\widehat{a}_{>0},\widehat{\overline{a}}_{>0}$ to the right and creation operators $\widehat{a}_{<0},\widehat{\overline{a}}_{<0}$ to the left (and then projects to the Weyl algebra).

For example, the normally ordered Hamiltonian (473) and total momentum operators (475) are

$$:\widehat{H}: \;=\; \widehat{\pi}_0^2+\sum_{n>0}\left(\widehat{a}_{-n}\widehat{a}_n+\widehat{\overline{a}}_{-n}\widehat{\overline{a}}_n\right), \tag{489}$$

$$:\widehat{P}: \;=\; \sum_{n>0}\left(\widehat{a}_{-n}\widehat{a}_n-\widehat{\overline{a}}_{-n}\widehat{\overline{a}}_n\right). \tag{490}$$

Acting with these normally ordered operators on basis vectors (486), we don't encounter any divergencies (unlike in (488)), and we have

$$:\widehat{H}:|\pi_0;\{n_i\},\{\overline{n}_j\}\rangle \;=\; \left(\pi_0^2+\sum_i n_i+\sum_j \overline{n}_j\right)|\pi_0;\{n_i\},\{\overline{n}_j\}\rangle, \tag{491}$$

$$:\widehat{P}:|\pi_0;\{n_i\},\{\overline{n}_j\}\rangle \;=\; \left(\sum_i n_i-\sum_j \overline{n}_j\right)|\pi_0;\{n_i\},\{\overline{n}_j\}\rangle. \tag{492}$$

In particular, all states $|\pi_0;\{n_i\},\{\overline{n}_j\}\rangle$ are eigenvectors of both $:\widehat{H}:$ and $:\widehat{P}:$. Interpreting the joint eigenvalue as the energy-momentum 2-vector, we see that:

- The pseudovacuum $|\pi_0\rangle$ has energy-momentum $(\pi_0^2,0)$.
- Applying $\widehat{a}_{-n}$ with $n>0$ to a state, we increase the energy-momentum by $(n,n)$ (which we interpret as adjoining a left-moving quantum to the system).
- Applying $\widehat{\overline{a}}_{-\overline{n}}$ with $n>0$ to a state, we increase the energy-momentum by $(\overline{n},-\overline{n})$ (which we interpret as adjoining a right-moving quantum).

*Remark* 4.16. There is a single (up to normalization) null-vector of $:\widehat{H}:$ in $\mathcal{H}$ – the vector

$$|\mathrm{vac}\rangle :=|\pi_0=0\rangle, \tag{493}$$

i.e. the pseudovacuum with $\pi_0=0$. We call this vector the vacuum vector (or vacuum state). It is a null-vector for both $:\widehat{H}:$ and $:\widehat{P}:$, which is interpreted as invariance of $|0\rangle$ under time-translations and rotation along $S^1$.[71]

Too long? Also, put links to later sections where central charge and $Z_{torus}$ are covered.

[71]Time-translation by time $t$ is represented on the space of states by the evolution operator $U(t)=e^{-it\widehat{H}}$. Rotation by angle $\theta$ along $S^1$ is similarly represented by $R(\theta)=e^{-i\theta\widehat{P}}$.

*Remark* 4.17. Later – after switching to Euclidean metric – we will see that the partition function of a torus defined using the normally ordered operators $:\widehat{H}:$ and $:\widehat{P}:$ does not have the expected modular invariance property (see Section 1.4.1). To restore it, one should replace $:\widehat{H}:$ with the operator $:\widehat{H}:-\frac{1}{12}$ (while $:\widehat{P}:=\widehat{P}$ does not have to be changed), which can be seen as the original operator $\widehat{H}$ (473) with the divergence regularized by Riemann zeta-function regularization:

$$\begin{aligned}(494)\quad \widehat{H} &= \frac{1}{2}\sum_{n\in\mathbb{Z}}\left(\widehat{a}_{-n}\widehat{a}_n+\widehat{\overline{a}}_{-n}\widehat{\overline{a}}_n\right)= \\ &= \frac{1}{2}\sum_{n>0}\left(\widehat{a}_{-n}\widehat{a}_n+\widehat{\overline{a}}_{-n}\widehat{\overline{a}}_n\right)+\frac{1}{2}\sum_{n<0}\left(-2n+\widehat{a}_{-n}\widehat{a}_n+\widehat{\overline{a}}_{-n}\widehat{\overline{a}}_n\right) \\ =: \widehat{H}: + \sum_{n>0} n &\underbrace{=}_{\text{zeta}-\text{regularization}} :\widehat{H}:+\lim_{s\to-1}\sum_{n>0}n^s =: \widehat{H}:+\zeta(-1) =: \widehat{H}:-\frac{1}{12}.\end{aligned}$$

At the moment this zeta-regularization prescription looks entirely ad hoc, and it is not clear why it should help with modular invariance. Note that with respect to this regularized $\widehat{H}$, the vacuum state $|\mathrm{vac}\rangle$ has energy $-\frac{1}{12}$ instead of zero. EDIT

The (somewhat surprising) take-home message for the moment is that the normal ordering breaks conformal invariance (in a mild way[72]) – in fact we will not see any problem with normal ordering in the genus zero theory (correlators of point observables on a cylinder/plane) – they do not contradict conformal invariance, but in genus one we have a problem.

4.2.5. *Aside: Schrödinger vs Heisenberg picture in quantum mechanics.* In the Scrödinger picture of quantum mechanics, time-evolution acts on states. I.e., one has time-dependent families of states linked by the evolution operator:

$$(495)\qquad |\psi\rangle_t = U(t-t_0)|\psi\rangle_{t_0}$$

where

$$(496)\qquad U(t)=e^{-i\widehat{H}t}$$

is the unitary evolution operator. Put another way, one has a family of the spaces of states $\mathcal{H}_t$ linked by isomorphisms $\mathcal{H}_t \xrightarrow{U(t'-t)} \mathcal{H}_{t'}$. Observables are operators $\widehat{O}$ acting on $\mathcal{H}_t$ at some particular $t$.

The infinitesimal version of (495) is the Schrödinger equation

$$(497)\qquad \frac{d}{dt}|\psi\rangle_t = -i\widehat{H}|\psi\rangle_t$$

(we mention it for comparison with the Heisenberg picture).

In the Heisenberg picture, evolution acts on observables instead of states. All states are thought of as elements of $\mathcal{H}_{t_0}$ for some fixed reference time $t_0$. But an observable is understood as a family $\widehat{O}_t$ arising as a pullback of some fixed ($t$-independent) operator $\widehat{O}$ acting on $\mathcal{H}_t$, along the evolution $U(t-t_0)\colon \mathcal{H}_{t_0}\to\mathcal{H}_t$:

$$(498)\qquad \widehat{O}_t \circlearrowright \mathcal{H}_{t_0} \xrightarrow{U(t-t_0)} \mathcal{H}_t \circlearrowleft \widehat{O}$$

[72] The change of the quantum Hamiltonian by a multiple of identity is a somewhat subtle effect: we usually need the commutators with $\widehat{H}$, not $\widehat{H}$ itself. E.g. time-dependence of observables in the Heisenberg picture (500) only depends on commutators with $\widehat{H}$.

I.e., one has

$$\widehat{O}_t = U(t-t_0)^{-1}\,\widehat{O}\,U(t-t_0). \tag{499}$$

The infinitesimal version of this equation is the Heisenberg equation

$$-i\frac{d}{dt}\widehat{O}_t = [\widehat{H},\widehat{O}_t]. \tag{500}$$

Below we will use the notation $\widehat{O}(t)\colon=\widehat{O}_t$ for the time-dependent operators of the Heisenberg picture.

Consider a correlator in Schrödinger picture (cf. Section 1.3.1) of quantum mechanics on the source interval (cobordism) $[t_{\mathrm{in}},t_{\mathrm{out}}]$, with in/out states $|\psi_{\mathrm{in}}\rangle$, $\langle\psi_{\mathrm{out}}|$,[73] of observables $\widehat{O}_1,\ldots,\widehat{O}_n$ inserted at times $t_{\mathrm{in}}<t_1<\cdots t_n<t_{\mathrm{out}}$.

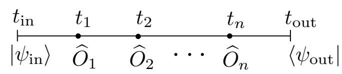



FIGURE 24. Correlator in quantum mechanics.

The correlator is given by

$$\langle\psi_{\mathrm{out}}|\,U(t_{\mathrm{out}}-t_n)\,\widehat{O}_n\cdots\widehat{O}_2\,U(t_2-t_1)\,\widehat{O}_1\,U(t_1-t_{\mathrm{in}})\,|\psi_{\mathrm{in}}\rangle \tag{501}$$

The same quantity can be equivalently written in Heisenberg picture, as

$$\langle\widetilde{\psi}_{\mathrm{out}}|\,\widehat{O}_n(t_n)\cdots\widehat{O}_2(t_2)\,\widehat{O}_1(t_1)\,|\widetilde{\psi}_{\mathrm{in}}\rangle \tag{502}$$

where $\widehat{O}_k(t_k)\colon=U(t_k-t_0)^{-1}\;\widehat{O}\;U(t_k-t_0)$ are the time-dependent observables (499) and

$$|\widetilde{\psi}_{\mathrm{in}}\rangle=U(t_0-t_{\mathrm{in}})|\psi_{\mathrm{in}}\rangle,\qquad |\widetilde{\psi}_{\mathrm{out}}\rangle=U(t_0-t_{\mathrm{out}})|\psi_{\mathrm{out}}\rangle \tag{503}$$

are the in-out states expressed as elements of the reference Hilbert space $\mathcal{H}_{t_0}$. Herethe reference time $t_0$ is chosen arbitrarily.

*Remark* 4.18. We remark that the product of time-dependent observables $\widehat{O}_n(t_n)\cdots\widehat{O}_1(t_1)$
clean it up a bit? in (502) is time-ordered – the times satisfy $t_n>\cdots>t_1$.

When we later consider field theory in Euclidean signature, this will correspond to setting $t=-it_{\mathrm{Eucl}}$ in the formulae above, with $t_{\mathrm{Eucl}}>0$. Then the evolution operator $U(T_{\mathrm{Eucl}})=e^{-T_{\mathrm{Eucl}}\widehat{H}}$ is non-invertible and only defined for positive $T_{\mathrm{Eucl}}$. In this situation, *only* time-ordered products of operators are defined. In this setting we should use (500) to define $T_{\mathrm{Eucl}}$-dependent observables.

4.2.6. *Back to free massless scalar field on a cylinder: time-dependent field operator.* Back to the quantum field theory on the cylinder, we think of it as a special model of quantum mechanics, where we understood the space of states (485) and we have a family of special operators

$$\widehat{\phi}(\sigma)=\sum_{n\in\mathbb{Z}}\widehat{\phi}_n e^{in\sigma}=\widehat{\phi}_0+\sum_{n\neq 0}\frac{i}{n}(-\widehat{a}_{-n}+\widehat{\overline{a}}_n)e^{in\sigma} \tag{504}$$

[73] We remind that in Dirac's notation $|\cdots\rangle$ are vectors in $\mathcal{H}$ and $\langle\cdots|$ are vectors in the linear dual $\mathcal{H}^*$.

(one operator for each $\sigma \in S^1$) acting on $\mathcal{H}$ and independent of $t$. We can treat these as special examples of observables in the Schrödinger picture.

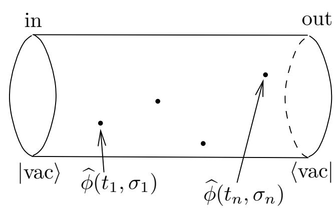
The corresponding time-dependent observables in the Heisenberg picture are obtained by solving the equation (500), which yields

$$\widehat{\phi}(t,\sigma) = \underbrace{e^{i\widehat{H}t}}_{U(t)^{-1}}\,\widehat{\phi}(\sigma)\,\underbrace{e^{-i\widehat{H}t}}_{U(t)} = \\ = \widehat{\phi}_0 + 2t\widehat{\pi}_0 + \sum_{n\neq 0} \frac{i}{n}\Big( - \widehat{a}_{-n} e^{in(t+\sigma)} + \widehat{\overline{a}}_n e^{in(-t+\sigma)}\Big). \tag{505}$$

Note the similarity of this formula with the formula for the general solution of the equations of motion in the classical theory (458).

Then we can consider, e.g., correlators of the form

(506)




$$= \quad \langle \mathrm{vac}|\widehat{\phi}(t_n,\sigma_n)\cdots\widehat{\phi}(t_1,\sigma_1)|\mathrm{vac}\rangle$$

with $t_n > \cdots > t_1$ and with $\sigma_n,\ldots,\sigma_1 \in S^1$. These correlators can be explicitly computed using (505) and using the commutation relations (472). We will discuss such correlators below, once we switch to Euclidean signature.

## 4.3. **Free massless scalar field on $\mathbb{C}$.**

4.3.1. *From Minkowski to Euclidean cylinder (via Wick rotation), and then to $\mathbb{C}^*$ (via exponential map).* Now let us switch the spacetime manifold of the free massless scalar field from Minkowski cylinder to the cylinder $\Sigma = \mathbb{R}\times S^1$ with Euclidean metric $g = (d\tau)^2 + (d\sigma)^2$. Here we will be denoting the Euclidean time – the coordinate on $\mathbb{R}$ – by $\tau$ (instead of $T_{\mathrm{Eucl}}$); $\sigma$ is the coordinate on $S^1$ as before.

Introducing a complex coordinate

$$\zeta = \tau + i\sigma, \quad \overline{\zeta} = \tau - i\sigma, \tag{507}$$

we can identify $\Sigma$ with $\mathbb{C}/2\pi i\mathbb{Z}$ (where $\zeta$ is the standard coordinate on $\mathbb{C}$).

Another useful model for $\Sigma$ for us is the punctured complex plane $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ with complex coordinate $z = e^{\zeta}$. This is in fact the model we will be using the most.

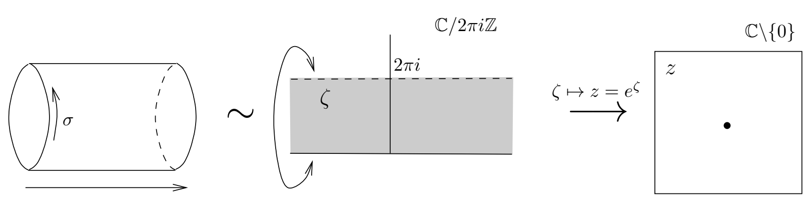



FIGURE 25. Three models of Euclidean cylinder.

The action functional of the classical theory is

$$\begin{aligned} S_{\mathrm{Eucl}}(\phi) &= \frac{\kappa}{2}\int_{\mathbb{R}\times S^1} d\tau d\sigma\, ((\partial_\tau\phi)^2+(\partial_\sigma\phi)^2) \\ &= 2\kappa \int_{\mathbb{C}/2\pi i\mathbb{Z}} \frac{i}{2} d\zeta\wedge d\bar\zeta\; \partial_\zeta\phi\,\partial_{\bar\zeta}\phi \\ &= 2\kappa\int_{\mathbb{C}\backslash\{0\}} \frac{i}{2} dz\wedge d\bar{z}\; \partial_z\phi\,\partial_{\bar z}\phi \end{aligned} \tag{508}$$

where $\kappa=\frac{1}{4\pi}$, as before (451).

The stress-energy tensor written in the complex coordinates $\zeta,\bar\zeta$ or $z,\bar z$ reads

$$\begin{aligned} T &= \underbrace{\kappa(\partial_\zeta\phi)^2}_{T_{\zeta\zeta}}(d\zeta)^2+\underbrace{\kappa(\partial^2_{\bar\zeta}\phi)}_{T_{\bar\zeta\bar\zeta}}(d\bar\zeta)^2 \\ &= \underbrace{\kappa(\partial_z\phi)^2}_{T_{zz}}(dz)^2+\underbrace{\kappa(\partial^2_{\bar z}\phi)}_{T_{\bar z\bar z}}(d\bar z)^2. \end{aligned} \tag{509}$$

The switch from Minkowski cylinder to Euclidean cylinder is achieved via "Wick rotation" – by substituting

$$t=-i\tau \tag{510}$$

in the formulae for the Minkowski cylinder with $\tau>0$ the Euclidean time. In particular, the evolution operator changes as

$$e^{i\widehat{H}t}\rightsquigarrow e^{-\widehat{H}\tau}. \tag{511}$$

The space of states $\mathcal{H}$ and the quantum Hamiltonian $\widehat{H}$ are the same in Minkowski and in Euclidean setting.[74]

The time-dependent (Heisenberg) field operator (505) in Euclidean setting becomes

$$\begin{aligned} \widehat{\phi}(\zeta) &= \widehat{\phi}_0 - i\widehat{\pi}_0(\zeta+\bar\zeta)+\sum_{n\neq 0}\frac{i}{n}\left(\widehat{a}_n e^{-n\zeta}+\widehat{\bar{a}}_n e^{-n\bar\zeta}\right) \\ &= \widehat{\phi}_0 - i\widehat{\pi}_0\log(z\bar z)+\sum_{n\neq 0}\frac{i}{n}\left(\widehat{a}_n z^{-n}+\widehat{\bar{a}}_n \bar{z}^{-n}\right). \end{aligned} \tag{512}$$

Lecture 19, 10/5/2022

4.3.2. *Aside: Wick's lemma (in the operator formalism).* Let

$$\mathcal{A}=\mathrm{Span}_{\mathbb{C}}\left(\{\widehat{b}_k,\widehat{b}_k^+\}_{k\in I},\mathbb{K}\right) \tag{513}$$

[74] If we were to retrace our steps and start from the Euclidean action functional, reinterpret it as Lagrangian mechanics, do the Legendre transform to obtain a Hamiltonian description and then canonically quantize, we would have obtained a different quantum Hamiltonian. This has to do with the fact that the rule of canonical quantization (375) is attuned to the unitary evolution; in Euclidean theory the canonical commutation relations have to be changed accordingly.

be the Heisenberg Lie algebra spanned by pairs of creation/annihilation operators indexed by some set $I$, and the central element $\mathbb{K}$, subject to commutation relations[75]

$$[\widehat{b}_i, \widehat{b}_j^+] = \delta_{ij}\mathbb{K}, \quad [\widehat{b}_i, \widehat{b}_j] = 0, \quad [\widehat{b}_i^+, \widehat{b}_j^+] = 0. \tag{514}$$

*Remark* 4.19. More abstractly, we think of a symplectic vector space $(V, \omega)$ equipped with a compatible complex structure $J\colon V \to V$, $J^2 = -\mathrm{id}$, with $g(x, y) = \omega(x, Jy)$ a positive-definite bilinear form. (Put another way, $(V, \omega, J, g)$ is a Kähler vector space.) Then one has a splitting $\mathbb{C} \otimes V = U \oplus \overline{U}$ of the complexified space $V$ into the $\pm i$-eigenspaces of $J$. Then the Lie algebra (513) is the Heisenberg Lie algebra of $(V, \omega)$ in the sense of Definition 4.2, where we have chosen some basis $\{b_i\}$ in $\overline{U}$ and the dual basis $\{b_i^+\}$ in $U$, which corresponds to creation/annihilation operators $\{\widehat{b}_i, \widehat{b}_i^+\}$ in $\mathcal{A}$.

Let $\{A_p\}_{p\in Y}$ be a collection of "preferred" elements of $\mathcal{A}$ which are some linear combinations of creation/annihilation operators,

$$A_p = \sum_{i\in I} c_{pi}\widehat{b}_i + d_{pi}\widehat{b}_i^+,$$

with $c_{pi}, d_{pi}$ complex coefficients. The indexing set $Y$ for the collection $\{A_p\}$ is arbitrary; it has no a priori relation to the set $I$ indexing the basis in $\mathcal{A}$.

We define the normal ordering $:\cdots:$ of an element of the free associative algebra generated by $\{\widehat{a}_k, \widehat{a}_k^+\}_{k\in I}$ as a $\mathbb{C}$-linear operation which reorders each word, putting the annihiliation operators $\widehat{a}_k$ to the right of the word and creation operators $\widehat{a}_k^+$ to the left of the word, and then projects the reordered word to the Weyl algebra of $\mathcal{A}$,[76]

$$\mathrm{Weyl}(\mathcal{A}) = U\mathcal{A}/(\mathbb{K} = 1). \tag{515}$$

For any pair $p, q$ one has the equality

$$A_pA_q - :A_pA_q: = g_{pq} \tag{516}$$

in the Weyl algebra, with $g_{pq} \in \mathbb{C}$ some complex numbers; we will (suggestively) refer to the matrix $(g_{pq})_{p,q\in Y}$ as the "propagator."

The reason for equality (516), with a multiple of identity on the right, is that it is clearly true if both $A_p$ and $A_q$ are creation or annihilation operators, due to the commutation relations (514); by linearity this property extends to $A_p, A_q$ any linear combinations of creation/annihilation operators.

*Remark* 4.20. Note that the normally ordered products satisfy the symmetry property

$$:A_{p_1}\cdots A_{p_n}: = :A_{p_{\sigma(1)}}\cdots A_{p_{\sigma(n)}}: \tag{517}$$

for $\sigma$ any permutation of the set $\{1, \ldots, n\}$. This property is obvious for $A_p$'s being just creation/annihilation operators, then one extends to general $A_p$'s by $\mathbb{C}$-linearity.

[75] We call the creation/annihilation operators here $\widehat{b}, \widehat{b}^+$ to avoid confusion with the operators $\widehat{a}, \widehat{\overline{a}}$ in the scalar field theory – which are also creation/annihilation operators, just with a different normalization convention.

[76] Cf. Definition 4.4. Unlike the setup of Section 4.1.3, here we are not thinking about $\hbar \to 0$ asymptotics (we are in purely quantum theory where we set $\hbar = 1$), so we don't consider coefficients in formal power series in $\hbar$.

The following is a very useful combinatorial statement allowing one to express any element of the Weyl algebra (or the subalgebra generated by the elements $\{A_p\}_{p\in Y}$) in terms of normally ordered elements.

**Lemma 4.21** (Wick)**.** *For $n>0$ and any sequence $p_1,\ldots,p_n\in Y$, one has the following equality in the Weyl algebra:*

$$A_{p_1}A_{p_2}\cdots A_{p_n}= \tag{518}$$

$$=\sum_{\substack{\{\alpha_1,\beta_1\}\sqcup\cdots\sqcup\{\alpha_s,\beta_s\}\subset\{1,\ldots,n\}\\ \text{a matching on }\{1,\ldots,n\}}} g_{p_{\alpha_1}p_{\beta_1}}\cdots g_{p_{\alpha_s}p_{\beta_s}}\,:\prod_{i\in\{1,\ldots,n\}\backslash\cup_k\{\alpha_k,\beta_k\}}A_{p_i}:.$$

*The sum here goes over matchings on the set $\{1,\ldots,n\}$ – collections of non-overlapping 2-element subsets considered up to permutation.*

**Examples:**

- For $n=2$, there are two matchings on the set $\{1,2\}$: $\{1,2\}$ and $\{\overbracket{1,2}\}$. We indicate by the bracket the matched elements, so in the first case, the set is completely unmatched, $s=0$. In the second case, both elements are matched, $s=1$. So, (518) yields

$$A_aA_b=g_{ab}+:A_aA_b: \tag{519}$$

(we are calling the indices $a,b$ instead of $p_1,p_2$ for convenience). In fact, this formula is just (516).

- For $n=3$, the possible matchings are $\{\overbracket{1,2},3\}$, $\{\overbracket{1,2,3}\}$, $\{1,\overbracket{2,3}\}$, $\{1,2,3\}$, thus the Wick's formula gives

$$A_aA_bA_c=g_{ab}A_c+g_{ac}A_b+g_{bc}A_a+:A_aA_bA_c:. \tag{520}$$

Note that $:A_p:=A_p$ for any $p\in Y$, so we don't have to write the normal ordering symbol for linear expressions in $A_p$'s.

- For $n=4$, we have the following possible matchings:

$$\begin{gathered}\{\overbracket{1,2},\overbracket{3,4}\},\ \{1,2,3,4\},\ \{1,2,3,4\},\\ \{\overbracket{1,2},3,4\},\ \{\overbracket{1,2,3},4\},\ \{\overbracket{1,2,3,4}\},\ \{1,\overbracket{2,3},4\},\ \{1,\overbracket{2,3,4}\},\ \{1,2,\overbracket{3,4}\},\\ \{1,2,3,4\}.\end{gathered} \tag{521}$$

In the first row here we have three *perfect* matchings (i.e. all of the set is matched). Wick's formula in this case gives

$$\begin{gathered}A_aA_bA_cA_d=\\ =g_{ab}g_{cd}+g_{ac}g_{bd}+g_{ad}g_{bc}+\\ +g_{ab}:A_cA_d:+g_{ac}:A_bA_d:+g_{ad}:A_bA_c:+g_{bc}:A_aA_d:+g_{bd}:A_aA_c:+g_{cd}:A_aA_b:+\\ +:A_aA_bA_cA_d:.\end{gathered} \tag{522}$$

Wick's lemma is proven by considering $A_1\cdots A_n$ to be a word comprised of only the creation and annihilation operators – in which case it is proven directly, by induction in $n$. Then the statement is extended to any $A_p$'s by $\mathbb{C}$-linearity.

4.3.3. *Propagator for the free massless scalar field on $\mathbb{C}^*$.* Going back to the free 2d massless scalar field on Euclidean cylinder (which we can parameterize by the complex coordinate $z \in \mathbb{C}^*$), we are in the setting of Wick's lemma: we have the Weyl algebra generated by creation/annihilation operators $\{\widehat{a}_n, \widehat{\overline{a}}_n\}_{n\neq 0} \cup \{\widehat{\phi}_0, \widehat{\pi}_0\}$ (we are thinking of $\widehat{\pi}_0$ as annihilation operator and of $\widehat{\phi}_0$ as creation operator w.r.t. the normal ordering) and a family of preferred linear elements

$$\widehat{\phi}(z) = \widehat{\phi}_0 - i\widehat{\pi}_0 \log(z\overline{z}) + \sum_{n\neq 0} \frac{i}{n}\left(\widehat{a}_n z^{-n} + \widehat{\overline{a}}_n\, \overline{z}^{-n}\right) \tag{523}$$

parametetrized by points $z \in \mathbb{C}^*$. I.e., in the notations of Section 4.3.2, we have $I = \mathbb{Z}$ (the indexing set for the basis of creation/annihilation operators) and $Y = \mathbb{C}^*$ (the indexing set for preferred linear combinations).

**Lemma 4.22.** *Assume $z, w \in \mathbb{C}^*$ two points satisfying $|z| \geq |w|$, $z \neq w$. Then one has*

$$\widehat{\phi}(z)\widehat{\phi}(w) - :\widehat{\phi}(z)\widehat{\phi}(w): = -2\log|z-w|. \tag{524}$$

The right hand side of (524) is the propagator in the sense of (516).

*Proof.* We compute

$$\widehat{\phi}(z)\widehat{\phi}(w) - :\widehat{\phi}(z)\widehat{\phi}(w): = \tag{525}$$

$$= \sum_{n,m\neq 0} \frac{i}{n}\cdot\frac{i}{m}\Big(\underbrace{(\widehat{a}_n z^{-n} + \widehat{\overline{a}}_n\,\overline{z}^{-n})(\widehat{a}_m w^{-m} + \widehat{\overline{a}}_m\,\overline{w}^{-m}) - :(\widehat{a}_n z^{-n} + \widehat{\overline{a}}_n\,\overline{z}^{-n})(\widehat{a}_m w^{-m} + \widehat{\overline{a}}_m\,\overline{w}^{-m}):}_{I}\Big) +$$
$$+\Big((\widehat{\phi}_0 - i\widehat{\pi}_0\log(z\overline{z}))(\widehat{\phi}_0 - i\widehat{\pi}_0\log(w\overline{w})) - :(\widehat{\phi}_0 - i\widehat{\pi}_0\log(z\overline{z}))(\widehat{\phi}_0 - i\widehat{\pi}_0\log(w\overline{w})):\Big).$$

We note that the expression $I$ vanishes if $n \neq m$, since in that case the elements $\widehat{a}_n z^{-n} + \widehat{\overline{a}}_n\,\overline{z}^{-n}$ and $\widehat{a}_m w^{-m} + \widehat{\overline{a}}_m\,\overline{w}^{-m}$ commute. Also, $I$ vanishes if $m > 0$, because then product $(\widehat{a}_n z^{-n} + \widehat{\overline{a}}_n\,\overline{z}^{-n})(\widehat{a}_m w^{-m} + \widehat{\overline{a}}_m\,\overline{w}^{-m})$ is already normally ordered. That leaves only the terms with $n = -m > 0$. So, continuing the computation, we have

$$\widehat{\phi}(z)\widehat{\phi}(w) - :\widehat{\phi}(z)\widehat{\phi}(w): = \tag{526}$$
$$= \sum_{n>0}\frac{1}{n^2}\Big(\underbrace{[\widehat{a}_n, \widehat{a}_{-n}]}_{n} z^{-n}w^n + \underbrace{[\widehat{\overline{a}}_n, \widehat{\overline{a}}_{-n}]}_{n}\overline{z}^{-n}\overline{w}^n\Big) - i\underbrace{[\widehat{\pi}_0, \widehat{\phi}_0]}_{-i}\log(z\overline{z})$$
$$= \sum_{n>0}\frac{1}{n}\left(\left(\frac{w}{z}\right)^n + \left(\frac{\overline{w}}{\overline{z}}\right)^n\right) - \log(z\overline{z}) = -\log\left(1-\frac{w}{z}\right) - \log\left(1-\frac{\overline{w}}{\overline{z}}\right) - \log(z\overline{z})$$
$$= -2\log|z-w|.$$

□

Note that the propagator (524) extends to a function on the configuration space of two points $z, w$ on $\mathbb{C}$ (allowing the point 0) and this extension is invariant under translations on $\mathbb{C}$, $(z, w) \mapsto (z + a, w + a)$.

Note also that the convergence behavior of the sum over $n$ in the computation (526) is as follows:

- it converges absolutely if $|z| > |w|$,
- converges conditionaly if $|z| = |w|$ and $z \neq w$,

- diverges if $|z|<|w|$ or if $z=w$.

4.3.4. *Correlators on the plane (in the radial quantization formalism).* One calls the canonical quantization formalism[77] for the theory on the cylinder mapped to $\mathbb{C}^*$ (see Figure 25) the "radial quantization" formalism.

We define the radial ordering of a product of local operators (observables) on $\mathbb{C}^*$ inserted at $n$ *distinct* points $z_1,\ldots,z_n\in\mathbb{C}^*$ as follows:

$$\mathcal{R}\left(\widehat{O}_1(z_1)\cdots\widehat{O}_n(z_n)\right):=\widehat{O}_{\sigma(1)}(z_{\sigma(1)})\cdots\widehat{O}_{\sigma(n)}(z_{\sigma(n)}), \tag{527}$$

where $\sigma\in S_n$ is a permutation of indices such that $|z_{\sigma(1)}|\geq\cdots\geq|z_{\sigma(n)}|$.

Examples of local operators $\widehat{O}_k(z)$ are:

- The field operator $\widehat{\phi}(z)$.
- Any derivative of the field operator $\partial_z^r\partial_{\bar{z}}^s\widehat{\phi}(z)$, with $r,s\geq 0$.
- Any normally ordered differential polynomial in $\widehat{\phi}(z)$, e.g., $:\partial_z\widehat{\phi}(z)\,\partial_{\bar{z}}^2\widehat{\phi}(z):$.

*Remark* 4.23. Local operators at the same radius commute:

$$[\widehat{O}_1(z),\widehat{O}_2(w)]=0\qquad\text{if }\ |z|=|w|,\ z\neq w. \tag{528}$$

This can be seen as the spacial locality property. In the example of free scalar field, for local operators as in the list above, (528) is a consequence of (524). This remark shows that the possible ambiguity of radial ordering arising when several of $z_i$'s have the same absolute value does not affect the right hand side of (527).

**Example 4.24.** If $z_1$, $z_2$, $z_3$ are three points on $\mathbb{C}^*$ with absolute values satisfying $|z_2|>|z_3|>|z_1|$ and $\widehat{O}_{1,2,3}$ are some local operators, then one has

$$\mathcal{R}\Big(\widehat{O}_1(z_1)\widehat{O}_2(z_2)\widehat{O}_3(z_3)\Big)=\widehat{O}_2(z_2)\widehat{O}_3(z_3)\widehat{O}_1(z_1). \tag{529}$$

In particular, one can consider the vacuum expectation value of this expression

$$\langle\mathrm{vac}|\mathcal{R}\Big(\widehat{O}_1(z_1)\widehat{O}_2(z_2)\widehat{O}_3(z_3)\Big)|\mathrm{vac}\rangle=\langle\mathrm{vac}|\widehat{O}_2(z_2)\widehat{O}_3(z_3)\widehat{O}_1(z_1)|\mathrm{vac}\rangle. \tag{530}$$

Note that only with this ordering in the right-hand side this is a well-defined expression.[78]

move the footnote into the main text? clean it up a bit?

[77] We say "formalism" where we should really say "approach to quantization" or "method of constructing a quantum field theory out of a classical one."

[78] One can see this e.g. by converting back from Heisenberg to Schrödinger pricture – the we will see that the operators $\widehat{O}_k$ are joined by the evolution operators $U(\log\frac{|z_2|}{|z_3|})$, $U(\log\frac{|z_3|}{|z_1|})$ and only for a positive Euclidean time $\tau$ the evolution operator $U(\tau)=e^{-\tau\widehat{H}}$ is well-defined.

The other way to see that radial ordering is necessary for convergence is to apply Wick's lemma to the product of local operators. Then we will have a computation similar to (526) where the infinite sum will converge if and only if the operators are radially ordered.

A related comment is that the vector $\prod_{i=1}^n\widehat{O}_i(z_i)|\mathrm{vac}\rangle$ (assuming that it exists) is in the domain of a local operator $\widehat{O}(z)$ if and only if $|z_i|\leq|z|$ and $z\neq z_i$ for $i=1,\ldots,n$. Using this argument inductively in $n$, one arrives to the necessity of radial ordering.

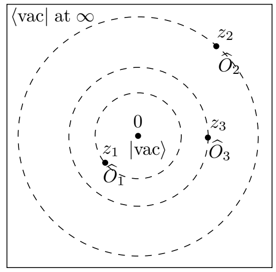



FIGURE 26. Radial ordering.

**Definition 4.25.** In operator formalism, we will understand the correlator of several local operators (point observables) $\widehat{O}_1,\ldots,\widehat{O}_n$ inserted at pairwise distinct points $z_1,\ldots,z_n\in\mathbb{C}^*$ as the expression[79]

$$\langle O_1(z_1)\cdots O_n(z_n)\rangle :=\langle\mathrm{vac}|\mathcal{R}\left(\widehat{O}_1(z_1)\cdots\widehat{O}_n(z_n)\right)|\mathrm{vac}\rangle. \tag{531}$$

**Example 4.26.** Lemma 4.22 implies

$$\mathcal{R}(\widehat{\phi}(z)\widehat{\phi}(w)) =: \widehat{\phi}(z)\widehat{\phi}(w) : -2\log|z-w| \tag{532}$$

for any $z\neq w\in\mathbb{C}^*$. Note that the normally ordered expression in the r.h.s. does is invariant under swapping $z$ and $w$ (cf. Remark 4.20).

**Example 4.27.** Two-point correlator of $\widehat{\phi}$. From (532) we find

$$\langle\phi(z)\phi(w)\rangle :=\langle\mathrm{vac}|\mathcal{R}\Big(\widehat{\phi}(z)\widehat{\phi}(w)\Big)|\mathrm{vac}\rangle=-2\log|z-w|+C \tag{533}$$

where Lecture 20, 10/7/2022

$$C=\langle\mathrm{vac}|:\widehat{\phi}(z)\widehat{\phi}(w):|\mathrm{vac}\rangle=\langle\mathrm{vac}|\widehat{\phi}_0^2|\mathrm{vac}\rangle \tag{534}$$

Here we expand $:\widehat{\phi}(z)\widehat{\phi}(w):$ using (523). All terms in the expansion (except the term $\widehat{\phi}_0^2$) contain $\widehat{a}_{\geq 0}$ or $\widehat{\overline{a}}_{\geq 0}$ on the right which yields zero when acting on $|\mathrm{vac}\rangle$, and/or contain $\widehat{a}_{<0}$, $\widehat{\overline{a}}_{<0}$ on the left, which vanishes when paired with $\langle\mathrm{vac}|$.

Note that (534) is an ill-defined expression formally independent of $z,w$ – an "infinite constant." This can be seen by examining the Schrödinger representation for the free particle (the zero-mode) where $|\mathrm{vac}\rangle=|\pi_0\rangle$ is represented by the Dirac delta-distribution $\delta(\pi_0)$ and $\widehat{\phi}_0=i\frac{\partial}{\partial\pi_0}$. Thus, the expression $\langle\mathrm{vac}|\widehat{\phi}_0^2|\mathrm{vac}\rangle$ in Schrödinger representation reads "the evaluation of distribution $\delta''(\pi_0)$ at $\pi_0=0$." This evaluation does not exist.

Put differently, $\widehat{\phi}_0$ is an unbounded operator on $\mathcal{H}$ and the vector $|\mathrm{vac}\rangle$ is not in its domain.

[79]We are not putting hats on observables in the left hand side, because there we think of them as classical objects (functions of jets of classical fields at a point), and we think of the symbol $\langle\cdots\rangle$ in the l.h.s. as averaging over the space of classical fields, cf. (49), (59).

Alternatively, we can say that $O_k$'s in the l.h.s. are elements of the abstract vector space $V$ of point observables in the sense of Section 1.7 (of which we don't think as operators acting on any Hilbert space and which do not depend on points $z_k$), placed at points $z_1,\ldots,z_n$. From that viewpoint, we are defining the correlator by the operator expression in the r.h.s.

**Notation.** In this section, we will be denoting the holomorphic derivative $\partial_z$ by $\partial$ and the antiholomorphic derivative $\partial_{\bar{z}}$ by $\bar{\partial}$. Thus, in this section symbols $\partial$ and $\bar{\partial}$ do not stand for the holomorphic/antiholomorphic Dolbeault operators $dz\,\partial_z$, $d\bar{z}\,\partial_{\bar{z}}$.

To summarize, correlators of the field $\phi$ are ill-defined due to the presence of the zero-mode $\widehat{\phi}_0$. However, correlators of the fields $\partial\phi$, $\bar{\partial}\phi$ are well-defined!

Note that from (523) one has the following nice expansions of the derivatives of the field in terms of creation/annihilation operators:

$$i\partial\widehat{\phi}(z) = \sum_{n\in\mathbb{Z}} \widehat{a}_n z^{-n-1}, \qquad i\bar{\partial}\widehat{\phi}(z) = \sum_{n\in\mathbb{Z}} \widehat{\bar{a}}_n \bar{z}^{-n-1}. \tag{535}$$

**Example 4.28.** For the two-point correlator of derivatives of the field we have

$$\begin{aligned}\langle\partial\phi(z)\partial\phi(w)\rangle :&= \langle\mathrm{vac}|\mathcal{R}\Big(\partial\widehat{\phi}(z)\partial\widehat{\phi}(w)\Big)|\mathrm{vac}\rangle = \langle\mathrm{vac}|\partial_z\partial_w \underbrace{\mathcal{R}\Big(\widehat{\phi}(z)\widehat{\phi}(w)\Big)}_{-2\log|z-w|+:\widehat{\phi}(z)\widehat{\phi}(w):}|\mathrm{vac}\rangle = \\ &= \langle\mathrm{vac}|\underbrace{-\frac{1}{(z-w)^2}}_{\partial_z\partial_w(-2\log|z-w|)} + :\partial\widehat{\phi}(z)\partial\widehat{\phi}(w):|\mathrm{vac}\rangle = -\frac{1}{(z-w)^2}.\end{aligned} \tag{536}$$

Here $z\neq w$ are any two distinct points in $\mathbb{C}\backslash\{0\}$. We used the fact that $:\partial\widehat{\phi}(z)\partial\widehat{\phi}(w):$, when expanded using (535), has only terms with $\widehat{a}_{\geq 0}$ or $\widehat{\bar{a}}_{\geq 0}$ on the right, and/or with $\widehat{a}_{\leq 0}$, $\widehat{\bar{a}}_{\leq 0}$ on the left. Hence the vacuum expectation value $\langle\mathrm{vac}|:\partial\widehat{\phi}(z)\partial\widehat{\phi}(w):|\mathrm{vac}\rangle$ is zero.

By similar reasoning one has

$$\langle\bar{\partial}\phi(z)\bar{\partial}\phi(w)\rangle = -\frac{1}{(\bar{z}-\bar{w})^2} \tag{537}$$

and

$$\langle\partial\phi(z)\bar{\partial}\phi(w)\rangle = 0. \tag{538}$$

We stress again that points $z$ and $w$ are assumed be distinct.[80]

One can proceed to compute several-point correlators of observables $\partial\phi$, $\bar{\partial}\phi$ using Wick's lemma.

**Example 4.29.** For the four-point correlator, one finds

$$\begin{aligned}&\langle\partial\phi(z_1)\partial\phi(z_2)\partial\phi(z_3)\partial\phi(z_4)\rangle :\, = \langle\mathrm{vac}|\mathcal{R}\Big(\partial\widehat{\phi}(z_1)\partial\widehat{\phi}(z_2)\partial\widehat{\phi}(z_3)\partial\widehat{\phi}(z_4)\Big)|\mathrm{vac}\rangle = \\ &\langle\mathrm{vac}|\Big(\partial\widehat{\phi}(z_1)\partial\widehat{\phi}(z_2)\partial\widehat{\phi}(z_3)\partial\widehat{\phi}(z_4) + \partial\widehat{\phi}(z_1)\partial\widehat{\phi}(z_2)\partial\widehat{\phi}(z_3)\partial\widehat{\phi}(z_4) + \partial\widehat{\phi}(z_1)\partial\widehat{\phi}(z_2)\partial\widehat{\phi}(z_3)\partial\widehat{\phi}(z_4) + \\ &+\partial\widehat{\phi}(z_1)\partial\widehat{\phi}(z_2)\cancel{:\partial\widehat{\phi}(z_3)\partial\widehat{\phi}(z_4):} + 5\text{ similar terms} + \cancel{:\partial\widehat{\phi}(z_1)\partial\widehat{\phi}(z_2)\partial\widehat{\phi}(z_3)\partial\widehat{\phi}(z_4):}\Big)|\mathrm{vac}\rangle\end{aligned} \tag{539}$$

[80]It is possible to sense of the correlator (538) as a distibution on $\mathbb{C}\times\mathbb{C}$ rather than as a function on the open configuration space $C_2(\mathbb{C}\backslash\{0\})$. Then the correlator becomes $\langle\partial\phi(z)\bar{\partial}\phi(w)\rangle = \pi\delta(z-w)$ – up to normalization, the Dirac delta-distribution supported on the diagonal $\mathrm{Diag}\subset\mathbb{C}\times\mathbb{C}$. This delta-distribution is an example of so-called "contact term."

$$= \frac{1}{z_{12}^2 z_{34}^2} + \frac{1}{z_{13}^2 z_{24}^2} + \frac{1}{z_{14}^2 z_{23}^2}.$$

Here we denoted $z_{ij} := z_i - z_j$. Note that in this computation only the three terms where all four operators are matched contribute.

By a similar computation one finds

$$\langle \partial\phi(z_1)\partial\phi(z_2)\bar\partial\phi(z_3)\bar\partial\phi(z_4)\rangle = \langle \partial\phi(z_1)\partial\phi(z_2)\bar\partial\phi(z_3)\bar\partial\phi(z_4)\rangle = \frac{1}{z_{12}^2 \bar z_{34}^2} \tag{540}$$

– only a single matching contributes.

More generally, by the same logic, the correlator

$$\langle \partial\phi(z_1)\cdots\partial\phi(z_n)\bar\partial\phi(w_1)\cdots\bar\partial\phi(w_m)\rangle \tag{541}$$

(all points $z_1,\ldots,z_n,w_1,\ldots,w_m$ are assumed to be distinct) vanishes unless both $n$ and $m$ are even, $n=2\nu$, $m=2\mu$. If they are even, the correlator is given by a sum over pairs (perfect matching of $z_i$'s, perfect matching of $w_j$'s) – thus, in total there are $(2\nu-1)!!\cdot(2\mu-1)!!$ terms. E.g. in the case $m=0$, one obtains a meromorphic function on $\mathbb{C}^n$ with second-order poles on principal diagonals. For instance,

$$\langle \phi(z_1)\cdots\phi(z_6)\rangle = \frac{-1}{z_{12}^2 z_{34}^2 z_{56}^2} + 14 \text{ similar terms}, \tag{542}$$

since one has $5!! = 5\cdot 3\cdot 1$ perfect matchings on the set of 6 elements.

Examining the terms contributing to the correlator (541) for general $m,n$, we can notice that it factorizes into a meromorphic part and an antimeromorphic part:

$$\langle \partial\phi(z_1)\cdots\partial\phi(z_n)\bar\partial\phi(w_1)\cdots\bar\partial\phi(w_m)\rangle = \langle \partial\phi(z_1)\cdots\partial\phi(z_n)\rangle\cdot\langle\bar\partial\phi(w_1)\cdots\bar\partial\phi(w_m)\rangle \tag{543}$$

**Example 4.30.** We have

$$\langle \partial\bar\partial\phi(z)\partial\phi(w)\rangle = \frac{\partial}{\partial\bar z}\underbrace{\langle\partial\phi(z)\partial\phi(w)\rangle}_{-\frac{1}{(z-w)^2}} = 0. \tag{544}$$

For the next example we need a slightly enhanced version of Wick's lemma, rearranging a product of normally-ordered words in terms of fully normally-ordered expressions.

**Lemma 4.31.** *In the notations of Lemma 4.21, for $n>0$, let $p_1,\ldots,p_n\in Y$ and let*

$$\{1,\ldots,n\} = S_1\sqcup\cdots\sqcup S_m \tag{545}$$

*be a partitioning of the set $\{1,\ldots,n\}$ into nonempty disjoint subsets $S_j$. Then one has the following equality in the Weyl algebra:*

$$:\prod_{p\in S_1} A_p:\cdots:\prod_{p\in S_m}A_p: = \tag{546}$$

$$= \sum_{\substack{\{\alpha_1,\beta_1\}\sqcup\cdots\sqcup\{\alpha_s,\beta_s\}\subset\{1,\ldots,n\}\\ \text{a matching on } \{1,\ldots,n\} \text{ s.t.}\\ \{\alpha_i,\beta_i\}\not\subset S_j\ \forall i,j}} \prod_{i=1}^s g_{p_{\alpha_i}p_{\beta_i}}\cdot :\prod_{i\in\{1,\ldots,n\}\setminus\cup_k\{\alpha_k,\beta_k\}} A_{p_i}: .$$

In other words, the right hand side is the sum over matchings, as in (518), except that now elements of each subset of $S_j$ of labels corresponding to one of the normally-ordered words in the l.h.s. are not allowed to be matched.

**Example:**

$$:A_aA_b:A_c = g_{ac}A_b + g_{bc}A_a + :A_aA_bA_c:\,. \tag{547}$$

Here the partitioning (545) is $\{1,2,3\} = \{1,2\} \sqcup \{3\}$ and the labels are $p_1 = a$, $p_2 = b$, $p_3 = c$. Notice that in comparison with (520), the term $g_{ab}A_c$ corresponds to a prohibited contraction $\{1,2,3\}$ and doesn't appear in the r.h.s.

**Example 4.32.** Consider the correlator

$$\begin{aligned}\langle \partial\phi(z_1)\big(:\partial\phi(z_2)\partial\phi(z_3):\big)\partial\phi(z_4)\rangle :&= \langle \mathrm{vac}|\mathcal{R}\Big(\partial\widehat{\phi}(z_1)\big(:\partial\widehat{\phi}(z_2)\partial\widehat{\phi}(z_3):\big)\partial\widehat{\phi}(z_4)\Big)|\mathrm{vac}\rangle = \\ = \langle \mathrm{vac}|\partial\widehat{\phi}(z_1)\big(:\partial\widehat{\phi}(z_2)\partial\widehat{\phi}(z_3):\big)\partial\widehat{\phi}(z_4) &+ \partial\widehat{\phi}(z_1)\big(:\partial\widehat{\phi}(z_2)\partial\widehat{\phi}(z_3):\big)\partial\widehat{\phi}(z_4)|\mathrm{vac}\rangle \\ &= \frac{1}{z_{12}^2 z_{34}^2} + \frac{1}{z_{13}^2 z_{24}^2}.\end{aligned} \tag{548}$$

Here the 2-point observable $:\partial\widehat{\phi}(z_2)\partial\widehat{\phi}(z_3):$ on the l.h.s. is a formal symbol defined by its correlators with other local fields, like in this example, where this observable is replaced in the operator language by a normally-ordered product of derivatives of field operators. Notice in comparison with (539) the absence of the term $\frac{1}{z_{23}^2 z_{14}^2}$ corresponding to a prohibited matching. In particular, (548) is a regular (in fact, holomorphic) function on the diagonal $z_2 \to z_3$ in $\mathbb{C}^4$.

This example illustrates that one can define a new point observable

$$:\partial\phi(z)\partial\phi(z):\ := \lim_{w\to z} :\partial\phi(w)\partial\phi(z): := \lim_{w\to z}\left(\partial\phi(w)\partial\phi(z) + \frac{1}{(w-z)^2}\right). \tag{549}$$

One sometimes calls point observables of this type – constructed as normally-ordered differential polynomials in the field – "composite fields." This definition is understood as an equality under a correlator with an arbitrary collection of other local observables ("test observables") inserted at points $\neq z$. In the operator language, we should replace $\phi \to \widehat{\phi}$ everywhere.

This new observable has well-defined correlators. E.g., taking the limit $z_2 \to z_3$ in (548), we obtain

$$\langle \partial\phi(z_1)\big(:\partial\phi(z_3)\partial\phi(z_3):\big)\partial\phi(z_4)\rangle = \frac{2}{z_{13}^2 z_{34}^2}. \tag{550}$$

**Definition 4.33.** We can define the (quantum) holomorphic/antiholomorphic stress-energy tensor in the massless scalar field theory as the composite fields

$$T_{zz} := -\frac{1}{2}:\partial\phi(z)\partial\phi(z): \quad , \quad T_{\bar{z}\bar{z}} := -\frac{1}{2}:\bar\partial\phi(z)\bar\partial\phi(z):\,. \tag{551}$$

Note that the conventional normalization factor in (551) is different than what we had in the classical theory (509). It is useful to also consider a local observable

$$T^{\mathrm{total}}(z) = T_{zz}(dz)^2 + T_{\bar{z}\bar{z}}(d\bar{z})^2 \tag{552}$$

valued in quadratic differentials – the total (quantum) stress-energy tensor. For instance, its correlator with, e.g. a collection of fields $\partial\phi(z_i)$ will be a section of the pullback of the bundle of quadratic differentials $K^{\otimes 2} \oplus \overline{K}^{\otimes 2} \to \Sigma$ to the space of configurations of points $(z, z_1, \ldots, z_n) \in \Sigma$, with $K = (T^{1,0})^* \Sigma$ the canonical line bundle. Here $\Sigma = \mathbb{C}\backslash\{0\}$.

**Notation.** From now on we will denote $T_{zz}$ by $T$ and $T_{\bar{z}\bar{z}}$ by $\bar{T}$. This is the standard convention in the literature on CFT. Lecture 21, 10/10/2022

4.4. **Operator product expansions.** Recall from Section 1.7.7 that the operator product expansions (OPEs) express the product of two local observables at points $z, w$ as a linear combination (with singular coefficients) of single local observables at $w$, in the asymptotics $z \to w$. These expressions are to be substituted in a correlator with an arbitrary collection of "test" local observables at points $z_1, \ldots, z_n \neq z, w$ and control the asymptotics of the correlator as $z \to w$.

**Example 4.34.** From Wick's lemma we have the equality

$$\mathcal{R}\,\partial\widehat{\phi}(z)\partial\widehat{\phi}(w) = -\frac{1}{(z-w)^2}\widehat{\mathbb{1}} + :\partial\widehat{\phi}(z)\partial\widehat{\phi}(w): \tag{553}$$

for any $z \neq w \in \mathbb{C}\backslash\{0\}$, as equality of linear operators on $\mathcal{H}$. Here for the moment we make the identity operator $\widehat{\mathbb{1}}$ explicit in the notations. Note that the second term is regular[81] (in fact, holomorphic) as $z \to w$. Thus, for any collection of point observables $O_1, \ldots, O_n$ at points $z_1, \ldots, z_n$ (distinct among themselves and distinct from $w$), one has

(554)
$$\langle \partial\phi(z)\partial\phi(w)O_1(z_1)\cdots O_n(z_n)\rangle = \langle \mathrm{vac}|\mathcal{R}\Big(\partial\widehat{\phi}(z)\partial\widehat{\phi}(w)\widehat{O}_1(z_1)\cdots\widehat{O}_n(z_n)\Big)|\mathrm{vac}\rangle \underset{z\to w}{\sim}$$
$$\underset{z\to w}{\sim} -\frac{1}{(z-w)^2}\langle \mathrm{vac}|\mathcal{R}\Big(\widehat{\mathbb{1}}(w)\widehat{O}_1(z_1)\cdots\widehat{O}_n(z_n)\Big)|\mathrm{vac}\rangle + \mathrm{reg.} = -\frac{1}{(z-w)^2}\langle \mathbb{1}(w)O_1(z_1)\cdots O_n(z_n)\rangle + \mathrm{reg.}$$

– this is an asymptotic expression for the correlator as $z \to w$ giving the principal part of its Laurent expansion in $z - w$; reg. stands for a term with regular behavior as $z \to r$. The identity operator operator $\widehat{\mathbb{1}}$ and identity field $\mathbb{1}$ do not affect the correlators in the r.h.s.

Thus, one has the operator product expansion

$\sim$ vs. $=$ is inconsistent throughout this section

$$\partial\phi(z)\partial\phi(w) \sim -\frac{\mathbb{1}}{(z-w)^2} + \mathrm{reg.} \tag{555}$$

The symbol $\sim$ means that one can trade the l.h.s. with the r.h.s. under a correlator with test observables, yielding the asymptotics as $z \to w$.

Am I being completely redundant here?

*Remark* 4.35. One can also be more explicit about the regular part: one can write the rightmost term in (553) as

$$:\partial\widehat{\phi}(z)\partial\widehat{\phi}(w): := \sum_{n\geq 0}\frac{1}{n!}(z-w)^n :\partial^{n+1}\widehat{\phi}(w)\,\partial\widehat{\phi}(w):. \tag{556}$$

The refined version of the OPE (555) is then

$$\partial\phi(z)\partial\phi(w) \sim -\frac{\mathbb{1}}{(z-w)^2} + \underbrace{\sum_{n\geq 0}\frac{1}{n!}(z-w)^n :\partial^{n+1}\phi(w)\,\partial\phi(w):}_{\mathrm{reg.}}. \tag{557}$$

[81] Generally, "regular" for us in the context of OPEs means just "continuous."

The r.h.s. is now a linear combination of local composite fields at the point $w$. Under a correlator with test observables, one has

$$(558)\quad \langle \partial\phi(z)\partial\phi(w)O_1(z_1)\cdots O_n(z_n)\rangle \underset{z\to w}{\sim}$$

$$\underset{z\to w}{\sim} -\frac{1}{(z-w)^2}\langle \mathbb{1}(w)O_1(z_1)\cdots O_n(z_n)\rangle + \sum_{n\geq 0}\frac{1}{n!}(z-w)^n\langle : \partial^{n+1}\phi(w)\partial\phi(w) : O_1(z_1)\cdots O_n(z_n)\rangle.$$

The sum on the right converges absolutely if and only if $|z-w| < \min\{|z_i - w|\}_{i=1}^n$ and in this convergence radius is equal to the l.h.s. Thus, the $\sim$ symbol here is actually equality, for $z$ sufficiently close to $w$ (closer than any of the test observables).

Similarly to (553) (or (555)), one finds

$$(559)\qquad \mathcal{R}\bar\partial\widehat{\phi}(z)\bar\partial\widehat{\phi}(w) \sim -\frac{\widehat{\mathbb{1}}}{(\bar z - \bar w)^2} + \mathrm{reg.},\qquad \mathcal{R}\partial\widehat{\phi}(z)\bar\partial\widehat{\phi}(w) \sim \mathrm{reg.}$$

These are again equalities of operators on $\mathcal{H}$; removing the hats and the radial ordering sign, we have the OPEs in the form similar to (555) – in the language of abstract correlators of observables as elements of $V$ (of Section 1.7).

**Example 4.36.** As the next example, consider the OPE between the stress-energy tensor and $\partial\phi$. From Wick's lemma we find

$$(560)\quad \mathcal{R}\underbrace{\widehat{T}(z)}_{:-\frac{1}{2}\partial\widehat{\phi}(z)\partial\widehat{\phi}(z):}\partial\widehat{\phi}(w) =$$

$$= -\frac{1}{2} : \partial\widehat{\phi}(z)\overbracket{\partial\widehat{\phi}(z) : \partial\widehat{\phi}(w)} + -\frac{1}{2} : \overbracket{\partial\widehat{\phi}(z)\partial\widehat{\phi}(z) : \partial\widehat{\phi}(w)} + \underbrace{: -\frac{1}{2}\partial\widehat{\phi}(z)\partial\widehat{\phi}(z)\partial\widehat{\phi}(w) :}_{\mathrm{reg.}}$$

$$\sim \frac{\partial\widehat{\phi}(z)}{(z-w)^2} + \mathrm{reg.}$$

This is not quite the desired OPE yet, as the operator in the r.h.s is at $z$ whereas we want to express the operator productin terms of local operators at $w$. This is remedied by expanding $\partial\widehat{\phi}(z)$ in Taylor series centered at $w$: $\partial\widehat{\phi}(z) = \partial\widehat{\phi}(w) + (z-w)\partial^2\widehat{\phi}(w) + O((z-w)^2)$.[82] Thus, one has

$$(561)\qquad \mathcal{R}\widehat{T}(z)\partial\widehat{\phi}(w) \sim \frac{\partial\widehat{\phi}(w)}{(z-w)^2} + \frac{\partial^2\widehat{\phi}(w)}{z-w} + \mathrm{reg.}$$

Similarly, one obtains

$$(562)$$
$$\mathcal{R}\widehat{\overline{T}}(z)\bar\partial\widehat{\phi}(w) \sim \frac{\bar\partial\widehat{\phi}(w)}{(\bar z-\bar w)^2} + \frac{\bar\partial^2\widehat{\phi}(w)}{\bar z - \bar w} + \mathrm{reg.},\quad \mathcal{R}\widehat{T}(z)\bar\partial\widehat{\phi}(w) \sim \mathrm{reg.},\quad \mathcal{R}\widehat{\overline{T}}(z)\partial\widehat{\phi}(w) \sim \mathrm{reg.}$$

**Example 4.37** ($TT$ OPE)**.** Let us calculate the OPE of the holomorphic component of the stress-energy tensor $T$ with itself:

$$(563)\quad \mathcal{R}\widehat{T}(z)\widehat{T}(w) = \mathcal{R} : -\frac{1}{2}\partial\widehat{\phi}(z)\partial\widehat{\phi}(z) : : -\frac{1}{2}\partial\widehat{\phi}(w)\partial\widehat{\phi}(w) : \underset{\mathrm{Wick}}{=}$$

[82]Here we used the fact that $\partial\widehat{\phi}(z)$ is holomorphic in $z$, see (535), thus, e.g., one does not have a term $(\bar z - \bar w)\partial\bar\partial\widehat{\phi}(w)$ in the Taylor expansion.

$$\underset{\text{Wick}}{=} \frac{1}{4} :\partial\widehat{\phi}(z)\partial\widehat{\phi}(z)\partial\widehat{\phi}(w)\partial\widehat{\phi}(w): \; + \frac{1}{4} :\partial\widehat{\phi}(z)\partial\widehat{\phi}(z)\partial\widehat{\phi}(w)\partial\widehat{\phi}(w): +$$

$$+ \frac{1}{4} :\partial\widehat{\phi}(z)\partial\widehat{\phi}(z)\partial\widehat{\phi}(w)\partial\widehat{\phi}(w): \; + \frac{1}{4} :\partial\widehat{\phi}(z)\partial\widehat{\phi}(z)\partial\widehat{\phi}(w)\partial\widehat{\phi}(w): +$$

$$+\frac{1}{4} :\partial\widehat{\phi}(z)\partial\widehat{\phi}(z)\partial\widehat{\phi}(w)\partial\widehat{\phi}(w): +\frac{1}{4} :\partial\widehat{\phi}(z)\partial\widehat{\phi}(z)\partial\widehat{\phi}(w)\partial\widehat{\phi}(w): +\frac{1}{4} :\partial\widehat{\phi}(z)\partial\widehat{\phi}(z)\partial\widehat{\phi}(w)\partial\widehat{\phi}(w):$$

$$\sim \frac{\frac{1}{2}\widehat{\mathbb{1}}}{(z-w)^4} - \frac{:\partial\widehat{\phi}(z)\partial\widehat{\phi}(w):}{(z-w)^2} + \text{reg.}$$

$$\underset{\text{expand } \partial\widehat{\phi}(z) \text{ at } w}{=} \frac{\frac{1}{2}\widehat{\mathbb{1}}}{(z-w)^4} - \frac{:\partial\widehat{\phi}(w)\partial\widehat{\phi}(w):}{(z-w)^2} - \frac{:\partial^2\widehat{\phi}(w)\partial\widehat{\phi}(w):}{z-w} + \text{reg.}$$

$$= \boxed{\frac{\frac{1}{2}\widehat{\mathbb{1}}}{(z-w)^4} + \frac{2\widehat{T}(w)}{(z-w)^2} + \frac{\partial\widehat{T}(w)}{z-w} + \text{reg.}}$$

Note the appearance of the fourth order pole here. As we will see later, it is linked to the phenomenon of central charge (and thus to projectivity of CFT as a Segal's functor).

By similar computations, one finds

$$\mathcal{R}\widehat{\overline{T}}(z)\widehat{\overline{T}}(w) \sim \frac{\frac{1}{2}\widehat{\mathbb{1}}}{(\bar{z}-\bar{w})^4} + \frac{2\widehat{\overline{T}}(w)}{(\bar{z}-\bar{w})^2} + \frac{\partial\widehat{\overline{T}}(w)}{\bar{z}-\bar{w}} + \text{reg.}, \qquad \mathcal{R}\widehat{T}(z)\widehat{\overline{T}}(w) \sim \text{reg.} \tag{564}$$

## 4.5. Digression: path integral formalism (in the example of free scalar field).

4.5.1. *Finite-dimensional Gaussian integral.* Let $F$ be an $N$-dimensional real vector space equipped with Euclidean metric $h$ and with a positive-definite bilinear form $B\colon \mathrm{Sym}^2 F \to \mathbb{R}$ and let $\underline{B} \in \mathrm{End}(F)$ be an endomorphism such that $B(u,v) = h(u, \underline{B}v)$. Then one has the following well-known Gaussian integral

$$\int_F \mu_h e^{-\frac{1}{2}B(u,u)} = (2\pi)^{\frac{N}{2}} (\det \underline{B})^{-\frac{1}{2}}. \tag{565}$$

Here $\mu_h$ is the Lebesgue measure on $F$ associated with the metric $h$ and $B(u,u)$ is the quadratic function on $F$ – the restriction of $B$ to the diagonal $\mathrm{Diag} \subset F \times F$.

4.5.2. *Wick's lemma for the moments of Gaussian measure.* For $f$ a polynomial function on $F$, consider its expectation value (average) with respect to the normalized Gaussian measure,

$$\langle f \rangle := \frac{1}{(2\pi)^{\frac{N}{2}} (\det \underline{B})^{-\frac{1}{2}}} \int_F \mu_h e^{-\frac{1}{2}B(u,u)} f(u). \tag{566}$$

Note that the normalization factor in the r.h.s. is chosen such that one has

$$\langle 1 \rangle = 1. \tag{567}$$

**Lemma 4.38** (Wick's lemma for the moments of Gaussian measure)**.** *Let $\theta_1, \ldots, \theta_n$ be some linear forms on $F$. Consider the Gaussian expectation value*

$$\langle \theta_1 \cdots \theta_n \rangle \tag{568}$$

*Then one has*

*(i) If $n$ is odd, the expectation value (568) is zero.*

*(ii) If $n = 2m$ is even, one has*

$$\langle\theta_1\cdots\theta_n\rangle = \sum_{\substack{\text{perfect matchings}\\ \{1,\ldots,n\}=\sqcup_{i=1}^m\{\alpha_i,\beta_i\}}} B^{-1}(\theta_{\alpha_1},\theta_{\beta_1})\cdots B^{-1}(\theta_{\alpha_m},\theta_{\beta_m}). \quad (569)$$

For example, for $n = 2$, one has

$$\langle\theta_1\theta_2\rangle = B^{-1}(\theta_1,\theta_2). \quad (570)$$

Here on the r.h.s., $B^{-1}$ is understood as a map $B^{-1}\colon F^*\otimes F^*\to\mathbb{R}$ which is adjoint to the map $F^*\to F$ – the inverse of the map $B^\#\colon F\to F^*$.

For $n = 4$, one has

(571)
$$\langle\theta_1\theta_2\theta_3\theta_4\rangle = B^{-1}(\theta_1,\theta_2)B^{-1}(\theta_3,\theta_4)+B^{-1}(\theta_1,\theta_3)B^{-1}(\theta_2,\theta_4)+B^{-1}(\theta_1,\theta_4)B^{-1}(\theta_2,\theta_3),$$

where the terms correspond to the three perfect matchings on the set $\{1,2,3,4\}$.

Note that the r.h.s. of (569) looks similar to the r.h.s. of (518) if we were to retain only the contributions of perfect matchings (and identify the propagator $g_{pq}$ with $B^{-1}(\theta_p,\theta_q)$).

*Sketch of proof of Lemma 4.38.* First note that part (i) of Lemma is obvious, since in this case the integrand in (566) changes sign under $u\to -u$.

For part (ii), consider the "generating functions for moments" – the following expectation value depending on the "source" parameter $J\in F^*$:

(572)
$$\langle e^{\langle J,u\rangle}\rangle = C\int_F \mu_h\, e^{-\frac12 B(u,u)+\langle J,u\rangle} = C\int_F \mu_h\, e^{-\frac12 B(u-B^{-1}J,u-B^{-1}J)+\frac12 B^{-1}(J,J)} = \\ \underset{v\colon = u-B^{-1}J}{=} C\int_F \mu_h e^{-\frac12 B(v,v)+\frac12 B^{-1}(J,J)} = e^{\frac12 B^{-1}(J,J)}$$

where $C = (2\pi)^{-\frac{N}{2}}\det(\underline{B})^{\frac12}$. Then we can obtain correlators of monomials (569) by taking multiple partial derivatives of (572) in $J$ and then setting $J = 0$.

More explicitly, consider an orthonormal basis in $F$ w.r.t. the metric $g$ and let $\{u^p\}$ be the corresponding coordinates on $F$. It suffices to prove (569) for $\theta_1 = u^{p_1},\ldots,\theta_n = u^{p_n}$ a collection of coordinate functions; the general result then follows by linearity. We have

(573)
$$\langle u^{p_1}\cdots u^{p_n}\rangle = \frac{\partial}{\partial J_{p_1}}\cdots\frac{\partial}{\partial J_{p_n}}\Big|_{J=0}\langle e^{\langle J,u\rangle}\rangle = \frac{\partial}{\partial J_{p_1}}\cdots\frac{\partial}{\partial J_{p_n}}\Big|_{J=0} e^{\frac12 B^{-1}(J,J)} = \\ = \frac{\partial}{\partial J_{p_1}}\cdots\frac{\partial}{\partial J_{p_n}}\Big|_{J=0}\frac{1}{2^m m!}\left(B^{-1}(J,J)\right)^m$$

where in the last step we selected the $m$-th term from the Taylor series of the exponential, since only it contributes to the $n = 2m$-th derivative in $J$ at $J = 0$ (note that in the last expression the restriction to $J = 0$ is irrelevant – the derivative is a constant). At this point we see that the answer is the sum over the ways to distribute the $2m$ derivatives in $J$ over $2m$ copies of $J$ in $\left(B^{-1}(J,J)\right)^m$. This results in the sum over perfect matchings in the r.h.s. of (569).[83] E.g., for $m = 1$ (i.e.

This is a bit rushed..

[83]Note that the set of perfect matchings on the set of $2m$ elements can be seen as a coset of the symmetric group, $S_{2m}/(S_m\ltimes\mathbb{Z}_2^m)$.

$n = 2$) we have

$$\langle u^{p_1} u^{p_2} \rangle = \frac{1}{2} \frac{\partial}{\partial J_{p_1}} \frac{\partial}{\partial J_{p_2}} (B^{-1})^{pq} J_p J_q = \tag{574}$$

$$= \frac{1}{2} \frac{\partial}{\partial J_{p_1}} \frac{\partial}{\partial J_{p_2}} (B^{-1})^{pq} J_p J_q + \frac{1}{2} \frac{\partial}{\partial J_{p_1}} \frac{\partial}{\partial J_{p_2}} (B^{-1})^{pq} J_p J_q = (B^{-1})^{p_1 p_2},$$

which is (569) specialized to the coordinate monomial $\theta^1 \theta^2$ with $\theta_1 = u^{p_1}$, $\theta_2 = u^{p_2}$. Here brackets show which derivatives hit which instances of $J$. □

Lecture 22, 10/12/2022

4.5.3. *Scalar field theory in the path integral formalism.* Let $\Sigma$ be a surface equipped with Riemannian metric $g$. In the path integral (or more appropriately, "functional integral") approach, the partition function of the scalar field on $\Sigma$ is given by a formal Gaussian integral

$$Z(\Sigma) = \text{“}\int_{\mathrm{Fields}_\Sigma} \mathcal{D}\phi\, e^{-\frac{1}{4\pi} S(\phi)}\text{”} \tag{575}$$

over the (infinite-dimesnional) space of functions $\mathrm{Fields}_\Sigma = C^\infty(\Sigma)$. Here

$$S(\phi) = \int_\Sigma \frac{1}{2} (d\phi \wedge * d\phi + \frac{m^2}{2} \phi^2 \mathrm{dvol}_g) = \int_\Sigma \frac{1}{2} \phi (\Delta + m^2) \phi \; \mathrm{dvol}_g \tag{576}$$

Here for the moment we are considering scalar field with mass $m \geq 0$; later we will want to set $m = 0$ to have a conformal theory. In (576) we assume that either $\Sigma$ is closed or else an appropriate boundary condition is imposed on fields $\phi$, so that the boundary term $\int_{\partial \Sigma} \frac{1}{2} \phi * d\phi$ vanishes – then the right equality in (576) is valid.

The expression (575) is similar to the l.h.s. of (565) if we make the identifications

$$\begin{gathered} F = \mathrm{Fields}_\Sigma, \quad u = \phi, \quad h(\phi_1, \phi_2) = \int_\Sigma \phi_1 \phi_2 \, \mathrm{dvol}_g, \\ B(\phi_1, \phi_2) = \frac{1}{4\pi} \int_\Sigma \phi_1 (\Delta + m^2) \phi_2 \, \mathrm{dvol}_g, \quad \underline{B} = \frac{1}{4\pi} (\Delta + m^2). \end{gathered} \tag{577}$$

Understanding the infinite-dimensional integral (575) as a measure-theoretic integral is problematic and we think of it as defined by the r.h.s. of (565):

$$Z(\Sigma) \colon = \det(c(\Delta + m^2))^{-\frac{1}{2}}, \tag{578}$$

where $c = \frac{1}{8\pi^2}$.

*Remark* 4.39. Determinants of differential operators are also nontrivial to make sense of, but there are viable solutions. One method is "zeta-regularization": for $D$ a differential operator with a discrete eigenvalue spectrum, one constructs the zeta-function of $D$ – a function of a complex variable $s$ defined as

$$\zeta_D(s) \colon = \sum_\lambda \lambda^{-s}. \tag{579}$$

The sum is over the eigenvalues of $D$ (in the case of continuum spectrum, the sum should be replaced by an integral). The sum converges to a holomorphic function for $\mathrm{Re}(s) > A$ for some $A$ and admits a unique meromorphic continuation to $\mathbb{C}$ with $s = 0$ a regular point. Then the zeta-regularized determinant is defined in terms of the derivative of the meromorphically continues zeta-function at $s = 0$ as

$$\det{}_{\zeta-\mathrm{reg}}(D) \colon = e^{-\zeta_D'(0)}. \tag{580}$$

Note that in (565) we wanted the quadratic form $B$ (and thus the operator $\underline{B}$) to be strictly positive. For the scalar field on a closed surface $\Sigma$ that forces $m > 0$; in the massless case the operator $\underline{B} = \Delta$ has a 1-dimensional kernel given by constant functions on $\Sigma$. Correspondingly, the determinant $\det \Delta$ is not well-defined even with zeta-regularization due to appearance of the eigenvalue $\lambda = 0$, which means that the zeta-function (579) is not defined. For $m > 0$, the partition function (578) is well-defined via zeta-regularization.

Moments of Gaussian measure. Correlators in the path integral formalism are given as Gaussian averages of products of fields and so are given by the Wick's lemma (569). For instance for $p_1 \neq p_2 \in \Sigma$ two points, one has

$$\langle \phi(p_1)\phi(p_2)\rangle = \text{“}\frac{1}{Z(\Sigma)}\int_{\mathrm{Fields}_\Sigma} \mathcal{D}\phi\, e^{-\frac{1}{4\pi}S(\phi)}\phi(p_1)\phi(p_2)\text{”} := G(p_1,p_2) \tag{581}$$

– the Green's function of the operator $\frac{1}{4\pi}(\Delta + m^2)$. Here the Green's function – the integral kernel of the operator $(\Delta + m^2)^{-1} = \underline{B}^{-1}$ is analogous to the matrix element of $\underline{B}^{-1}$ appearing in (570), (574). One should think of the r.h.s. of (581) as the mathematical definition of the l.h.s., motivated by Wick's lemma in the finite-dimensional case. Put another way, in the context of infinite-dimensional Gaussian integrals, Wick's lemma becomes not a lemma (equality between two well-defined objects), but a *definition* of the moments of the infinite-dimensional Gaussian measure.

Likewise, for the four-point correlator one has

$$\begin{aligned}\langle \phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)\rangle = \text{“}\frac{1}{Z(\Sigma)}\int_{\mathrm{Fields}_\Sigma} \mathcal{D}\phi\, e^{-\frac{1}{4\pi}S(\phi)}\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)\text{”} :=\\ := G(p_1,p_2)G(p_3,p_4) + G(p_1,p_3)G(p_2,p_4) + G(p_1,p_4)G(p_2,p_3)\end{aligned} \tag{582}$$

We note that formulae (581), (582) make sense for any closed surface, for $m > 0$ (for $m = 0$, the operator $\Delta$ is non-invertible and hence the Green's function does not exist).

Case $\Sigma = \mathbb{C}$. Let us restrict to the case $\Sigma = \mathbb{C}$ – the complex plane. The Green's function $G(z,w)$ can be explicitly found in terms of Bessel's function $K_0$,[84]

$$G(z,w) = 2K_0(m\cdot|z-w|), \tag{583}$$

In particular for $m \to 0$ and $z \neq w$ fixed one has the asymptotic behavior

$$G(z,w) \underset{m\to 0}{\sim} -2\log|z-w| + C(m) \tag{584}$$

where $C(m) = -2\log m + c$ is a constant (in $z,w$) which diverges as $m \to 0$; here $c = 2(\log 2 - \gamma)$. Thus, we find that the two-point correlator

$$\langle \phi(z)\phi(w)\rangle = G(z,w) \tag{585}$$

computed in the path integral formalism does not exist in the conformal limit $m \to 0$. Recall that its counterpart in the radial quantization picture (533) is also problematic due to the appearance of an “infinite constant” $\langle \mathrm{vac}|\widehat{\phi_0^2}|\mathrm{vac}\rangle$.

[84] Bessel's function $K_0(r)$ is a solution of the ODE $\left(\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} + 1\right)y = 0$; it has logarithmic asymptotics $K_0(r) \sim -\log r + (\log 2 - \gamma) + o(r)$ as $r \to 0$ (where $\gamma = 0.5772\ldots$ is the Euler's constant). At $r \to +\infty$ the function $K_0$ is exponentially decaying, $K_0(r) \sim \sqrt{\frac{\pi}{2r}}e^{-r}$.

Next, if we consider the two-point correlator of derivatives of the field

$$\langle \partial\phi(z)\partial\phi(w)\rangle = \partial_z\partial_w G(z,w) \underset{m\to 0}{\to} -\frac{1}{|z-w|^2}, \tag{586}$$

we see that it has a well-defined limit $m \to 0$, which also agrees with our earlier result obtained in the radial quantization picture (536).

One can apply this method to construct similar correlators of derivatives of fields on any surface – the Green's function itself does not exist in the limit $m \to 0$ but its derivatives do have a limit.[85]

As an example of a more complicated local observable, we can consider the following quadratic polynomial on $\mathrm{Fields}_\Sigma$:

$$:\partial\phi(z)\partial\phi(z): \ := \lim_{w\to z}\left(\partial\phi(w)\partial\phi(z) + \frac{1}{(w-z)^2}\right) \tag{587}$$

When computing the correlator of this observable with a collection of other other observables by Wick's lemma, the correction $\frac{1}{(z-w)^2}$ cancels the contribution of Wick contraction $\partial\phi(w)\partial\phi(z)$ – so effectively one can say this contraction is prohibited when computing correlators involving $:\partial\phi(z)\partial\phi(z):$.[86]

As an illustration, let us compute the correlator of the stress-energy tensor with itself (in the path integral formalism):

$$\begin{aligned} \langle T(z)T(w)\rangle &= \langle : -\frac{1}{2}\partial\phi(z)\partial\phi(z) :: -\frac{1}{2}\partial\phi(w)\partial\phi(w) :\rangle = \\ &= \langle : -\frac{1}{2}\partial\phi(z)\partial\phi(z) :: -\frac{1}{2}\partial\phi(w)\partial\phi(w) :\rangle + \langle : -\frac{1}{2}\partial\phi(z)\partial\phi(z) :: -\frac{1}{2}\partial\phi(w)\partial\phi(w) :\rangle \\ &= \frac{2}{4}\frac{-1}{(z-w)^2}\,\frac{-1}{(z-w)^2} = \frac{1}{2}\frac{1}{(z-w)^4}. \end{aligned} \tag{588}$$

Note that contractions inside $:\cdots:$ are prohibited.

OPEs. We remark that one can also find OPEs within the path integral formalism (from Wick's lemma). For example, consider the correlator

$$\langle \partial\phi(z)\partial\phi(w) \underbrace{O_1(z_1)\cdots O_n(z_n)}_{\text{test observables}}\rangle \tag{589}$$

in the asymptotics $z \to w$. The correlator is given by a sum over perfect matchings of constituent fields, where we should distinguish two subclasses of matchings:

(i) Matchings where $\partial\phi(z)$ and $\partial\phi(w)$ are paired (Wick-contracted) – these terms sum up to $-\frac{1}{(z-w)^2}\langle O_1(z_1)\cdots O_n(z_n)\rangle$.
(ii) Matchings where $\partial\phi(z)$ and $\partial\phi(w)$ are not paired (rather, each is paired with one of $O_i$'s.) These terms are regular as $z \to w$.

Thus, one obtains

$$\langle \partial\phi(z)\partial\phi(w)O_1(z_1)\cdots O_n(z_n)\rangle \underset{z\to w}{\sim} -\frac{1}{(z-w)^2}\langle O_1(z_1)\cdots O_n(z_n)\rangle + \text{reg.} \tag{590}$$

[85] Of course, on a general surface we don't have the radial quantization picture to compare to – that one is specific to $\Sigma = \mathbb{C}$. So on general $\Sigma$ it makes sense to take the path integral prescription as the definition of CFT correlators.

[86] In the path integral formalism we cannot talk about normal ordering of operators – since we don't have operators – so the limiting process in the r.h.s. of (587) becomes the definition of the "normally-ordered" differential polynomial in the l.h.s.

This corresponds to the OPE (555) which we previously obtained from the radial quantization picture.

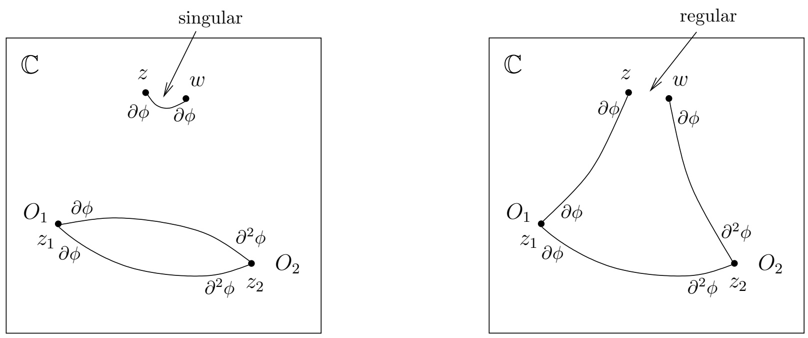



FIGURE 27. An example of a singular and a regular (as $z \to w$) contribution to the correlator (589). In this example, the two test observables are $O_1 =: \partial\phi\partial\phi :$ and $O_2 =: \partial^2\phi\partial^2\phi :$; we depict observables as corollas with the number of prongs being the degree of the differential monomial in $\phi$; edges correspond to Wick contractions. Thus each picture is one summand in the computation of the correlator via Wick's lemma.

*Remark* 4.40. When studying the theory on $\mathbb{C}$ we introduced a small positive mass $m$ in order to have well-defined Green's function (and then we let $m \to 0$ in correlators). Another possibility, instead of introducing a mass, is to have a massless theory, but replace $\mathbb{C}$ with a disk $D_R = \{z \in \mathbb{C} \mid |z| \leq R\}$ of large radius $R$, where one imposes Dirichlet boundary condition $\phi|_{\partial D_R} = 0$. Then one can write an explicit Green's function

$$G(z,w) = -2\log\frac{|z-w|}{|R - \frac{z\bar{w}}{R}|} \underset{R\to\infty}{\sim} -2\log|z-w| + C \tag{591}$$

with $C = 2\log R$.

<u>Summary: path integral vs. radial quantization.</u> The path integral formalism allows one another way to compute the same quantities as the radial quantization (or "operator formalism") does – correlators and OPEs. The two formalisms should be seen as complementing each other: path integral formalism has the benefit that it can be applied to general surfaces, not just $\mathbb{C}$. The benefit of the operator formalism is that it also recovers the space of states (and extra structure it might have, e.g., in the case of scalar field, the action of the Heisenberg Lie algebra). So, ultimately, the path integral formalism is better suited for handling global geometry (nontrivial surfaces) while the operator formalism gives a good handle of the local picture of CFT near a puncture (where $\Sigma$ can be approximated by $\mathbb{C}^*$).

## 5. General CFT on $\mathbb{C}$: Belavin-Polyakov-Zamolodchikov axiomatic picture

In this Section we will be talking about Belavin-Polyakov-Zamolodchikov [6] picture of a general CFT on $\mathbb{C}$, sometimes using the scalar field as an illustration.

### 5.1. **Virasoro algebra.**

**Definition 5.1.** Virasoro algebra is the central extension $\mathbb{C} \to \mathrm{Vir} \to \mathcal{W}$ of the Witt algebra $\mathcal{W}$ (the Lie algebra of meromorphic vector fields on $\mathbb{C}$ with only pole at 0 allowed, see Section 2.5.1), defined by the Lie brackets

$$\left[f(z)\frac{\partial}{\partial z}, g(z)\frac{\partial}{\partial z}\right]^{\mathrm{Vir}} = (fg' - gf')\frac{\partial}{\partial z} + \frac{c}{12}\mathbb{K}\oint_\gamma \frac{dz}{2\pi i} f'''(z)g(z), \tag{592}$$

where $\mathbb{K}$ is the central element, $c \in \mathbb{C}$ is a complex number (a parameter of the central extension) – the "central charge," $\gamma$ is a closed simple curve going around 0 counterclockwise.[87]

Virasoro algebra has the standard set of generators $\{L_n\}_{n\in\mathbb{Z}}, \mathbb{K}$ subject to commutation relations

$$[L_n, L_m] = (n-m)L_{n+m} + \delta_{n,-m}\frac{c}{12}(n^3 - n)\mathbb{K}, \quad n, m \in \mathbb{Z} \tag{593}$$

and $[\mathbb{K}, \cdots] = 0$; $L_n$ are the lifts of the standard generators $l_n = -z^{n+1}\partial_z$ of the Witt algebra.

**Exercise:** check that the Lie brackets (592) or equivalently (593) satisfy the Jacobi identity.

In fact, Virasoro algebra is the *unique* (up to a choice of the value of the parameter $c$) central extension of the Witt algebra, which is the content of the following theorem.

Feigin-Fuchs?

**Theorem 5.2.** *One has*

$$H^2_{\mathrm{Lie}}(\mathcal{W}, \mathbb{C}) = \mathbb{C} \tag{594}$$

*– the second Lie algebra (Chevalley-Eilenberg) cohomology of the Witt algebra (with coefficients in the trivial module) has rank* 1*. This cohomology is generated by the cohomology class of the Lie 2-cocycle*

$$\lambda(f(z)\partial_z, g(z)\partial_z) = \frac{1}{12}\oint_\gamma \frac{dz}{2\pi i} f'''(z)g(z). \tag{595}$$

Lecture 23, 10/14/2022

### 5.2. **Axiomatic CFT on $\mathbb{C}$. Action of Virasoro algebra on $\mathcal{H}$.**

We will start setting up general conformal field theory on $\mathbb{C}$ as an axiomatic picture, following [6].

In this picture, a CFT is the following collection of data.

(I) **Space of states.** One has a complex vector space – the space of states $\mathcal{H}$ – with a distinguished vector $|\mathrm{vac}\rangle \in \mathcal{H}$.

(II) **Space of fields, local operators.** One has a complex vector space of local observables (or "space of composite fields") $V$. For $z \in \mathbb{C}$ we will denote $V_z$ a copy of $V$ placed at $z$;[88] we denote a copy of an element $\Phi \in V$ placed at a point $z$ by $\Phi(z) \in V_z$.

---

[87] The conventional normalization factor $\frac{1}{12}$ in (592) is chosen in such a way that the central charge of the free massless scalar field is $c = 1$.

[88] I.e. we are thinking of a trivial vector bundle $\mathcal{V} = V \times \mathbb{C}$ over $\mathbb{C}$ with typical fiber $V$ and $V_z$ the fiber over a specific point $z$.

For $z \neq 0$, $\Phi(z) \in V_z$ is represented by a (possibly unbounded) operator $\widehat{\Phi}(z) \in \mathrm{End}(\mathcal{H})$.[89]

(III) **Field-state correspondence.** One has a linear isomorphism

$$\mathsf{s}\colon V \xrightarrow{\sim} \mathcal{H} \tag{596}$$

mapping a field $\Phi \in V$ to the state $\lim_{z\to 0} \widehat{\Phi}(z)|\mathrm{vac}\rangle$. (In particular, such a limit is required to exist for any $\Phi$ and determine an isomorphism between fields and states.) See Section 5.3 below for an example.

(IV) **Inner product.** Both $\mathcal{H}$ and $V$ carry real structures and hermitian inner products (intertwined by $\mathsf{s}$). For the hermitian conjugate of a local operator one has

$$(\widehat{\Phi}(z))^+ = \bar{z}^{-2h} z^{-2\bar{h}} \widehat{\Phi^*}(1/\bar{z}). \tag{597}$$

Here $*$ denotes the complex conjugation in $V$ and $(h, \bar{h})$ is the conformal weight of the field $V$ (see Definition 5.12 below).

(V) **Radial ordering, domains of field operators, same-time commutativity.** For any $n$-tuple of elements $\Phi_1, \ldots, \Phi_n \in V$, the vector

$$\widehat{\Phi}_1(z_1) \cdots \widehat{\Phi}_n(z_n)|\mathrm{vac}\rangle \tag{598}$$

is assumed to be well-defined if $z_i$'s are radially ordered, $|z_1| \geq \cdots \geq |z_n|$. As a consequence (using the same axiom for a string of $n+1$ local operators) the vector (598) is in the domain of $\widehat{\Phi}(z)$ if $|z| \geq |z_1|$ and $z \neq z_i$ for $i = 1, \ldots, n$. Operators $\widehat{\Phi}_1(z)$, $\widehat{\Phi}_2(w)$ are assumed to commute if $z \neq w$ and $|z| = |w|$.

(VI) **Correlators.** For an $n$-tuple of elements $\Phi_1, \ldots, \Phi_n \in V$, the correlator is defined as

$$\langle \Phi_1(z_1) \cdots \Phi_n(z_n)\rangle \colon = \langle \mathrm{vac}|\mathcal{R}\Big(\widehat{\Phi}_1(z_1) \cdots \widehat{\Phi}_n(z_n)\Big)|\mathrm{vac}\rangle. \tag{599}$$

where $\langle \mathrm{vac}| \colon = \big\langle |\mathrm{vac}\rangle, -\big\rangle_{\mathcal{H}} \in \mathcal{H}^*$ is the covector dual to the vector $|\mathrm{vac}\rangle$. The correlator (599) is a smooth function on $C_n(\mathbb{C})$ – the open configuration space of $n$ points on $\mathbb{C}$, depending linearly on the fields $\Phi_1, \ldots, \Phi_n$.[90]

(VII) **Identity field and stress-energy tensor.** $V$ contains a element $\mathbb{1}$ acting on $\mathcal{H}$ by identity and special elements $T$, $\overline{T}$ satisfying holomorphicity/antiholomorphicity

$$\bar\partial \widehat{T}(z) = 0, \quad \partial \widehat{\overline{T}}(z) = 0 \tag{600}$$

and the OPEs

$$\mathcal{R}\widehat{T}(z)\widehat{T}(w) \quad \sim \quad \frac{\frac{c}{2}\widehat{\mathbb{1}}}{(z-w)^4} + \frac{2\widehat{T}(w)}{(z-w)^2} + \frac{\partial\widehat{T}(w)}{z-w} + \mathrm{reg.} \tag{601}$$

$$\mathcal{R}\widehat{\overline{T}}(z)\widehat{\overline{T}}(w) \quad \sim \quad \frac{\frac{\bar{c}}{2}\widehat{\mathbb{1}}}{(\bar{z}-\bar{w})^4} + \frac{2\widehat{\overline{T}}(w)}{(\bar{z}-\bar{w})^2} + \frac{\bar\partial\,\widehat{\overline{T}}(w)}{\bar{z}-\bar{w}} + \mathrm{reg.} \tag{602}$$

$$\mathcal{R}\widehat{T}(z)\widehat{\overline{T}}(w) \quad \sim \quad \mathrm{reg.} \tag{603}$$

with $c, \bar{c}$ some complex numbers (the holomorphic and antiholomorphic central charges).

[89]In other words, there is a map $Y\colon V \times \mathbb{C}^* \to \mathrm{End}(\mathcal{H})$, linear in $V$ and smooth on $\mathbb{C}^*$. We denote $Y(\Phi, z)$ by $\widehat{\Phi}(z)$.

[90]If one of the points $z_i$ in the l.h.s. of (599) is zero, one understands the r.h.s. as a limit $z_i \to 0$.

Elements $\mathbb{1}, T, \overline{T} \in V$ are real (with respect to the real structure on $V$).

(VIII) **Projective action of conformal vector fields on states.** One has a projective representation $\rho$ of the Lie algebra of conformal vector fields on $\mathbb{C}^*$ on $\mathcal{H}$, where the conformal vector field $v = u(z)\partial_z + \overline{u(z)}\partial_{\bar{z}}$ on $\mathbb{C}^*$ (with $u$ a meromorphic function on $\mathbb{C}$ with pole allowed only at $z=0$) is represented by the operator

$$\rho(u\partial + \bar{u}\bar{\partial}) := -\frac{1}{2\pi i}\oint_\gamma \left(dz\, u(z)\, \widehat{T}(z) - d\bar{z}\, \overline{u(z)}\, \widehat{\overline{T}}(z)\right) \qquad \in \mathrm{End}(\mathcal{H}) \tag{604}$$

where $\gamma \in \mathbb{C}^*$ is a closed contour going around zero once counterclockwise.[91] In particular the standard generators of the Witt algebra $\mathcal{W}$, $l_n = -z^{n+1}\partial_z$, are represented by

$$\widehat{L}_n := \rho(-z^{n+1}\partial_z) = \frac{1}{2\pi i}\oint_\gamma dz\, z^{n+1}\widehat{T}(z) \qquad \in \mathrm{End}(\mathcal{H}) \tag{605}$$

and likewise for the generators of the antiholomorphic copy $\overline{\mathcal{W}}$ of the Witt algebra:

$$\widehat{\overline{L}}_n := \rho(-\bar{z}^{n+1}\partial_{\bar{z}}) = \frac{1}{2\pi i}\oint_\gamma d\bar{z}\, \bar{z}^{n+1}\widehat{\overline{T}}(\bar{z}) \qquad \in \mathrm{End}(\mathcal{H}). \tag{606}$$

We remark that the inverse formulae for (605) and (606), expressing the stress-energy tensor in terms of operators $\widehat{L}_n$, $\widehat{\overline{L}}_n$ are:

$$\widehat{T}(z) = \sum_{n\in\mathbb{Z}} z^{-n-2}\widehat{L}_n, \quad \widehat{\overline{T}}(z) = \sum_{n\in\mathbb{Z}} \bar{z}^{-n-2}\widehat{\overline{L}}_n. \tag{607}$$

I.e., essentially (and up to a shift in numbering), operators $\widehat{L}_n$ are the Fourier modes of the field $\widehat{T}(z)$ restricted to a circle.

**Lemma 5.3.** *(i) As a consequence of the $TT$ OPE (601), operators $\widehat{L}_n$ satisfy the Virasoro commutation relation (593) with central charge $c$ (the coefficient in the fourth order pole in (601)).*

*(ii) Similarly, as a consequence of $\overline{T}\,\overline{T}$ OPE (602), operators $\widehat{\overline{L}}$ satisfy the Virasoro commutation relation with central charge $\bar{c}$.*

*(iii) As a consequence of $T\overline{T}$ OPE (603), the generators of the holomorphic and antiholomorphic copies of the Virasoro algebra commute: $[\widehat{L}_n, \widehat{\overline{L}}_m] = 0$.*

We will prove this lemma in Section 5.2.2 below.

*Remark* 5.4. In the axioms above, it would be more correct to distinguish two versions of the space of states:

- $\mathcal{H} = \mathcal{H}^{\mathrm{small}}$ – the one identified with $V$ by the field-state correspondence (596), containing the vector $|\mathrm{vac}\rangle$ and carrying a hermitian inner product.

[91] If we combine $\widehat{T}(z)$ and $\widehat{\overline{T}}(z)$ into a single object – the total stress-energy operator $\widehat{T}^{\mathrm{total}}(z) = \widehat{T}(z)(dz)^2 + \widehat{\overline{T}}(z)(d\bar{z})^2$ – a quadratic differential on $\mathbb{C}^*$ valued in $\mathrm{End}(\mathcal{H})$, we can phrase (604) as

$$\rho(v) = -\frac{1}{2\pi i}\oint_\gamma \iota_v \widehat{T}^{\mathrm{total}}.$$

Here the contraction with the vector field $v$ converts the total stress-energy tensor from a quadratic differential into an (operator-valued) 1-form, which can then be integrated over the 1-cycle $\gamma$.

- A completion $\mathcal{H}^{\mathrm{big}}$ of $\mathcal{H}^{\mathrm{small}}$ on which the field operators $\widehat{\Phi}(z)$ act (it should be a completion containing all vectors of the form (598)).

E.g. in the scalar field theory, we can define $\mathcal{H}^{\mathrm{small}}$ to be the set of all *finite* linear combinations of basis vectors (486), while $\mathcal{H}^{\mathrm{big}}$ is spanned by the same basis but has to contain certain infinite linear combinations.

We note that neither $\mathcal{H}^{\mathrm{small}}$ nor $\mathcal{H}^{\mathrm{big}}$ is a Hilbert space: $\mathcal{H}^{\mathrm{small}}$ carries a hermitian form, but is not complete with respect to it, while $\mathcal{H}^{\mathrm{big}}$ contains vectors (598) which generally have infinite $L^2$ norm if $|z_1| \geq 1$.

One can also consider the $L^2$ completion $\mathcal{H}^{\mathrm{Hilb}}$ of $\mathcal{H}^{\mathrm{small}}$. As follows from the axioms (IV) and (V), $\mathcal{H}^{\mathrm{Hilb}}$ is guaranteed to contain the vector (598) only if $z_i$'s are distinct, radially-ordered *and are contained in the open unit disk* $\{z \in \mathbb{C} \mid |z| < 1\}$.[92]

*Remark* 5.5. The hermitian conjugate of the Virasoro generator $\widehat{L}_n$ (605) is readily computed from (597):

$$\widehat{L}_n^+ = \frac{-1}{2\pi i}\oint_\gamma d\bar{z}\, \bar{z}^{n+1} \underbrace{\bar{z}^{-4}\widehat{T}(1/\bar{z})}_{\widehat{T}(z)^+} \underset{w=1/\bar{z}}{=} \frac{1}{2\pi i}\oint_{\gamma'} dw\, w^{-n+1}\widehat{T}(w) = \widehat{L}_{-n} \tag{608}$$

Here $\gamma'$ is the image of the contour $\gamma$ under the inversion map $z \mapsto 1/\bar{z}$, which is again a contour going once around zero in positive direction. We have used the fact that $T$ has conformal weight $(2,0)$, see Example 5.13. Similarly, one proves

$$\widehat{\bar{L}}_n^+ = \widehat{\bar{L}}_{-n}. \tag{609}$$

The following is also an immediate consequence of (597).

**Lemma 5.6.** *For any fields $\Phi_1, \ldots, \Phi_n$ and $z_1, \ldots, z_n \in \mathbb{C}\backslash\{0\}$ an $n$-tuple of distinct points one has*

$$\overline{\langle \Phi_1(z_1)\cdots\Phi_n(z_n)\rangle} = \prod_{i=1}^n \bar{z}_i^{-2h_i} z_i^{-2\bar{h}_i} \cdot \langle \Phi_1^*(1/\bar{z}_1)\cdots\Phi_n^*(1/\bar{z}_n)\rangle \tag{610}$$

*where $(h_i, \bar{h}_i)$ is the conformal weight of $\Phi_i$. The bar over the correlator in the l.h.s. stands for complex conjugation.*

*Proof.* Without loss of generality we may assume that points $z_i$ are radially ordered, $|z_1| \geq \cdots \geq |z_n|$. We have

$$\begin{aligned}\overline{\langle \Phi_1(z_1)\cdots\Phi_n(z_n)\rangle} &= \overline{\langle \mathrm{vac}|\widehat{\Phi}_1(z_1)\cdots\widehat{\Phi}_n(z_n)|\mathrm{vac}\rangle} = \\ &= \langle \mathrm{vac}|\Big(\widehat{\Phi}_1(z_1)\cdots\widehat{\Phi}_n(z_n)\Big)^+|\mathrm{vac}\rangle = \langle \mathrm{vac}|\widehat{\Phi}_n(z_n)^+\cdots\widehat{\Phi}_1(z_1)^+|\mathrm{vac}\rangle \\ &\underset{(597)}{=} \prod_{i=1}^n \bar{z}_i^{-2h_i} z_i^{-2\bar{h}_i}\cdot \langle \mathrm{vac}|\widehat{\Phi}_n^*(1/\bar{z}_n)\cdots\widehat{\Phi}_1^*(1/\bar{z}_1)|\mathrm{vac}\rangle\end{aligned} \tag{611}$$

[92]Indeed, for the square of the $L^2$ norm of the vector (598) we have $||\widehat{\Phi}_1(z_1)\cdots\widehat{\Phi}_n(z_n)|\mathrm{vac}\rangle||^2 = \langle \mathrm{vac}|(\widehat{\Phi}_n(z_n))^+\cdots(\widehat{\Phi}_1(z_1))^+\widehat{\Phi}_1(z_1)\cdots\widehat{\Phi}_n(z_n)|\mathrm{vac}\rangle = \prod_{i=1}^n \bar{z}_i^{-2h_i} z_i^{-2\bar{h}_i}\cdot\langle \mathrm{vac}|\widehat{\Phi}_n^*(1/\bar{z}_n)\cdots\widehat{\Phi}_1^*(1/\bar{z}_1)\widehat{\Phi}_1(z_1)\cdots\widehat{\Phi}_n(z_n)|\mathrm{vac}\rangle$. The correlator on the right is certain to exist only if the insertion points of the operators $1/\bar{z}_n, \ldots, 1/\bar{z}_1, z_1, \ldots, z_n$ are distinct and the sequence is radially ordered. This implies that all $z_i$'s must be in the open unit disk. (Note that if $|z_1| = 1$ then $1/\bar{z}_1 = 1/z_1$, thus the sequence is radially ordered but not all points are distinct.)

$$= \prod_{i=1}^{n} \bar{z}_i^{-2h_i} z_i^{-2\bar{h}_i} \cdot \langle \Phi_1^*(1/\bar{z}_1) \cdots \Phi_n^*(1/\bar{z}_n) \rangle$$

□

5.2.1. *Example: action of Virasoro algebra on $\mathcal{H}$ in the scalar field theory. Abelian Sugawara construction.* In the example of the free scalar field theory we know the stress-energy tensor (551):

$$\widehat{T}(z) = -\frac{1}{2} : \partial\widehat{\phi}(z)\partial\widehat{\phi}(z) := \frac{1}{2} \sum_{j,k\in\mathbb{Z}} z^{-j-k-2} : \widehat{a}_j \widehat{a}_k : \tag{612}$$

where we used the expansion (535) of $\partial\widehat{\phi}(z)$ in terms of creation-annihilation operators. In particular, $\widehat{T}(z)$ has no dependence on $\bar{z}$, i.e. the holomorphicity axiom (600) holds (we skip the computations for $\widehat{\overline{T}}(z)$ – they are similar). OPEs (601), (602), (603) hold with the central charges $c = \bar{c} = 1$ – we know this from the explicit computation in Example 4.37. From (605) and (612) we find the operators $\widehat{L}_n$ to be

$$\widehat{L}_n = \frac{1}{2} \sum_{k\in\mathbb{Z}} : \widehat{a}_k \widehat{a}_{n-k} : \tag{613}$$

and similarly

$$\widehat{\overline{L}}_n = \frac{1}{2} \sum_{k\in\mathbb{Z}} : \widehat{\bar{a}}_k \widehat{\bar{a}}_{n-k} : \tag{614}$$

Note that the normal ordering is only relevant for $\widehat{L}_n$, $\widehat{\overline{L}}_n$ with $n = 0$, as for $n \neq 0$ the operators $\widehat{a}_k, \widehat{a}_{n-k}$ commute for any $k$, and likewise for $\widehat{\bar{a}}_k, \widehat{\bar{a}}_{n-k}$.

**Exercise:** Show by a direct computation that the operators (613) satisfy Virasoro commutation relations with $c = 1$, from the commutation relations (472) for the creation/annihilation operators.

Equality (613) expresses the generators of Virasoro algebra with central charge $c = 1$ as quadratic polynomials in generators of the Heisenberg Lie algebra (476). Thus, we have an inclusion

$$\mathrm{Vir}_{c=1} \hookrightarrow U^{(2)}\mathrm{Heis}, \tag{615}$$

where $U^{(2)}$ means the subspace of (at most) quadratic elements in the universal enveloping algebra (of the Heisenberg Lie algebra). This inclusion is the abelian version of the Sugawara construction, realizing Virasoro algebra (at certain other values of $c$) inside the quadratic part of the universal enveloping algebra of the affine Lie algebra (a.k.a. Kac-Moody algebra) $\widehat{\mathfrak{g}}$. We will come to the non-abelian Sugawara construction later, when talking about Wess-Zumino-Witten model.

*Remark* 5.7. Comparing (613) and (614) with (489), (490), we observe the equalities

$$\widehat{L}_0 + \widehat{\overline{L}}_0 = \widehat{H}, \qquad \widehat{L}_0 - \widehat{\overline{L}}_0 = \widehat{P}, \tag{616}$$

expressing the quantum Hamiltonian and the total momentum operators in terms of Virasoro generators $\widehat{L}_0$, $\widehat{\overline{L}}_0$. In a general CFT, formulae (616) become the *definitions* of the Hamiltonian and the total momentum operators.

Note that due to (604), the operator $\widehat{H} = \widehat{L}_0 + \widehat{\overline{L}}_0$ represents on $\mathcal{H}$ the vector field $-z\partial_z - \bar{z}\partial_{\bar{z}}$ or, in terms of coordinates $\tau, \sigma$ on the cylinder, the vector field

$-\partial_\tau$. Likewise, the operator $\widehat{P} = \widehat{L}_0 - \widehat{\overline{L}}_0$ represents the vector field $-z\partial_z + \bar{z}\partial_{\bar{z}}$ or, in terms of the cylinder, $i\partial_\sigma$. Ultimately, the operators represent infinitesimal translations along the cylinder and rotations of the cylinder, as the Hamiltonian and total momentum should, cf. Remark 4.12.

5.2.2. *Virasoro commutation relations from TT OPE (contour integration trick).* Let us prove Lemma 5.3. We will focus on the case (i): assuming that the $TT$ OPE (601) is known, let us calculate the commutator of operators $\widehat{L}_n$, $\widehat{L_m}$ using their definition via the stress-energy tensor (605):

$$(617)\quad [\widehat{L}_n, \widehat{L}_m] = \widehat{L}_n\widehat{L}_m - \widehat{L}_m\widehat{L}_n =$$
$$= \oint_{\gamma_{0,R}} \frac{dz}{2\pi i} \oint_{\gamma_{0,r}} \frac{dw}{2\pi i} z^{n+1} w^{m+1} \widehat{T}(z)\widehat{T}(w) - \oint_{\gamma_{0,R}} \frac{dw}{2\pi i} \oint_{\gamma_{0,r}} \frac{dz}{2\pi i} w^{m+1} z^{n+1} \widehat{T}(w)\widehat{T}(z)$$
$$= \oint_{\gamma_{0,R}} \frac{dw}{2\pi i} \oint_{\Gamma} \frac{dz}{2\pi i} z^{n+1} w^{m+1} \mathcal{R}\left(\widehat{T}(z)\widehat{T}(w)\right).$$

Here we denoted $\gamma_{z,r}$ the circle of radius $r$ centered at $z$, with counterclockwise orientation; we assume the two radii to satisfy $0 < r < R$; $\Gamma$ is the 1-cycle $\gamma_{R'} - \gamma_r$ with $R' > R$. We are exploiting the freedom to deform the integration contour, due to holomorphicity of the integrand for $z \neq w$ and $z, w \neq 0$ (in particular, the property (600)). We can then further deform the contour $\Gamma$ to the circle $\gamma_{w,\epsilon}$ centered at $w$, of radius $0 < \epsilon < R$.

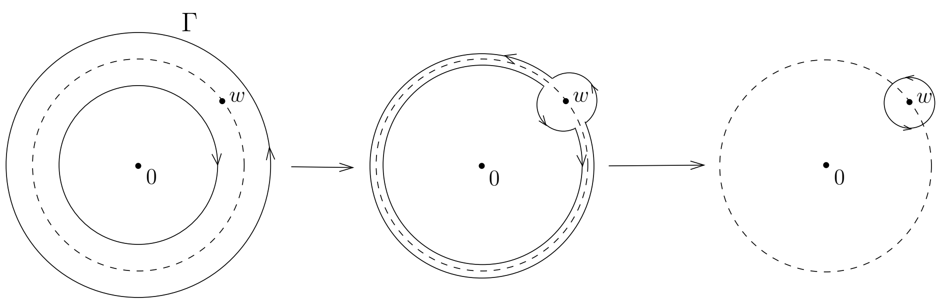



FIGURE 28. Deformation of the integration contour for the integral over $z$ (solid curve). The dashed circle is the (fixed) integration contour for $w$.

Replacing the radially ordered product of stress energy tensors with the OPE (601), we have then

(618)
$$[\widehat{L}_n, \widehat{L}_m] = \oint_{\gamma_{0,R}} \frac{dw}{2\pi i} \oint_{\gamma_{w,\epsilon}} \frac{dz}{2\pi i} z^{n+1} w^{m+1} \left( \frac{\frac{c}{2}\widehat{\mathbb{1}}}{(z-w)^4} + \frac{2\widehat{T}(w)}{(z-w)^2} + \frac{\partial\widehat{T}(w)}{z-w} + \text{reg.} \right)$$
$$\underset{\substack{z = w+\alpha,\\ \text{expand in } \alpha}}{=} \oint_{\gamma_{0,R}} \frac{dw}{2\pi i} \oint_{\gamma_{0,\epsilon}} \frac{d\alpha}{2\pi i} \left( w^{n+1} + (n+1)w^n\alpha + \frac{(n+1)n}{2} w^{n-1}\alpha^2 + \frac{(n+1)n(n-1)}{6}\alpha^3 + \cdots \right) \cdot$$
$$\cdot\, w^{m+1} \left( \frac{\frac{c}{2}\widehat{\mathbb{1}}}{\alpha^4} + \frac{2\widehat{T}(w)}{\alpha^2} + \frac{\partial\widehat{T}(w)}{\alpha} + \text{reg.} \right).$$

Here the integral over $\alpha$ simply computes the residue at $\alpha = 0$ of the integrand, i.e., the coefficient of $\alpha^{-1}$. Thus, continuing the computation we have

(619)
$$[\widehat{L}_n, \widehat{L}_m] = \oint_{\gamma_{0,R}} \frac{dw}{2\pi i}\Big( \underbrace{w^{n+m+2}\partial\widehat{T}(w)}_{\text{integrate by parts}} + 2(n+1)w^{n+m+1}\widehat{T}(w) + \frac{(n+1)n(n-1)}{6}\frac{c}{2}w^{n+m-1}\widehat{\mathbb{1}}\Big)$$
$$= \oint_{\gamma_{0,R}} \frac{dw}{2\pi i}\Big( \underbrace{(2(n+1)-(n+m+2))}_{n-m} w^{n+m+1}\widehat{T}(w) + \frac{c}{12}(n^3-n)w^{n+m-1}\widehat{\mathbb{1}}\Big)$$
$$\underset{(605)}{=} (n-m)\widehat{L}_{n+m} + \frac{c}{12}(n^3-n)\delta_{n,-m}\widehat{\mathbb{1}}.$$

This is indeed the Virasoro commutation relation (593). This proves case (i) of Lemma 5.3. The other two cases are proved similarly. Lecture 24, 10/22/2022

5.3. **Field-state correspondence in the example of the scalar field CFT.** Let us examine the field-state correspondence is the map (596),

(620)
$$\begin{array}{rccc} \mathsf{s}\colon & V & \to & \mathcal{H} \\ & \Phi & \mapsto & \lim_{z\to 0} \widehat{\Phi}(z)|\mathrm{vac}\rangle \end{array}$$

in the case of the scalar field theory. We start with simple examples.

For $\Phi = i\partial\phi$, we have

(621)
$$\mathsf{s}(i\partial\phi)\colon = \lim_{z\to 0} i\partial\widehat{\phi}(z)|\mathrm{vac}\rangle = \lim_{z\to 0}\sum_{n\in\mathbb{Z}} z^{-n-1}\widehat{a}_n|\mathrm{vac}\rangle$$

where we used (535) to express the derivative of the field operator in terms of creation/annihilation operators. Notice that for $n \geq 0$ one has $\widehat{a}_n|\mathrm{vac}\rangle = 0$, while for $n \leq -2$ one has $z^{-n-1} \underset{z\to 0}{\to} 0$. So, the only surviving term in the r.h.s. of (621) is $n = -1$:

(622)
$$\mathsf{s}(i\partial\phi) = \widehat{a}_{-1}|\mathrm{vac}\rangle$$

– a state with a single left-mover of energy-momentum $(1, 1)$.

For higher derivatives of the fundamental field $\phi$ we find

(623)
$$\mathsf{s}(i\partial^p\phi) = \lim_{z\to 0} i\partial^p\widehat{\phi}(z)|\mathrm{vac}\rangle = \lim_{z\to 0}\sum_{n\in\mathbb{Z}}(-n-1)(-n-2)\cdots(-n-p+1)z^{-n-p}\widehat{a}_n|\mathrm{vac}\rangle$$

where $p \geq 1$. In the r.h.s. the summand satisfies the following:

- vanishes for $n \geq 0$, since then $\widehat{a}_n|\mathrm{vac}\rangle = 0$,
- vanishes as $z \to 0$ for $n \leq -p-1$, since then $\lim_{z\to 0} z^{-n-p} = 0$,
- vanishes for $n = -1, -2, \ldots, -p+1$, since then the product $(-n-1)(-n-2)\cdots(-n-p+1)$ vanishes.

Thus, the only surviving term is $n = -p$:

(624)
$$\mathsf{s}(i\partial^p\phi) = (p-1)!\,\widehat{a}_{-p}|\mathrm{vac}\rangle$$

– a state with a single left-mover of energy-momentum $(p, p)$.

*Remark* 5.8. Note that

(625)
$$\mathsf{s}(\phi) = \lim_{z\to 0}\widehat{\phi}(z)|\mathrm{vac}\rangle \underset{(523)}{=} \lim_{z\to 0}\widehat{\phi}_0|\mathrm{vac}\rangle$$

is ill-defined. This absence of the image of $\phi$ under field-state correspondence (together with the fact that correlators of $\phi$ are ill-defined) reinforces the point that $\phi$ should not be considered as an element of $V$ (while derivatives of $\phi$ are in $V$).

As a more complicated example, consider the normally ordered differential monomial $\Phi =: i\partial\phi\, \partial\phi :$,

(626)
$$\mathsf{s}(: i\partial\phi\, i\partial\phi :) = \lim_{z\to 0} : i\partial\widehat{\phi}(z)\, i\partial\widehat{\phi}(z) : |\mathrm{vac}\rangle = \sum_{n,m\in\mathbb{Z}} z^{-n-m-2} : \widehat{a}_n \widehat{a}_m : |\mathrm{vac}\rangle = \widehat{a}_{-1}\widehat{a}_{-1}|\mathrm{vac}\rangle$$

– the state with two left-moving quanta of energy-momentum $(1,1)$. Here the only surviving term in the double sum is $n = m = -1$, similarly to the situations above.

In particular, since the quantum stress-energy tensor (as an element of $V$) $T = -\frac{1}{2} : \partial\phi\partial\phi :$, we have

$$\mathsf{s}(T) = \frac{1}{2}\widehat{a}_{-1}\widehat{a}_{-1}|\mathrm{vac}\rangle. \tag{627}$$

Note that using (613), we can write the r.h.s. as $\widehat{L}_{-2}|\mathrm{vac}\rangle$.

*Remark* 5.9. In fact, in any CFT one has

$$\mathsf{s}(T) = \widehat{L}_{-2}|\mathrm{vac}\rangle. \tag{628}$$

This, together with properties $\widehat{L}_{\geq -1}|\mathrm{vac}\rangle = 0$ and $\widehat{L}_{-2-p} = \mathsf{s}(\frac{1}{p!}\partial^p T)$ for $p \geq 0$, is a consequence of (607).

A generalization of the examples above is the case where $\Phi$ is a general normally-ordered differential monomial in $\phi$:

$$\begin{aligned}\mathsf{s}\Big( : \left(\prod_{j=1}^{r} \frac{i\partial^{n_j}\phi}{(n_j-1)!}\right)\left(\prod_{k=1}^{s} \frac{i\bar\partial^{\bar{n}_k}\phi}{(\bar{n}_k-1)!}\right) : \Big) = \\ = \widehat{a}_{-n_1}\cdots\widehat{a}_{-n_r}\widehat{\bar{a}}_{-\bar{n}_1}\cdots\widehat{\bar{a}}_{-\bar{n}_s}|\mathrm{vac}\rangle \underset{(486)}{=} |0;\{n_i\};\{\bar{n}_j\}\rangle\end{aligned} \tag{629}$$

where $1 \leq n_1 \leq \cdots \leq n_r$, $1 \leq \bar{n}_1 \leq \cdots \leq \bar{n}_s$. The computation is similar to the computations above (only a single term in the $(r+s)$-fold sum survives). Note that we identified all basis vectors (486) of $\mathcal{H}$ with $\pi_0 = 0$ as images of particular vectors in $V$ (differential monomials), under the field-state correspondence.

Since the map (620) is supposed to be an isomorphism, this means that $V$ should contain some more elements in addition to differential polynomials in $\phi$,[93] with images of these extra elements giving the states with $\pi_0 \neq 0$.

5.3.1. *Vertex operators (in the scalar field theory).* A vertex operator is defined as

$$\widehat{V}_\alpha(z) : = : e^{i\alpha\widehat{\phi}(z)} : \tag{630}$$

where $\alpha \in \mathbb{R}$ is a parameter ("charge"). We emphasize that the vertex operator is a construction specific to the scalar field theory. We understand the operator (630) as a local operator acting on $\mathcal{H}$, corresponding to an abstract field $V_\alpha \in V$ placed at a point $z \in \mathbb{C}$.

---

[93]We mean normally-ordered differential polynomials, where $\phi$ is not allowed to appear without derivatives, cf. Remark 5.8.

Let us find the state corresponding to the vertex operator $V_\alpha$:

(631) $$\mathsf{s}(V_\alpha) = \lim_{z\to 0} \widehat{V}_\alpha|\mathrm{vac}\rangle = \lim_{z\to 0} :e^{i\alpha\widehat{\phi}(z)}:|\mathrm{vac}\rangle = \\ \underset{(523)}{=} e^{i\alpha\sum_{n<0}\frac{i}{n}(\widehat{a}_n z^{-n}+\widehat{\bar{a}}_n\bar{z}^{-n})} e^{i\alpha\sum_{n>0}\frac{i}{n}(\widehat{a}_n z^{-n}+\widehat{\bar{a}}_n\bar{z}^{-n})} e^{i\alpha\widehat{\phi}_0} \cancel{e^{\alpha\widehat{\pi}_0\log(z\bar{z})}}|\mathrm{vac}\rangle$$

Here the last exponential acting on $|\mathrm{vac}\rangle$ acts as identity, since $\widehat{\pi}_0|\mathrm{vac}\rangle = 0$. The next observation is that in Schrödinger representation of the quantum free particle system (corresponding to the zero-mode $\phi_0, \pi_0$), with states being $L^2$ functions of $\pi_0$, one has $\widehat{\pi}_0 = \pi_0\cdot$ a multiplication operator and $\widehat{\phi}_0 = i\frac{\partial}{\partial\pi_0}$ a derivation operator (cf. the discussion in Example 4.27). Thus, the exponential $e^{i\alpha\widehat{\phi}_0}\colon \psi(\pi_0) \mapsto \psi(\pi_0-\alpha)$ is the shift operator. In particular, it maps the vacuum $|\pi_0 = 0\rangle$ represented by the delta-function centered at zero to the delta-function centered at $\alpha$, i.e., the vector $|\pi_0 = \alpha\rangle$. In other words, $e^{i\alpha\widehat{\phi}_0}$ maps the vacuum $|\mathrm{vac}\rangle$ to the pseudo-vacuum $|\pi_0 = \alpha\rangle$ with zero-mode momentum $\alpha$. Thus, continuing the computation (631), we have

(632) $$\mathsf{s}(V_\alpha) = e^{i\alpha\sum_{n<0}\frac{i}{n}(\widehat{a}_n z^{-n}+\widehat{\bar{a}}_n\bar{z}^{-n})} e^{i\alpha\sum_{n>0}\frac{i}{n}(\widehat{a}_n z^{-n}+\widehat{\bar{a}}_n\bar{z}^{-n})}|\pi_0 = \alpha\rangle$$

Here the right exponential acts by identity, since the annihilation operators in the exponent kill the pseudovacuum. The left exponential becomes identity as $z \to 0$, thus one has

(633) $$\mathsf{s}(V_\alpha) = |\pi_0 = \alpha\rangle$$

So, the image of a vertex operator under the field-state correspondence is a pseudovacuum. Combining this computation with the computation with (629), we have

(634) $$\mathsf{s}(:(\text{differential monomial in } \phi)\cdot V_\alpha:) = |\alpha; \{n_i\}; \{\bar{n}_j\}\rangle$$

with differential monomial as in the l.h.s. of (629). Thus are recovering all basis vectors of $\mathcal{H}$ as images of elements of $V$, once we have adjoined the vertex operators. Put another way, for the field-state correspondence to be an isomorphism, we should set the space of local fields in the scalar field theory to be

(635) $$V = \mathrm{span}_\mathbb{C}\{:(\text{differential polynomials in } \phi)\cdot V_\alpha: \;\;|\;\; \alpha\in\mathbb{R}\}$$

where as usual differential polynomials are not allowed to contain $\phi$ without derivatives.

Lecture 25, 10/26/2022

5.4. **Local Virasoro action at a puncture.** We continue with the CFT data/axioms list:

(IX) **Local projective action of conformal vector fields on fields at a point $z$.** Similarly to the projective action (604) of conformal vector fields on states, one has a projective action of conformal vector fields with singularities (vector fields of the form $v = u(w)\partial_w + \overline{u(w)}\partial_{\bar{w}}$ with $u$ a meromorphic function) on fields at a point $z \in \mathbb{C}\backslash\{0\}$,

(636) $$\rho^{(z)}\colon \mathrm{conf}_{\mathrm{sing}}(\mathbb{C}) \to \mathrm{End}(V_z)$$

given by

(637) $$\rho^{(z)}(u\partial + \bar{u}\bar{\partial})\circ\Phi(z)\colon = -\frac{1}{2\pi i}\oint_{\gamma_z}\Big(dw\, u(w)\, T(w)\,\Phi(z) + d\bar{w}\,\overline{u(w)}\,\overline{T}(w)\,\Phi(z)\Big),$$

for any field $\Phi(z) \in V_z$. Here $\gamma_z$ is a contour going around $z$ once in a positive direction (and small enough so that it does not enclose any poles of $u$ apart from $z$). We understand the r.h.s. of (637) as defining a new local field at $z$. Equality (637) is understood either (a) as an equality under a correlator with an arbitrary collection of test field, or (b) as equality of local operators (then we put hats on $T$, $\Phi$ and the l.h.s., and we radially order the operator product in the r.h.s.).

In particular, the vector fields $-(w-z)^{n+1}\partial_w$ (standard meromorphic vector fields generating the Witt algebra, centered at $z$ instead of the origin) correspond to certain operators $L_n^{(z)}$ acting on $V_z$:

$$L_n^{(z)}\Phi(z) := \rho^{(z)}(-(w-z)^{n+1}\partial_w)\Phi(z) = \frac{1}{2\pi i}\oint_{\gamma_z} dw\,(w-z)^{n+1}T(w)\,\Phi(z) \tag{638}$$

We will also write the l.h.s. as $(L_n\Phi)(z)$. Calculating the integral on the right as a residue, we observe that the fields $L_n\Phi$ are the coefficients of the OPE $T(w)\Phi(z)$:

$$T(w)\Phi(z) \sim \sum_{n\in\mathbb{Z}} (w-z)^{-n-2}(L_n\Phi)(z) = \tag{639}$$
$$= \cdots + \frac{(L_1\Phi)(z)}{(w-z)^3} + \frac{(L_0\Phi)(z)}{(w-z)^2} + \frac{(L_{-1}\Phi)(z)}{w-z} + \underbrace{(L_{-2}\Phi)(z) + (w-z)(L_{-3}\Phi)(z) + \cdots}_{\text{reg.}}$$

By an argument similar to Lemma 5.3 (and the computation of Section 5.2.2), operators $L_n^z$ acting on $V_z$ satisfy Virasoro commutation relations.

Similarly to (638), one defines operators $\overline{L}_n^{(z)}$ acting on $V_z$, corresponding to the terms in the OPE $\overline{T}(w)\Phi(z)$.

Explain more/prove?

Remark 5.5 has an analog for the hermitian conjugates of the operators $L_n^{(z)}, \overline{L}_n^{(z)}$:

$$(L_n^{(z)})^+ = L_{-n}^{(z)}, \quad (\overline{L}_n^{(z)})^+ = \overline{L}_{-n}^{(z)}. \tag{640}$$

This follows from Remark 5.5 by field-state correspondence.

*Remark* 5.10. Consider the OPE (639) for $\Phi = \mathbb{1}$ the identity field. One has

$$T(w)\mathbb{1}(z) = T(w) = \sum_{n\geq 0} \frac{1}{n!}(w-z)^n \partial^n T(z) \tag{641}$$

Where on the right we have the Taylor expansion of $T(w)$ centered at $z$; the sum is convergent for $w$ sufficiently close to $z$.[94] Comparing the coefficients in (641) and in (639) with $\Phi = \mathbb{1}$, we obtain

$$\ldots, L_1\mathbb{1} = 0,\ L_0\mathbb{1} = 0,\ L_{-1}\mathbb{1} = 0,\ \boxed{L_{-2}\mathbb{1} = T},\ L_{-3}\mathbb{1} = \partial T,\ L_{-4}\mathbb{1} = \frac{1}{2!}\partial^2 T\ \ldots \tag{642}$$

One has similar formulae for $\overline{L}_n\mathbb{1}$, in particular, one has $\overline{L}_{-2}\mathbb{1} = \overline{T}$.

(X) $L_{-1}$ **axiom.**[95] For any $\Phi \in V$ one has

$$(L_{-1}\Phi)(z) \quad = \quad \partial\Phi(z), \tag{643}$$

[94] More precisely, under a correlator with test fields $\Phi_1(z_1), \ldots, \Phi_n(z_n)$, the field $T(w)$ can be replaced with the r.h.s. of (641) – and the sum is convergent – if $|w-z| < |z_i - z|$ for all $i$.

[95] Informally, the axiom can be phrased as "$L_{-1}$ acts by infinitesimally moving the puncture $z$."

$$(\overline{L}_{-1}\Phi)(z) \quad = \quad \bar{\partial}\Phi(z). \tag{644}$$

Here one understands that the field $\partial\Phi(z)$ is defined by its behavior under a correlator with test fields: $\langle\partial\Phi(z)\Phi_1(z_1)\cdots\Phi_n(z_n)\rangle = \partial_z\langle\Phi(z)\Phi_1(z_1)\cdots\Phi_n(z_n)\rangle$ (or in the language of field operators, $\widehat{\partial\Phi}(z) := \frac{\partial}{\partial z}\widehat{\Phi}(z)$). The case of $\bar{\partial}\Phi$ is similar.

*Remark* 5.11. If $v = u\partial + \bar{u}\bar{\partial}$ is a conformal vector field on $\mathbb{C}$ without singularities (except possibly at zero), then the field operator corresponding to (637) is

$$\widehat{\rho^{(z)}(v) \circ \Phi}(z) = [\rho(v), \widehat{\Phi}(z)] \tag{645}$$

where the r.h.s. is the commutator of the field operator with the operator (604) representing the vector field $v$ on the space of states $\mathcal{H}$. Equality (645) is proven by a contour integration trick similar to one of Section 5.2.2: the r.h.s. of (645) is an integral over a cycle $\Gamma$ – the difference of two circles, one of radius $R > |z|$ and one of radius $r < |z|$; this contour can be deformed to a single circle centered at $z$, which yields the l.h.s. of (645).

**Definition 5.12.** We say that a field $V \in \Phi$ has *conformal weight* (or *conformal dimension*) $(h, \bar{h}) \in \mathbb{R}^2$ if one has

$$(L_0\Phi)(z) = h\Phi(z), \qquad (\overline{L}_0\Phi)(z) = \bar{h}\Phi(z), \tag{646}$$

i.e., $\Phi$ is an eigenvector of operators $L_0, \overline{L}_0$ simultaneously, with eigenvalues $h, \bar{h}$.

**Example 5.13.** Consider (639) for $\Phi = T$ and compare with the standard $TT$ OPE (601). We obtain

$$L_{\geq 3}T = 0, \quad L_2 T = \frac{c}{2}\mathbb{1}, \quad L_1 T = 0, \quad L_0 T = 2T, \quad L_{-1}T = \partial T \tag{647}$$

Likewise, from $\overline{T}T$ OPE (603) we have

$$\overline{L}_{\geq -1}T = 0. \tag{648}$$

In particular, we see that $T$ has conformal weight $(h, \bar{h}) = (2, 0)$. Similarly, $\overline{T}$ has conformal weight $(0, 2)$.

We will be assuming that $L_0, \overline{L}_0$ are simultaneously diagonalizable on $V$,[96] thus the space $V$ is bi-graded by conformal weight:

$$V = \bigoplus_{h,\bar{h}\subset\Delta} V^{h,\bar{h}} \tag{649}$$

where $\Delta \subset \mathbb{R}^2$ is some set of admissible conformal weights.

The action of a Virasoro generator $L_{-n}$ changes the conformal weight of a field as[97]

$$(h, \bar{h}) \to (h + n, \bar{h}). \tag{650}$$

Similarly, the action of $\overline{L}_{-n}$ changes the conformal weight as

$$(h, \bar{h}) \to (h, \bar{h} + n). \tag{651}$$

[96]There are interesting examples of CFTs where this diagonalizability assumption fails. Such CFTs are called "logarithmic."

[97]This is a consequence of the relation $[L_0, L_{-n}] = nL_{-n}$ in Virasoro algebra: if $L_0\Phi = h\Phi$, then one has $L_0(L_{-n}\Phi) = L_{-n}(L_0 + n)\Phi = (h + n)L_{-n}\Phi$. Likewise, $[\overline{L}_0, L_{-n}] = 0$ implies that the eigenvalue of $\overline{L}_0$ does not change under the action of $L_{-n}$.

The following is a standard assumption on admissible conformal weights.

**Assumption 5.14.**
(a) $\Delta \subset \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$,[98]
(b) If $(h, \bar{h}) \in \Delta$ then[99]

$$h - \bar{h} \in \mathbb{Z}, \tag{652}$$

(c) $V^{0,0} = \mathrm{Span}(\mathbb{1})$.

5.5. **Primary fields.**

**Definition 5.15.** A field $V \in \Phi$ is said to be primary, of conformal weight $(h, \bar{h})$ if it satisfies the OPE

$$T(w)\Phi(z) \sim \frac{h\Phi(z)}{(w-z)^2} + \frac{\partial\Phi(z)}{w-z} + \text{reg.}, \quad \overline{T}(w)\Phi(z) \sim \frac{\bar{h}\Phi(z)}{(\bar{w}-\bar{z})^2} + \frac{\bar{\partial}\Phi(z)}{\bar{w}-\bar{z}} + \text{reg.} \tag{653}$$

Equivalently, $\Phi \in V$ is primary, with conformal weight $(h, \bar{h})$, if

$$\begin{array}{ll} L_{>0}\Phi = 0, & \overline{L}_{>0}\Phi = 0, \\ L_0\Phi = h\Phi, & \overline{L}_0\Phi = \bar{h}\Phi. \end{array} \tag{654}$$

Put another way, a primary field is a highest weight vector of $V$ as a module over Virasoro $\oplus$ $\overline{\text{Virasoro}}$, of weight $(h, \bar{h})$.

For $\Phi$ a primary field, fields obtained from it by repeated application of negative Virasoro generators $L_{<0}, \overline{L}_{<0}$, i.e., fields of the form

$$L_{-k_r} \cdots L_{-k_1} \overline{L}_{-l_s} \cdots \overline{L}_{-l_1} \Phi \tag{655}$$

with $k_1, \ldots, k_r, l_1, \ldots, l_s \geq 1$, are called "descendants" of $\Phi$. If $\Phi$ has conformal weight $(h, \bar{h})$ then the descendant (655) has conformal weight $(h + \sum_i k_i, \bar{h} + \sum_j l_j)$ (cf. (650), (651)). The subspace of $V$ spanned by all descendants of a primary field $\Phi$ is called the "conformal family" of $\Phi$.

add a table of first descendants?

The space of fields $V$ splits as a direct sum of irreducible highest weight modules of the Lie algebra $\mathrm{Vir} \oplus \overline{\mathrm{Vir}}$ with primary fields being the highest weight vectors:

$$V = \bigoplus_\alpha V^{(\Phi_\alpha)}. \tag{656}$$

Here the sum is over species of primary fields (i.e. over a basis in the subspace of primary fields in $V$); $V^{(\Phi_\alpha)}$ is the conformal family of $\Phi_\alpha$.

*Remark* 5.16. There can be linear dependencies between descendants of a given $\Phi_\alpha$.[100] More precisely, one can consider a *Verma module* $\mathbb{V}^{h,\bar{h}}$ (free highest weight module) of the Lie algebra $\mathrm{Vir}\oplus\overline{\mathrm{Vir}}$ – the span of formal expressions $L_{-k_r} \cdots L_{-k_1} \overline{L}_{-l_s} \cdots \overline{L}_{-l_1} \Phi_\alpha$ with $1 \leq k_1 \leq \cdots \leq k_r$, $1 \leq l_1 \leq \cdots \leq l_s$ (i.e. all ordered descendants are considered to be independent), with $\Phi_\alpha$ a vector of weight $(h, \bar{h})$ and annihilated by $L_{>0}, \overline{L}_{>0}$. Then $V^{(\Phi_\alpha)}$ is the quotient of the Verma module $\mathbb{V}^{h,\bar{h}}$ by a submodule,

$$V^{(\Phi_\alpha)} \simeq \mathbb{V}^{(h,\bar{h})}/\mathsf{N} \tag{657}$$

[98]Otherwise the 2-point correlator $\langle \Phi(z)\Phi(w) \rangle$ can grow as points $z$ and $w$ become farther and farther apart, which contradicts the physical intuition of local interactions.

[99]Needed for single-valuedness of correlators, cf. Remark 1.18.

[100]For instance, in any CFT one has $L_{-1}\mathbb{1} = 0$. Also, see (667) below for a nontrivial example in scalar field theory.

The submodule $\mathsf{N}$ that one quotients out is the kernel of the sesquilinear form $\langle,\rangle$ on $\mathbb{V}^{h,\bar{h}}$, defined in such a way that one has $L_n^+ = L_{-n}, \overline{L}_n^+ = \overline{L}_{-n}$ and the highest vector has norm 1 (in particular, vectors in $\mathsf{N}$ have zero norm). Also, $\mathsf{N} \subset \mathbb{V}^{h,\bar{h}}$ is the submodule generated by "singular vectors" $\chi \in \mathbb{V}^{h,\bar{h}}$ – vectors with with the property $L_{>0}\chi = 0, \overline{L}_{>0}\chi = 0$.

5.5.1. *Transformation property of a primary field.* Let us fix a conformal vector field $v = u(w)\partial_w + \overline{u(w)}\partial_{\bar{w}}$ regular at $z$. For $\Phi \in V$ a primary of conformal weight $(h,\bar{h})$, by (637) and (653) we have

$$\rho^{(z)}(u\partial + \bar{u}\bar{\partial})\Phi(z) = \tag{658}$$
$$= -\frac{1}{2\pi i}\oint_{\gamma_z} dw \underbrace{u(w)}_{u(z)+(w-z)\partial u(z)+\cdots} \underbrace{T(w)\Phi(z)}_{\frac{h\Phi(z)}{(w-z)^2}+\frac{\partial\Phi(z)}{w-z}+\text{reg.}} + d\bar{w} \underbrace{\overline{u(w)}}_{\overline{u(z)}+(\bar{w}-\bar{z})\bar{\partial}\overline{u(z)}+\cdots} \underbrace{\overline{T}(w)\Phi(z)}_{\frac{\bar{h}\Phi(z)}{(\bar{w}-\bar{z})^2}+\frac{\bar{\partial}\Phi(z)}{\bar{w}-\bar{z}}+\text{reg.}}$$
$$= -u(z)\partial\Phi(z) - \overline{u(z)}\bar{\partial}\Phi(z) - h\partial u(z)\Phi(z) - \bar{h}\bar{\partial}\,\overline{u(z)}\,\Phi(z)$$

– a computation of the contour integral as a residue.

Finite version, interpretation #1: "active transformations." Formula (658) expresses the change of a field under an infinitesimal conformal map. For a finite conformal (holomorphic) map $z \mapsto w(z)$, it implies that the field transforms as

$$\Phi(z) \mapsto \Phi'(w) = \left(\frac{\partial w}{\partial z}\right)^{-h} \left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{-\bar{h}} \Phi(z) \tag{659}$$

As a check of compatibility with (658), take a map close to identity, $w(z) = z+\epsilon u(z)$. Then in the first order in $\epsilon$ we have

$$\delta\Phi(z) = \Phi'(z) - \Phi(z) = \epsilon \cdot (\text{r.h.s. of (658)}) \tag{660}$$

In Section 5.6.1 below we will see that (659) will become an equivariance property of correlators of primary fields under the diagonal action of a global conformal map on all field insertion points.

Finite version, interpretation #2: "passive transformations" Instead of moving points on the surface $\Sigma = \mathbb{C}$, we can think about $z \mapsto w(z)$ as a change of local coordinate. We will use $z, w$ as names of local coordinate charts and call $p$ (previously $z$) the point on $\Sigma$. Think of the vector bundle $\mathcal{V}$ of fields over $\Sigma$; it has typical fiber $V$ and its local trivialization at a point $p$ depends on a choice of local coordinate $z$ or $w$ around $p$. Thus there is an isomorphism $V \to V_p$ from the standard fiber to the particular fiber over the point depending on a choice of a local coordinate near $p$. Fix $\Phi \in V$ a field and denote its image in $V_p$ using the chart $z$ by $\Phi_{(z)}(p)$. Then we have

$$\Phi_{(w)}(p) = \left(\left.\frac{\partial w}{\partial z}\right|_p\right)^{-h} \left(\left.\frac{\partial \bar{w}}{\partial \bar{z}}\right|_p\right)^{-\bar{h}} \Phi_{(z)}(p) \tag{661}$$

Thus, the Jacobian on the right hand side is the transition function of the vector bundle.

Put another way, the combination

$$\underline{\Phi}(z) = \Phi(z)(dz)^h(d\bar{z})^{\bar{h}} \tag{662}$$

is a coordinate-independent object valued in the line bundle

$$K^{h,\bar{h}} :\, = K^{\otimes h} \otimes \overline{K}^{\otimes \bar{h}} \tag{663}$$

over $\Sigma$. Here $K = (T^{1,0})^*\Sigma$ is the line bundle of $(1,0)$-forms and $\overline{K} = (T^{0,1})^*\Sigma$ is the line bundle of $(0,1)$-forms. For instance (see below), a correlation function of primary fields is a section of the product of several line bundles (663) pulled back to the configuration space of points on $\Sigma$.

5.5.2. *Examples of primary fields in scalar field theory.* In the scalar field theory, we have the following.

- The field $\partial\phi(z)$ is primary, with $(h,\bar{h}) = (1,0)$. This follows from the OPEs (561), (562). Similarly, $\bar{\partial}\phi(z)$ is $(0,1)$-primary.
- The stress-energy tensor $T$ is a field of conformal weight $(2,0)$, but it is not primary, since $T(w)T(z)$ OPE contains a fourth-order pole (563).
- The field $\partial^2\phi$ has conformal weight $(2,0)$ but is not primary: differentiating (561) we have

$$T(w)\partial^2\phi(z) \sim \frac{2\partial\phi(z)}{(w-z)^3} + \frac{2\partial^2\phi(z)}{(w-z)^2} + \frac{\partial^3\phi(z)}{w-z} + \text{reg.} \tag{664}$$

– contains a third-order pole.

Here is another example.

**Lemma 5.17.** *The vertex operator $V_\alpha =: e^{i\alpha\phi} :$, with $\alpha$ any real number, is primary, of conformal weight $(h,\bar{h}) = (\frac{\alpha^2}{2}, \frac{\alpha^2}{2})$.*

Note that $h, \bar{h}$ are (generally) not integers! (Thus, in particular, we really do need real tensor powers of the line bundles in (663)).

*Proof.* Let us calculate the OPE $T(w)V_\alpha(z)$ in the language of field operators:

$$\begin{aligned}
\mathcal{R}\widehat{T}(w)\widehat{V}_\alpha(z) &= \mathcal{R} : -\frac{1}{2}\partial\widehat{\phi}(w)\partial\widehat{\phi}(w) : \sum_{n\geq 0}\frac{(i\alpha)^n}{n!} : \underbrace{\widehat{\phi}(z)\widehat{\phi}(z)\cdots\widehat{\phi}(z)}_{n} := \\
&\underset{\text{Wick}}{=} : -\frac{1}{2}\partial\widehat{\phi}(w)\partial\widehat{\phi}(w) \sum_{n\geq 0}\frac{(i\alpha)^n}{n!} n(n-1)\widehat{\phi}(z)\widehat{\phi}(z)\underbrace{\widehat{\phi}(z)\cdots\widehat{\phi}(z)}_{n-2} : + \\
&\quad + : -\partial\widehat{\phi}(w)\partial\widehat{\phi}(w) \sum_{n\geq 0}\frac{(i\alpha)^n}{n!} n\widehat{\phi}(z)\underbrace{\widehat{\phi}(z)\cdots\widehat{\phi}(z)}_{n-1} : + \underbrace{: \widehat{T}(w)\widehat{V}_\alpha(z) :}_{\text{reg.}} \\
&\sim -\frac{1}{2}\frac{1}{(w-z)^2}(i\alpha)^2\widehat{V}_\alpha(z) + \frac{1}{w-z} : \partial\widehat{\phi}(w)(i\alpha)e^{i\alpha\widehat{\phi}(z)} : + \text{reg.} \\
&\sim \frac{\frac{\alpha^2}{2}\widehat{V}_\alpha(z)}{(w-z)^2} + \frac{\partial\widehat{V}_\alpha(z)}{w-z} + \text{reg.}
\end{aligned} \tag{665}$$

The OPE $\overline{T}(w)V_\alpha(z)$ is computed similarly. Comparing with (653) we see that $V_\alpha$ is primary (no cubic or higher poles in the OPE with $T$, $\overline{T}$), and the conformal weight is $h = \bar{h} = \frac{\alpha^2}{2}$ as claimed. $\square$

**Exercise 5.18.** (a) Show that the field

$$: 2(\partial\phi)^4 - 3(\partial^2\phi)^2 + 2\partial\phi\,\partial^3\phi : \tag{666}$$

is primary, of conformal weight $(4,0)$. Or, equivalently, check that the corresponding by (629) state $(2\widehat{a}_{-1}^4 + 3\widehat{a}_{-2}^2 - 4\widehat{a}_{-3}\widehat{a}_{-1})|\mathrm{vac}\rangle \in \mathcal{H}$ is annihilated by operators $\widehat{L}_{>0}$ and has eigenvalue 4 w.r.t. $\widehat{L}_0$.

(b) Show that one has

$$(2L_{-3} - 4L_{-2}L_{-1} + L_{-1}^3)\partial\phi = 0, \tag{667}$$

i.e., this particular Virasoro descendant of the primary field $\partial\phi$ in free scalar theory vanishes (in terms of Remark 5.16, this descendant belongs to $\mathsf{N}$ – the quotiented-out submodule).

EDIT

*Remark* 5.19. Classification of all primary fields in scalar field theory is a nontrivial problem; the answer is known as a corollary of a theorem by Feigin-Fuchs [12].

First note that the space of fields (or space of states) of the free scalar theory is

$$V = \bigoplus_{\alpha\in\mathbb{R}} \mathbb{V}_\alpha^{\mathrm{Heis}} \otimes \mathbb{V}_\alpha^{\overline{\mathrm{Heis}}} \tag{668}$$

where $\mathbb{V}_\alpha^{\mathrm{Heis}}$ be the Verma module of the Heisenberg Lie algebra, with highest vector of $\widehat{a}_0$-weight $\alpha$.

Let $M_h$ be the highest weight irreducible Virasoro module with $L_0$-highest weight $h$ and central charge $c = 1$ and let $\mathbb{V}_h^{\mathrm{Vir}}$ be the highest weight Verma module (possibly reducible) of the Virasoro algebra with $L_0$-highest weight $h$ and central charge $c = 1$. One has:

(1) If $\alpha \notin \frac{1}{\sqrt{2}}\mathbb{Z}$ then

$$\mathbb{V}_\alpha^{\mathrm{Heis}} \simeq M_{\frac{\alpha^2}{2}} = \mathbb{V}^{\mathrm{Vir}}_{\frac{\alpha^2}{2}} \tag{669}$$

is a single irreducible representation of Virasoro and contains no singular vectors.

(2) If $\alpha = \pm\frac{N}{\sqrt{2}}$ for some $N = 0, 1, 2, \ldots$, then one has

$$\mathbb{V}_\alpha^{\mathrm{Heis}} \simeq M_{\frac{N^2}{4}} \oplus M_{\frac{(N+2)^2}{4}} \oplus M_{\frac{(N+4)^2}{4}} \oplus \cdots \tag{670}$$

For instance, $\mathbb{V}_0^{\mathrm{Heis}}$ contains an infinite sequence of Virasoro-highest weight (primary) vectors $\chi_0 = \mathbb{1}, \chi_1 = i\partial\phi, \chi_2, \chi_3, \ldots$, with $\chi_n$ having conformal weight $h = n^2$; $\chi_2$ is given explicitly by (666).

In the full scalar field theory, the Verma module $\mathbb{V}_{0,0}^{\mathrm{Heis}\oplus\overline{\mathrm{Heis}}} = \mathbb{V}_0^{\mathrm{Heis}} \otimes \mathbb{V}_0^{\overline{\mathrm{Heis}}}$ of the two copies of Heisenberg algebra contains a two-parameter family of Virasoro-highest weight vectors (primary fields) $\chi_{n,\bar{n}}$ with $n, \bar{n} = 0, 1, 2, \ldots$, with conformal weights $(h = n^2, \bar{h} = \bar{n}^2)$.

(3) A related point to the above is that the Virasoro Verma module $\mathbb{V}_h^{\mathrm{Vir}}$ for $h = \frac{N^2}{4}$ is reducible and contains singular vectors at levels $\frac{(N+2k)^2}{4} - \frac{N^2}{4}$ with $k = 1, 2, \ldots$. Vanishing descendant (667) above gives an example of a singular vector at level 3 in $\mathbb{V}_{h=1}^{\mathrm{Vir}}$; here $N = 2$, $k = 1$. Also, $\mathbb{V}_h^{\mathrm{Vir}}$ fits (depending on parity of $N$) into one of the two sequences of maps between

Virasoro Verma modules

$$\begin{gathered}\mathbb{V}_0^{\mathrm{Vir}} \leftarrow \mathbb{V}_1^{\mathrm{Vir}} \leftarrow \mathbb{V}_{2^2}^{\mathrm{Vir}} \leftarrow \mathbb{V}_{3^2}^{\mathrm{Vir}} \leftarrow \cdots \\ \mathbb{V}_{(\frac{1}{2})^2}^{\mathrm{Vir}} \leftarrow \mathbb{V}_{(\frac{3}{2})^2}^{\mathrm{Vir}} \leftarrow \mathbb{V}_{(\frac{5}{2})^2}^{\mathrm{Vir}} \leftarrow \mathbb{V}_{(\frac{7}{2})^2}^{\mathrm{Vir}} \leftarrow \cdots\end{gathered} \tag{671}$$

For each map here, the image of the highest vector or a singular vector is a singular vector in the target module (and each singular vector arises that way – ultimately comes from the highest vector of one of the modules to the right in the sequence). Also, one has that the irreducible Virasoro module

$$M_{\frac{N^2}{4}} = \mathbb{V}_{\frac{N^2}{4}}^{\mathrm{Vir}} / \mathbb{V}_{\frac{(N+2)^2}{4}}^{\mathrm{Vir}} \tag{672}$$

is the quotient of the Verma module by the submodule generated by the first singular vector (all subsequent singular vectors are already in that submodule).

Lecture 26, 10/28/2022

5.5.3. *More on vertex operators.* Here are some other interesting properties of vertex operators in free scalar theory.

- The 2-point correlator of vertex operators is

$$\langle V_\alpha(w) V_\beta(z)\rangle = \begin{cases} |w-z|^{-2\alpha^2} & \text{if } \beta=-\alpha, \\ 0 & \text{otherwise}\end{cases} \tag{673}$$

More generally, the $n$-point correlator of vertex operators is

$$\langle V_{\alpha_1}(z_1)\cdots V_{\alpha_n}(z_n)\rangle = \begin{cases} \prod_{1\leq j<k\leq n} |z_j-z_k|^{2\alpha_j\alpha_k} & \text{if } \alpha_1+\cdots+\alpha_n=0, \\ 0 & \text{otherwise}\end{cases} \tag{674}$$

- Vertex operators satisfy the OPE

$$V_\alpha(w)V_\beta(z) \sim |w-z|^{2\alpha\beta} V_{\alpha+\beta}(z) + (\text{less singular terms}). \tag{675}$$

- One has the OPE

$$i\partial\phi(w) V_\alpha(z) \sim \frac{\alpha}{w-z} V_\alpha(z) + \text{reg.} \tag{676}$$

All these properties follow from the explicit formula for the vertex operator (630) and Wick's lemma. For instance, let us prove (673). We apply Wick's lemma to the product of two vertex operators (as operators on $\mathcal{H}$: For simplicity, assume $|w|>|z|$. We have

$$\begin{aligned}\widehat{V}_\alpha(w)\widehat{V}_\beta(z) &= \sum_{n,m\geq 0} \frac{1}{n!m!}(i\alpha)^n (i\beta)^m : \widehat\phi(w)^n : : \widehat\phi(z)^m : = \\ &\underset{\mathrm{Wick}}{=} \sum_{k\geq 0}\sum_{n,m\geq k} \frac{1}{n!m!} \underbrace{\binom{n}{k}\binom{m}{k} k!}_{\#(k\text{-fold Wick contractions})} (i\alpha)^n (i\beta)^m (-2\log|w-z|)^k : \widehat\phi(w)^{n-k}\widehat\phi(z)^{m-k} : \\ &= \sum_{k\geq 0}\sum_{n,m\geq k} \frac{1}{(n-k)!(m-k!)k!}(i\alpha)^n(i\beta)^m(-2\log|w-z|)^k : \widehat\phi(w)^{n-k}\widehat\phi(z)^{m-k} : \\ &\underset{n'=n-k,\, m'=m-k}{=} \sum_{k\geq 0} \frac{(2\alpha\beta)^k}{k!}(\log|w-z|)^k \sum_{n',m'\geq 0} \frac{(i\alpha)^{n'}(i\beta)^{m'}}{n'!m'!} : \widehat\phi(w)^{n'}\widehat\phi(z)^{m'} :\end{aligned} \tag{677}$$

$$= e^{2\alpha\beta \log|w-z|} : e^{i\alpha\widehat{\phi}(w)} e^{i\beta\widehat{\phi}(z)} := \boxed{|w-z|^{2\alpha\beta} : \widehat{V}_\alpha(w)\widehat{V}_\beta(z) :}$$

The normally ordered product of vertex operators on the right can be written as $e^{i(\alpha+\beta)\widehat{\phi}_0}(1+\cdots)$ where $\cdots$ are normally ordered terms with zero VEV (vacuum expectation value). The operator $e^{i(\alpha+\beta)\widehat{\phi}_0}$ shift the vacuum $|\mathrm{vac}\rangle$ to a pseudovacuum $|\pi_0 = \alpha+\beta\rangle$, so it has expectation value zero unless $\alpha+\beta=0$, and in the latter case the VEV is 1. Thus,

$$\langle \mathrm{vac}|\widehat{V}_\alpha(w)\widehat{V}_\beta(z)|\mathrm{vac}\rangle = |w-z|^{2\alpha\beta} \cdot \begin{cases} 1 & \text{if } \alpha+\beta=0, \\ 0 & \text{otherwise} \end{cases} \tag{678}$$

This finishes the proof of (673).

Note that the computation (677) also implies the OPE (675):

$$\begin{aligned} \mathcal{R}\widehat{V}_\alpha(w)\widehat{V}_\beta(z) &= |w-z|^{2\alpha\beta} : \underbrace{\widehat{V}_\alpha(w)}_{\text{expand around } z} \widehat{V}_\beta(z) := \\ &= |w-z|^{2\alpha\beta} \sum_{k,l\geq 0} \frac{(w-z)^k(\bar{w}-\bar{z})^l}{k!l!} : \partial^k\bar{\partial}^l \widehat{V}_\alpha(z)\, \widehat{V}_\beta(z) : \\ &= |w-z|^{2\alpha\beta}\widehat{V}_{\alpha+\beta}(z) + O(|w-z|^{2\alpha\beta+1}), \end{aligned} \tag{679}$$

where we used the property $: \widehat{V}_\alpha(z)\widehat{V}_\alpha(z) := \widehat{V}_{\alpha+\beta}(z)$, obvious from the definition of the vertex operator (630).

The correlator (674) is also obtained from Wick's lemma, see [8, section 9.1.1].

The OPE (676) is obtained by a computation similar to (665) (actually simpler, as there are only single Wick contractions).

### 5.6. **Conformal Ward identity (via contour integration trick).** 

In any CFT on $\mathbb{C}$ one the following.

**Theorem 5.20** (Conformal Ward identity)**.** *Fix a collection of fields $\Phi_1,\ldots,\Phi_n \in V$, a collection of distinct points $z_1,\ldots,z_n \in \mathbb{C}$, a conformal vector field $v = u(w)\partial_w + \overline{u(w)}\partial_{\bar{w}}$ with $u(w)\partial_w$ a meromophic vector field on $\mathbb{CP}^1$ with poles allowed only at the points $z_1,\ldots,z_k$ (in particular we are assuming that $w=\infty$ is a regular point of $u\partial$). Then one has*

$$\sum_{k=1}^n \langle \Phi_1(z_1)\cdots\rho^{(z_k)}(v)\circ\Phi_k(z_k)\cdots\Phi_n(z_n)\rangle = 0 \tag{680}$$

*where $\rho^{(z_k)}(v)\circ\Phi_k(z_k)$ is the action of the vector field $v$ on the field $\Phi_k$ defined via (637).*

We denote the l.h.s. of (680) by $\delta_v\langle\Phi_1(z_1)\cdots\Phi_n(z_n)\rangle$ – the action of the vector fields on the correlator (via acting on individual fields). Thus, the Ward identity says that the action of a conformal vector field on a correlator vanishes.

Note that (by complexification) we can treat $u(w)\partial_w$ and $\overline{u(w)}\partial_{\bar{w}}$ in (680) as independent meromorphic and antimeromorphic vector fields (not complex conjugate to one another).

*Proof.* Consider the action of a meromorphic vector field $u(w)\partial_w$ on a correlator. Let $\Gamma = C_{0,R}$ be a circle centered at the origin of a large radius $R$ (in particular, large enough that it encloses all $z_i$'s). Then we have

$$(681)\qquad -\frac{1}{2\pi i}\oint_\Gamma dw\, u(w)\Big\langle T(w)\Phi_1(z_1)\cdots\Phi_n(z_n)\Big\rangle = \\ \underset{\text{deformation of contour}}{=} \sum_{k=1}^n -\frac{1}{2\pi i}\oint_{\gamma_k} dw\, u(w)\Big\langle T(w)\Phi_1(z_1)\cdots\Phi_n(z_n)\Big\rangle \\ \underset{(637)}{=} \sum_{k=1}^n \langle \Phi_1(z_1)\cdots \rho^{(z_k)}(u\partial)\circ\Phi_k(z_k)\cdots\Phi_n(z_n)\rangle = \delta_{u\partial}\langle\Phi_1(z_1)\cdots\Phi_n(z_n)\rangle,$$

Here $\gamma_k = C_{z_k,r_k}$ is a circle around $z_k$ of radius $r_k$ small enough that $\gamma_k$ does not enclose any $z_i$ with $i\neq k$. We used the fact that the correlator with $T(w)$ is meromorphic in $w$, with possible poles at $w=z_1,\ldots,z_n$, to deform the integration contour $\Gamma$ to $\gamma_1\cup\cdots\cup\gamma_n$.

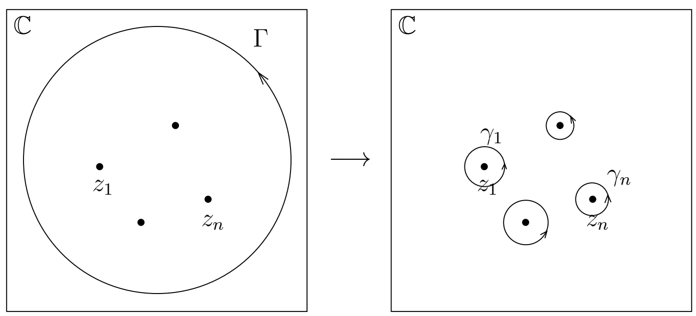



FIGURE 29. Deformation of the integration contour $\Gamma$ (large circle) into a collection of small circles $\gamma_1,\ldots,\gamma_n$ around punctures $z_1,\ldots,z_n$.

It remains to show that the l.h.s. of (681) vanishes. For that, let us use Lemma 5.6:

$$(682)\qquad \overline{-\frac{1}{2\pi i}\oint_\Gamma dw\, u(w)\Big\langle T(w)\Phi_1(z_1)\cdots\Phi_n(z_n)\Big\rangle} = \\ = \frac{1}{2\pi i}\oint_{\Gamma\ni w} d\bar{w}\overline{u(w)}\bar{w}^{-4}\langle T(1/\bar{w})\Phi_1^*(1/\bar{z}_1)\cdots\Phi_n^*(1/\bar{z}_n)\rangle\cdot\prod_{i=1}^n \bar{z}_i^{-2h_i} z_i^{-2\bar{h}_i} \\ \underset{y=1/\bar{w}}{=} -\frac{1}{2\pi i}\oint_{\Gamma'\ni y} dy\, u_y(y)\langle T(y)\Phi_1^*(1/\bar{z}_1)\cdots\Phi_n^*(1/\bar{z}_n)\rangle\cdot\prod_{i=1}^n \bar{z}_i^{-2h_i} z_i^{-2\bar{h}_i}$$

where $u_y(y)=\overline{u(w)}/\bar{w}^2$ is regular at $y=0$, since the vector field $u(w)\partial_w$ was required to be regular at $w=\infty$; $\Gamma'$ is a circle around zero of small radius $1/R$. The integrand in the r.h.s. of (682) is a meromorphic function in $y$ and $\Gamma'$ does not enclose any poles (in particular $y=0$ is a regular point), hence (682) vanishes. This proves that the r.h.s. of (681) is zero.

The case of the action of an antimeromorphic vector field on a correlator is similar. □

Informally, the argument is: take the integral in the l.h.s. (681) over a contour around $w=\infty$ in $\mathbb{CP}^1$. One the one hand the integral vanishes, since integrand is holomorphic around $w = \infty$. On the other hand, the contour can be deformed into a union of small circles around field insertions $z_i$, which yields $\delta_v$ of the correlator.

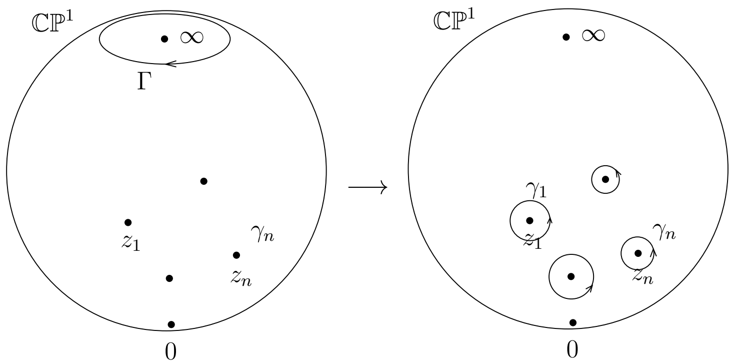



FIGURE 30. Deformation of the integration contour on $\mathbb{CP}^1$.

**Example 5.21.** Let $u(w)\partial_w = \frac{-\partial_w}{w-z_0}$ – a meromorphic vector field with a simple pole at $z_0$. Assume that $\Phi_1, \ldots, \Phi_n$ are *primary* fields with conformal weights $(h_i, \bar{h}_i)$. Applying (680) to the correlator $\langle \mathbb{1}(z_0)\Phi_1(z_1)\cdots\Phi_n(z_n)\rangle$,[101] we obtain

(683)
$$0 = \langle \rho^{z_0}\left(\frac{-\partial_w}{w-z_0}\right)\circ \mathbb{1}(z_0)\ \Phi_1(z_1)\cdots\Phi_n(z_n)\rangle +$$

$$+\sum_{k=1}^{n} \langle \cancel{\mathbb{1}(z_0)}\Phi_1(z_1)\cdots \rho^{z_k}\Big(\underbrace{\frac{-\partial_w}{w-z_0}}_{\text{expand at } z_k}\Big)\circ\Phi_k(z_k)\cdots\Phi_n(z_n)\rangle$$

$$\underset{(638)}{=} \langle (L_{-2}\mathbb{1})(z_0)\Phi_1(z_1)\cdots\Phi_n(z_n)\rangle+$$

$$+\sum_{k=1}^{n}\langle\Phi_1(z_1)\cdots\rho\left(-\frac{1}{z_k-z_0}\partial_w + \frac{w-z_k}{(z_k-z_0)^2}\partial_w - \frac{(w-z_k)^2}{(z_k-z_0)^3}\partial_w+\cdots\right)\circ\Phi_k(z_k)\cdots\Phi_n(z_n)\rangle$$

$$= \langle\underbrace{(L_{-2}\mathbb{1})(z_0)}_{T(z_0)}\Phi_1(z_1)\cdots\Phi_n(z_n)\rangle+$$

$$+\sum_{k=1}^{n}\langle\Phi_1(z_1)\cdots\left(\frac{1}{z_k-z_0}L_{-1}-\frac{1}{(z_k-z_0)^2}L_0+\underbrace{\cancel{\frac{1}{(z_k-z_0)^3}L_1-\cdots}}_{\text{since }\Phi_k\text{ is primary}}\right)\circ\Phi_k(z_k)\cdots\Phi_n(z_n)\rangle$$

$$=\langle T(z_0)\Phi_1(z_1)\cdots\Phi_n(z_n)\rangle+\sum_{k=1}^{n}\langle\Phi_1(z_1)\cdots\left(\frac{1}{z_k-z_0}\frac{\partial}{\partial z_k}-\frac{h_k}{(z_k-z_0)^2}\right)\Phi_k(z_k)\cdots\Phi_n(z_n)\rangle.$$

Or, written another way:
(684)
$$\langle T(z_0)\Phi_1(z_1)\cdots\Phi_n(z_n)\rangle = \left(\sum_{k=1}^{n}\frac{h_k}{(z_k-z_0)^2}-\frac{1}{z_k-z_0}\frac{\partial}{\partial z_k}\right)\circ\langle\Phi_1(z_1)\cdots\Phi_n(z_n)\rangle.$$

[101]We inserted $\mathbb{1}(z_0)$, which does not affect the correlator, since we required that the vector field only has poles at the points where fields are inserted.

Thus, the correlator of the stress-energy with a collection of primary fields is expressed as a certain differential operator acting on the correlator of just the primary fields.

**Example 5.22.** If the correlator of primary fields $\Phi_1,\ldots,\Phi_n$ is known then any correlator of their descendants can be recovered as a certain differential operator acting on $\langle\Phi_1\cdots\Phi_n\rangle$. Such an expression is obtained from Ward identity by repeatedly applying meromorphic vector fields of the form $-(w-z_k)^{-r+1}\partial_w$ to the correlator of the primary fields.

For instance applying the vector field $u\partial=-(w-z_1)^{-r+1}\partial_w$ (for some $r\geq 1$) to $\langle\Phi_1(z_1)\cdots\Phi_n(z_n)\rangle$ we find

$$\begin{aligned}0=\delta_{u\partial}\langle\Phi_1(z_1)\cdots\Phi_n(z_n)\rangle &= \\ &=\langle(L_{-r}\Phi_1)(z_1)\,\Phi_2(z_2)\cdots\Phi_n(z_n)\rangle+ \\ +\underbrace{\left(\sum_{k=2}^n(z_k-z_1)^{-r+1}\partial_{z_k}-(r-1)(z_k-z_1)^{-r}h_k\right)}_{-\mathcal{D}}&\circ\langle\Phi_1(z_1)\Phi_2(z_2)\cdots\Phi_n(z_n)\rangle.\end{aligned}\tag{685}$$

Thus, one has

$$\langle(L_{-r}\Phi_1)(z_1)\,\Phi_2(z_2)\cdots\Phi_n(z_n)\rangle=\mathcal{D}\langle\Phi_1(z_1)\Phi_2(z_2)\cdots\Phi_n(z_n)\rangle\tag{686}$$

with $\mathcal{D}$ the differential operator appearing in (685). Here we were assuming that $\Phi_1,\ldots,\Phi_n$ are primary.

5.6.1. *Constraints on correlators from global conformal symmetry.* Let us explore the consequences of the Ward identity (680) with $v$ a conformal vector field on $\mathbb{CP}^1$ without singularities.

For $\Phi_1,\ldots,\Phi_n\in V$ primary and $v=u\partial+\bar{u}\bar{\partial}$ a conformal vector field without singularities, the Ward identity reads

$$\begin{aligned}&0=\delta_v\langle\Phi_1(z_1)\cdots\Phi_n(z_n)\rangle= \\ &=\sum_{k=1}^n\langle\Phi_1(z_1)\cdots\Big(-u(z_k)\partial_{z_k}-\overline{u(z_k)}\partial_{\bar{z}_k}-h_k\partial u(z_k)-\bar{h}_k\overline{\partial u(z_k)}\Big)\Phi_k(z_k)\cdots\Phi_n(z_n)\rangle\end{aligned}\tag{687}$$

The "finite" (or "integrated") version is then as follows: for $z\mapsto w(z)$ a holomorphic map $\mathbb{CP}^1\to\mathbb{CP}^1$ (i.e., a Möbius transformation) one has

$$\langle\Phi_1(w(z_1))\cdots\Phi_n(w(z_n))\rangle=\prod_{i=1}^n\left(\frac{\partial w}{\partial z}(z_i)\right)^{-h_i}\left(\frac{\partial\bar{w}}{\partial\bar{z}}(z_i)\right)^{-\bar{h}_i}\cdot\langle\Phi_1(z_1)\cdots\Phi_n(z_n)\rangle\tag{688}$$

Put another way, one has an equality

$$\begin{aligned}&\langle\Phi_1(z_1)(dz_1)^{h_1}(d\bar{z}_1)^{\bar{h}_1}\cdots\Phi_n(z_n)(dz_n)^{h_n}(d\bar{z}_n)^{\bar{h}_n}\rangle= \\ &\qquad=\langle\Phi_1(w_1)(dw_1)^{h_1}(d\bar{w}_1)^{\bar{h}_1}\cdots\Phi_n(w_n)(dw_n)^{h_n}(d\bar{w}_n)^{\bar{h}_n}\rangle\end{aligned}\tag{689}$$

Using the notation (662),

$$\underline{\Phi}(z):=\Phi(z)(dz)^h(d\bar{z})^{\bar{h}}\quad\in V\otimes K_z^{h,\bar{h}},\tag{690}$$

the $n$-point correlator of primary fields is a section of a certain line bundle on the open configuration space of points on $\mathbb{CP}^1$, invariant under the Möbius group (the latter being the statement of the Ward identity):

$$\langle \underline{\Phi}_1(z_1)\cdots\underline{\Phi}_n(z_n)\rangle \in \Gamma\left(C_n(\mathbb{CP}^1), \bigotimes_{i=1}^n \pi_i^* K^{h_i,\bar{h}_i}\right)^{PSL_2(\mathbb{C})} \tag{691}$$

where $\pi_i\colon C_n(\mathbb{CP}^1)\to\mathbb{CP}^1$ is the map selecting the $i$-th point of the $n$-tuple; $K^{h_i,\bar{h}_i}$ is the line bundle (663) on $\mathbb{CP}^1$.

*Remark* 5.23. If the vector field $v$ is at most linear in coordinates, then (687) holds without assuming that fields $\Phi_1,\ldots,\Phi_n$ are primary. At the level of "finite" conformal maps, the identity (688) holds for $z\mapsto w(z)$ translations, rotations and dilations, without assuming that the fields are primary.

**Lemma 5.24.** *If the OPE of fields $\Phi_1,\Phi_2\in V$ contains the term*

$$\frac{C}{(w-z)^{\alpha}(\bar{w}-\bar{z})^{\bar{\alpha}}}\Phi(z) \tag{692}$$

*with some field $\Phi\in V$ and $C$ a constant, then the exponents in (692) satisfy*

$$h(\Phi_1)+h(\Phi_2)=\alpha+h(\Phi),\quad \bar{h}(\Phi_1)+\bar{h}(\Phi_2)=\bar{\alpha}+\bar{h}(\Phi), \tag{693}$$

*where $h,\bar{h}$ are the conformal weights of the fields involved.*

*Proof.* This is a consequence of (688) and Remark 5.23: one considers the correlator $\langle\Phi_1(w)\Phi_2(z)\Phi_3(z_3)\cdots\Phi_n(z_n)\rangle$, with $\Phi_3,\ldots,\Phi_n\in V$ arbitrary test fields, and acts on it with rotation and dilation around $z$. For simplicity, set $z=0$ and consider the map $z\mapsto\lambda z$ with $\lambda\in\mathbb{C}^*$. Then we have, in the asymptotics $w\to 0$,

(694)
$$\begin{tikzcd}
\langle\Phi_1(\lambda w)\Phi_2(0)\Phi_3(\lambda z_3)\cdots\Phi_n(\lambda z_n)\rangle \arrow[r, equal] \arrow[d, equal, "\mathrm{OPE}"'] & \prod_{i=1}^n \lambda^{-h_i}\bar{\lambda}^{-\bar{h}_i}\langle\Phi_1(w)\Phi_2(0)\Phi_3(z_3)\cdots\Phi_n(z_n)\rangle \arrow[d, equal, "\mathrm{OPE}"'] \\
\frac{C}{\lambda^{\alpha}\bar{\lambda}^{\bar{\alpha}}w^{\alpha}\bar{w}^{\bar{\alpha}}}\langle\Phi(0)\Phi_3(\lambda z_3)\cdots\Phi_n(\lambda z_n)\rangle+\cdots \arrow[d, equal, "(688)"'] & \frac{C}{w^{\alpha}\bar{w}^{\bar{\alpha}}}\prod_{i=1}^n \lambda^{-h_i}\bar{\lambda}^{-\bar{h}_i}\langle\Phi(0)\Phi_3(z_3)\cdots\Phi_n(z_n)\rangle+\cdots \arrow[d, equal] \\
\frac{C\lambda^{-h(\Phi)}\bar{\lambda}^{-\bar{h}(\Phi)}}{\lambda^{\alpha}\bar{\lambda}^{\bar{\alpha}}w^{\alpha}\bar{w}^{\bar{\alpha}}}\prod_{i=3}^n \lambda^{-h_i}\bar{\lambda}^{-\bar{h}_i}\langle\Phi(0)\Phi_3(z_3)\cdots\Phi_n(z_n)\rangle+\cdots & \frac{C}{w^{\alpha}\bar{w}^{\bar{\alpha}}}\prod_{i=1}^n \lambda^{-h_i}\bar{\lambda}^{-\bar{h}_i}\langle\Phi(0)\Phi_3(z_3)\cdots\Phi_n(z_n)\rangle+\cdots
\end{tikzcd}$$

Here $\cdots$ stands for the other terms in the OPE. Equality in the last row implies the claimed relation on the OPE exponents (693). □

<u>One-point correlators.</u>

**Lemma 5.25.** *Let $\Phi\in V$ be a field (not necessarily primary) of conformal weight $(h,\bar{h})$. Then*

$$\langle\Phi(z)\rangle=\left\{\begin{array}{ll} C_\Phi & \text{if } h=\bar{h}=0,\\ 0 & \text{otherwise}\end{array}\right. \tag{695}$$

*where $C_\Phi$ is a constant function. (the value of the constant depends on $\Phi$).*

*Proof.* Using the Ward identity with $v$ a constant vector field $a\partial_w+\bar{a}\partial_{\bar{w}}$ (with arbitrary coefficients $a,\bar{a}\in\mathbb{C}$), we find that the one-point correlator satisfies $(a\partial_z+$

$\bar{a}\partial_{\bar{z}})\langle\Phi(z)\rangle = 0$, i.e., the correlator is a constant function. Applying the vector field $v = b(w-z)\partial_w + \bar{b}(\bar{w}-\bar{z})\partial_{\bar{w}}$ to the correlator, we see that it satisfies

$$(bh + \bar{b}\bar{h})\langle\Phi(z)\rangle = 0 \tag{696}$$

for any $b, \bar{b} \in \mathbb{C}$. Thus, the one-point correlator must vanish unless $h = \bar{h} = 0$. □

<u>Two-point correlators.</u>

**Lemma 5.26.** *Let $\Phi_1, \Phi_2 \in V$ be two fields of conformal weights $(h_i, \bar{h}_i)$, $i = 1, 2$.*

*(a) One has*

$$\langle\Phi_1(z_1)\Phi_2(z_2)\rangle = C_{\Phi_1\Phi_2}\frac{1}{(z_1-z_2)^{h_1+h_2}(\bar{z}_1-\bar{z}_2)^{\bar{h}_1+\bar{h}_2}} \tag{697}$$

*with $C_{\Phi_1\Phi_2}$ some constant depending on $\Phi_1, \Phi_2$.*

*(b) If $\Phi_1, \Phi_2$ are primary, then the constant $C_{\Phi_1,\Phi_2}$ in (697) vanishes unless one has*

$$h_1 = h_2,\ \ \bar{h}_1 = \bar{h}_2. \tag{698}$$

*(c) For $\Phi_1, \Phi_2$ two fields satisfying condition (698) on conformal weights, the constant $C_{\Phi_1\Phi_2}$ in (697) is related to the hermitian inner product on $V$ (cf. Axiom (IV)) by*

$$C_{\Phi_1\Phi_2} = \langle\Phi_1^*, \Phi_2\rangle_V. \tag{699}$$

*Proof.* Part (a) follows from (688) for translations, rotations and dilations (we exploit Remark 5.23).

For (b), let us fix the two points at $z_1 = 0$ and $z_2 = 1$ and act on the correlator with the vector field $u\partial_w = w(1-w)\partial_w$ – a holomorphic vector field on the entire $\mathbb{CP}^1$. The Ward identity (687) in this case reads

$$0 = \langle -h_1\Phi_1(z_1)\Phi_2(z_2)\rangle + \langle\Phi_1(z_1)h_2\Phi_2(z_2)\rangle = (h_2-h_1)\langle\Phi_1(z_1)\Phi_2(z_2)\rangle. \tag{700}$$

Thus unless $h_1 = h_2$, the 2-point correlator vanishes. Likewise, acting with the vector field $\bar{w}(1-\bar{w})\partial_{\bar{w}}$, we find that unless $\bar{h}_1 = \bar{h}_2$, the correlator also has to vanish.

For (c), we calculate the r.h.s. of (699) exploiting the state-field correspondence:

$$\begin{aligned}\langle\Phi_1^*, \Phi_2\rangle_V &= \lim_{w,z\to 0}\left\langle\widehat{\Phi}_1^*(w)|\mathrm{vac}\rangle, \widehat{\Phi}_2(z)|\mathrm{vac}\rangle\right\rangle_{\mathcal{H}} = \lim_{w,z\to 0}\langle\mathrm{vac}|\widehat{\Phi}_1^*(w)^+\widehat{\Phi}_2(z)|\mathrm{vac}\rangle \\ &\underset{(597)}{=} \lim_{w,z\to 0}\bar{w}^{-2h_1}w^{-2\bar{h}_1}\langle\mathrm{vac}|\widehat{\Phi}_1(1/\bar{w})\widehat{\Phi}_2(z)|\mathrm{vac}\rangle \\ &\underset{(697)}{=} C_{\Phi_1\Phi_2}\lim_{w,z\to 0}\bar{w}^{-2h_1}w^{-2\bar{h}_1}\left(1/\bar{w}-z\right)^{-h_1-h_2}\left(1/w-\bar{z}\right)^{-\bar{h}_1-\bar{h}_2} \\ &= C_{\Phi_1\Phi_2}.\end{aligned} \tag{701}$$

Here in the last step we used the condition (698). □

**Example 5.27.** In scalar field theory, the correlators

$$\langle\partial\phi(w)\partial\phi(z)\rangle = -\frac{1}{(w-z)^2}, \qquad \langle V_\alpha(w)V_\beta(z)\rangle = \begin{cases}\frac{1}{|w-z|^{2\alpha^2}}, & \alpha = \beta \\ 0, & \alpha \neq \beta\end{cases} \tag{702}$$

(cf. (536), (673)) are examples of two-point correlators of primary fields (of weight $(1,0)$ in the first case and of weight $(\frac{\alpha^2}{2},\frac{\alpha^2}{2})$ in the second case). They are clearly consistent with the general ansatz (697).

**Example 5.28.** The $TT$ OPE (601) and the ansatz (697) imply that the two-point correlator of the stress-energy tensor is

$$\langle T(w)T(z)\rangle = \frac{c/2}{(w-z)^4}. \tag{703}$$

With (699) this implies

$$\langle T,T\rangle_V = \frac{c}{2}. \tag{704}$$

Since the inner product on $V$ is assumed to be positive definite, this means that the central charge $c$ must be a positive number.[102]

Three-point correlators of primary fields. Lecture 27, 10/31/2022

**Lemma 5.29.** *For any three primary fields $\Phi_1,\Phi_2,\Phi_3\in V$, with $\Phi_i$ of conformal weights $(h_i,\bar{h}_i)$, one has*

$$\langle\Phi_1(z_1)\Phi_2(z_2)\Phi_3(z_3)\rangle = C_{\Phi_1\Phi_2\Phi_3}\prod_{1\leq i<j\leq 3}\frac{1}{(z_i-z_j)^{2\alpha_{ij}}(\bar{z}_i-\bar{z}_j)^{2\bar{\alpha}_{ij}}}, \tag{705}$$

*where $C_{\Phi_1\Phi_2\Phi_3}$ is a constant (depending on the fields but not on the points $z_1,z_2,z_3$) and the exponents are expressed in terms of conformal weights of the fields:*

$$\begin{aligned} \alpha_{12}&=\frac{1}{2}(h_1+h_2-h_3),\ \alpha_{13}=\frac{1}{2}(h_1+h_3-h_2),\ \alpha_{23}=\frac{1}{2}(h_2+h_3-h_1),\\ \bar{\alpha}_{12}&=\frac{1}{2}(\bar{h}_1+\bar{h}_2-\bar{h}_3),\ \bar{\alpha}_{13}=\frac{1}{2}(\bar{h}_1+\bar{h}_3-\bar{h}_2),\ \bar{\alpha}_{23}=\frac{1}{2}(\bar{h}_2+\bar{h}_3-\bar{h}_1). \end{aligned} \tag{706}$$

*Proof #1 (idea).* Take the unique Möbius transformation $f\colon\mathbb{CP}^1\to\mathbb{CP}^1$ that maps points $z_1,z_2,z_3$ to $0,1,2$. Then the Ward identity (688) allows one to write the 3-point correlator as

$$\langle\Phi_1(z_1)\Phi_2(z_2)\Phi_3(z_3)\rangle = \prod_{i=1}^{3}(\partial f(z_i))^{h_i}(\overline{\partial f(z_i)})^{\bar{h}_i}\cdot\underbrace{\langle\Phi_1(0)\Phi_1(1)\Phi_3(2)\rangle}_{\widetilde{C}} \tag{707}$$

with $\widetilde{C}$ some constant. Computing explicitly the derivatives in the r.h.s., one obtains (705). □

Let us introduce the notation

$$\mu=\frac{dz_1\wedge dz_2}{(z_1-z_2)^2}\qquad\in\Gamma(C_2(\mathbb{CP}^1),\pi_1^*K\otimes\pi_2^*K)\subset\Omega^2(C_2(\mathbb{CP}^1)). \tag{708}$$

with $\pi_i$ as in (691). We will call $\mu$ the *Szegö kernel.*[103]

**Lemma 5.30.** *The Szegö kernel defined by (708) is the unique (up to normalization) nowhere vanishing* Möbius-invariant *holomorphic* 2*-form on the configuration space of two points on* $\mathbb{CP}^1$.

[102]Positivity of the inner product on $V$ is a part of the unitarity assumption for a CFT. There are interesting examples (e.g. the so-called ghost system or $bc$ system) where it fails, and the central charge can be negative. For instance, in the $bc$ system one has $c=-26$.

[103]In the standard terminology, it is the square root of $\mu$ that is called the Szegö kernel.

*Proof.* To check that $\mu$ is Möbius-invariant, we observe that it is invariant under (a) translations $z \mapsto z + a$, (b) rotation and dilation $z \mapsto \lambda z$ (since $\mu$ is homogeneous of degree zero), (c) the map $i\colon z \mapsto 1/z$ (indeed, $i^*\mu = \frac{\frac{-dz_1}{z_1^2}\frac{-dz_2}{z_2^2}}{(\frac{1}{z_1}-\frac{1}{z_2})^2} = \mu$). These transformation generate all Möbius transformations, thus $\mu$ is Möbius-invariant. The fact that $\mu$ is nowhere vanishing is obvious if $z_1, z_2 \neq \infty$. For $z_1 = \infty$ we switch for the point $z_1$ to the coordinate chart $w_1 = 1/z_1$ near the point $\infty \in \mathbb{CP}^1$. We have then $\mu = -\frac{dw_1 \wedge dz_2}{(1-w_1 z_2)^2}$ – it is nonvanishing at $w_1 = 0$. The case $z_2 = \infty$ is similar.

If $\nu$ is some other Möbius-invariant section of the line bundle $\pi_1^* K \otimes \pi_2^* K$ over $C_2(\mathbb{CP}^1)$, we must have $\nu = f\mu$ for some Möbius-invariant function $f$ on $C_2(CP^1)$. Such a function has to be constant, since any two points on $\mathbb{CP}^1$ can be moved to $0, 1$ by a Möbius transformation (and thus $f(z_1, z_2) = f(0, 1)$ for any $z_1 \neq z_2 \in \mathbb{CP}^1$). This proves uniqueness of $\mu$ up to a multiplicative constant. $\square$

In terms of the Szegö kernel, the three-point function of primary fields (705) admits an equivalent expression:

$$\langle \underline{\Phi}_1(z_1)\underline{\Phi}_2(z_2)\underline{\Phi}_3(z_3)\rangle = C_{\Phi_1\Phi_2\Phi_3} \prod_{1\leq i<j\leq 3} (\pi_{ij}^*\mu)^{\alpha_{ij}} (\pi_{ij}^*\bar{\mu})^{\bar{\alpha}_{ij}} \tag{709}$$

where $\pi_{ij}\colon C_3(\mathbb{CP}^1) \to C_2(\mathbb{CP}^1)$ maps $(z_1, z_2, z_3) \mapsto (z_i, z_j)$ and we used the notation (690). The exponents (706) are chosen in such a way that the r.h.s. of (709) is the section of the same line bundle over $C_3(\mathbb{CP}^1)$ as the l.h.s., i.e., so that the power of $dz_i$ is the same on both sides for $i = 1, 2, 3$:

$$h_1 = \alpha_{12} + \alpha_{13},\ h_2 = \alpha_{12} + \alpha_{23},\ h_3 = \alpha_{13} + \alpha_{23}, \tag{710}$$

and similarly for powers of $d\bar{z}_i$.

*Proof #2 of Lemma 5.29.* Denote the r.h.s. of (709) without the factor $C_{\Phi_1\Phi_2\Phi_3}$ by $A$. The l.h.s. of (709) and $A$ both are sections of the line bundle $\bigotimes_{i=1}^3 \pi_i^* K^{h_i, \bar{h}_i}$ over $C_3(\mathbb{CP}^1)$. Moreover, both are Möbius invariant (the ll.h.s by Ward identity and $A$ by Möbius-invariance of Szegö kernel) and $A$ is nonvanishing. Therefore, one has

$$(\text{l.h.s. of (709)}) = f \cdot A \tag{711}$$

where $f$ is a Möbius-invariant function on $C_3(\mathbb{CP}^1)$. Since Möbius group acts 3-transitively on $\mathbb{CP}^1$, such a function has to be constant. $\square$

Correlators of $n \geq 4$ primary fields

**Lemma 5.31.** *For $\Phi_1, \ldots, \Phi_n \in V$ a collection of $n \geq 4$ primary fields, with $\Phi_i$ of conformal dimension $(h_i, \bar{h}_i)$, one has*

$$\langle \underline{\Phi}_1(z_1)\cdots\underline{\Phi}_n(z_n)\rangle = \prod_{1\leq i<j\leq n} (\pi_{ij}^*\mu)^{\alpha_{ij}} (\pi_{ij}^*\bar{\mu})^{\bar{\alpha}_{ij}} \cdot \mathcal{F}_{\Phi_1\cdots\Phi_n}(\lambda_1, \ldots, \lambda_{n-3}), \tag{712}$$

*where $\mu$ is the Szegö kernel (708), $\lambda_i = [z_1, z_2 : z_3, z_{i+3}]$ for $i = 1, \ldots, n-3$ are cross-ratios, the exponents $\alpha_{ij}$, $\bar{\alpha}_{ij}$ are*

$$\alpha_{ij} = \frac{1}{n-2}(h_i + h_j - \frac{1}{n-1}\sum_{k=1}^n h_k), \quad \bar{\alpha}_{ij} = \frac{1}{n-2}(\bar{h}_i + \bar{h}_j - \frac{1}{n-1}\sum_{k=1}^n \bar{h}_k) \tag{713}$$

*and $\mathcal{F}_{\Phi_1\cdots\Phi_n}$ is some smooth function on $C_{n-3}(\mathbb{CP}^1\backslash\{0,1,\infty\})$ (it cannot be determined from the global conformal symmetry).*

Put another way, the result is that any Möbius-invariant section of the line bundle in the r.h.s. of (691) is built out of two types of "building blocks" – cross-ratios and Szegö kernels.

*Proof.* The proof is similar to the proof #2 of Lemma 5.29 above: the l.h.s. of (712) and $B\colon=\prod_{1\leq i<j\leq n}(\pi_{ij}^*\mu)^{\alpha_{ij}}(\pi_{ij}^*\bar{\mu})^{\bar{\alpha}_{ij}}$ are both Möbius-invariant sections of the line bundle[104] $\bigotimes_{i=1}^n \pi_i^* K^{h_i,\bar{h}_i}$ over $C_n(\mathbb{CP}^1)$ and $B$ is nonvanishing, therefore one has

$$(\text{l.h.s. of } (712)) = g\cdot B \tag{714}$$

with $g$ a Möbius-invariant function on $C_n(\mathbb{CP}^1)$. Choosing a Möbius transformation that maps $(z_1,\ldots,z_n)$ to $(1,0,\infty,\lambda_1,\ldots,\lambda_{n-3})$, we obtain

$$g(z_1,\ldots,z_n)=g(1,0,\infty,\lambda_1,\ldots,\lambda_{n-3})=:\mathcal{F}(\lambda_1,\ldots,\lambda_{n-3}). \tag{715}$$

□

## 5.7. **Holomorphic fields, mode operators.**

### 5.7.1. *Holomorphic fields.*

**Definition 5.32.** We call a (not necessarily primary) field $\Phi\in V$ "holomorphic" if it satisfies $\bar{\partial}\Phi=0$. Then in particular, correlation functions of the form $\langle\Phi(z)\Phi_1(z_1)\cdots\Phi_n(z_n)\rangle$ are holomorphic in $z$ (for $z$ away ). Similarly, we call $\Phi\in V$ "antiholomorphic" if it satisfies $\partial\Phi=0$.

**Lemma 5.33.** *If a field $\Phi\in V$ has conformal weight of the form $(h,0)$ (i.e. $\bar{h}=0$) then it is holomorphic. Similarly, if $\Phi$ has conformal weight $(0,\bar{h})$ then it is antiholomorphic.*

*Proof.* Consider a field $\Phi\in V$ of conformal weight $(h,\bar{h}=0)$. Computing the square of the norm of $\overline{L}_{-1}\Phi$ we find

$$\left\langle\overline{L}_{-1}\Phi,\overline{L}_{-1}\Phi\right\rangle \underset{(640)}{=} \left\langle\Phi,\overline{L}_1\overline{L}_{-1}\Phi\right\rangle=\left\langle\Phi,(2\overline{L}_0+\overline{L}_{-1}\overline{L}_1)\Phi\right\rangle=2\bar{h}\langle\Phi,\Phi\rangle=0. \tag{716}$$

Here we used that $\overline{L}_1\Phi=0$, since if it were nonzero it would be a field of conformal weight $(h,-1)$, and by Assumption 5.14 (a) negative conformal weights are inadmissible. Since the hermitian form on $V$ is assumed to be nondegenerate, this implies

$$\overline{L}_{-1}\Phi=\bar{\partial}\Phi=0, \tag{717}$$

i.e., $\Phi$ is a holomorphic field. □

For example, in any CFT, the stress-energy tensor $T$ is a $(2,0)$-field and therefore is holomorphic.[105] In the scalar field theory, $\partial\phi$ is a $(1,0)$-field and thus is holomorphic.

---

[104] Note that the exponents (713) are chosen in such a way that one has $\alpha_{ij}=\alpha_{ji}$ and $\sum_{j\neq i}\alpha_{ij}=h_i$ (and similarly for $\bar{\alpha}_{ij}$), which implies that both sides of (712) are sections of the same line bundle.

[105] We already included holomorphicity of $T$ as a part of axiomatics in (600). Lemma 5.33 provides another explanation why $T$ should be holomorphic.

5.7.2. *Mode operators.*

**Definition 5.34.** Let $\Xi \in V$ be a holomorphic field of conformal weight $(h, 0)$, with $h \in \mathbb{Z}$. One defines the "mode operators" associated with $\Xi$ as the operators $\Xi_{(n)} \in \mathrm{End}(V)$, with $n \in \mathbb{Z}$, defined by

$$\Xi_{(n)}\Phi(z) = \frac{1}{2\pi i}\oint_{\gamma_z} dw(w-z)^{n+h-1}\Xi(w)\Phi(z) \tag{718}$$

for any test field $\Phi \in V$, with $\gamma_z$ the contour going around $z$. Put another way, operators $\Xi_{(n)}$ yield terms in the OPE of $\Xi$ with the test field:

$$\Xi(w)\Phi(z) \sim \sum_{n\in\mathbb{Z}} \frac{\Xi_{(n)}\Phi(z)}{(w-z)^{n+h}}. \tag{719}$$

For instance, the mode operators for the stress-energy tensor $T$ are the Virasoro generators $L_n$, cf. (638). Another example: mode operators for the identity field $\mathbb{1}$ are $\mathbb{1}_{(n)} = \delta_{n,0}\,\mathrm{id}_V$.

The shift by $h$ in the definition (718) is designed in such a way that the operator $\Xi_{(-n)}$ shifts the conformal weight by $(n, 0)$.

5.7.3. *The Lie algebra of mode operators.*

**Lemma 5.35.** *Assume that the CFT contains a collection of holomorphic fields $\{\Phi_i\}_{i\in I}$ (with $I$ an indexing set) of conformal weights $(h_i, 0)$ satisfying the OPEs*

$$\Phi_i(w)\Phi_j(z) \sim \sum_{k\in I} f_{ijk}\frac{\Phi_k(z)}{(z-w)^{h_i+h_j-h_k}} + \text{reg.} \tag{720}$$

*with $f_{ijk}$ some constants (note that the exponents in the OPE are fixed by Lemma 5.24). Then the mode operators of fields $\Phi_i$ satisfy the commutation relations*

$$[\Phi_{i(n)}, \Phi_{j(m)}] = \sum_{k\in I} f_{ijk}\begin{pmatrix} n+h_i-1 \\ h_i+h_j-h_k-1\end{pmatrix}\Phi_{k(n+m)}. \tag{721}$$

The proof is similar to the proof of Virasoro commutation relations from $TT$ OPE in Section 5.2.2.

*Remark* 5.36. Similarly to Definition 5.34, one also has the "centered-at-zero version" of mode operators: for $\Xi \in V$ a holomorphic field, one has mode operators $\widehat{\Xi}_{(n)}$ acting on the space of states $\mathcal{H}$ defined by

$$\widehat{\Xi}_{(n)} = \frac{1}{2\pi i}\oint_{\gamma_0} dw\, w^{n+h-1}\widehat{\Xi}(w) \tag{722}$$

with $\gamma_0$ a contour around zero, or equivalently:

$$\widehat{\Xi}(w) = \sum_{n\in\mathbb{Z}} \frac{\widehat{\Xi}_{(n)}}{w^{n+h}}. \tag{723}$$

For example, in the scalar field theory, for the holomorphic field $\Xi = i\partial\phi$, the corresponding mode operators acting on states are the creation/annihilation operators:

$$\widehat{(i\partial\phi)}_{(n)} = \widehat{a}_n, \tag{724}$$

as follows from (535).

5.7.4. *Ward identity associated with a holomorphic field.*

**Lemma 5.37.** *Assume that the CFT contains a holomorphic $\Xi$ of conformal weight $(h,0)$. Then one has the corresponding Ward identity: for any collection of fields $\Phi_1,\ldots,\Phi_n \in V$ and meromorphic section $f = f(w)(\partial_w)^{h-1}$ of the line bundle $K^{\otimes(1-h)}$ over $\mathbb{CP}^1$ with singularities allowed at $z_1,\ldots,z_n$, one has*

$$\sum_{k=1}^{n} \langle \Phi_1(z_1)\cdots \rho_\Xi^{(z_k)}(f)\circ\Phi_k(z_k)\cdots\Phi_n(z_n)\rangle = 0 \tag{725}$$

*where the action of $f$ on $V_z$ is given by the contour integral around $z$,*

$$\rho_\Xi^{(z)}(f)\circ\Phi(z) := \frac{1}{2\pi i}\oint_{\gamma_z} \underbrace{dw\, f(w)\Xi(w)}_{\iota_f(\Xi(w)(dw)^h)}\Phi(z). \tag{726}$$

The proof is completely analogous to the proof of the conformal Ward identity (680).

**Example 5.38.** In the scalar field theory, take $\Xi = i\partial\phi$ and $\Phi_1 = V_{\alpha_1},\ldots,\Phi_n = V_{\alpha_n}$ vertex operators, and set $f=1$. Then the Ward identity (725) reads

$$(\alpha_1+\cdots+\alpha_n)\langle V_{\alpha_1}(z_1)\cdots V_{\alpha_n}(z_n)\rangle = 0 \tag{727}$$

where we used the OPE (676). This implies the result that the correlator of vertex operators can be nonzero only if the sum of their charges $\alpha_i$ vanishes (cf. (674)).

5.8. **Transformation law for the stress-energy tensor.** The action of a holomorphic vector field $u(w)\partial_w$ on the stress-energy tensor is given by

$$\begin{aligned}\rho^{(z)}(u\partial)T(z) &\underset{(637)}{=} -\frac{1}{2\pi i}\oint_{\gamma_z} dwu(w)T(w)T(z) = \\ = -\frac{1}{2\pi i}\oint_{\gamma_z} dw&(u(z)+(w-z)\partial u(z)+\frac{1}{2}(w-z)^2\partial^2u(z)+\frac{1}{6}(w-z)^3\partial^3u(z)+\cdots)\cdot \\ &\cdot\left(\frac{c/2}{(w-z)^4}+\frac{2T(z)}{(w-z)^2}+\frac{\partial T(z)}{w-z}+\text{reg}\right) \\ &= -u(z)\partial T(z)-2\partial u(z)T(z)-\frac{c}{12}\partial^3u(z)\end{aligned} \tag{728}$$

If not for the last term, this would have been the transformation law of a $(2,0)$-primary field (cf. (658)). The last term in (728) is a certain correction due to the projective property of CFT (a manifestation of conformal anomaly). We note that the action of an antiholomorphic vector field on $T$ is zero,

$$\rho^{(z)}(\bar{u}\bar{\partial})T(z) = 0, \tag{729}$$

since $\bar{T}T$ OPE is regular.

The calculation (728) expresses the infinitesimal transformation of $T$ under a conformal vector field (seen as an infinitesimal conformal map). Its counterpart for a "finite" conformal (holomorphic) transformation $z\mapsto w(z)$ is:

$$T_{(z)}(z)\mapsto T_{(w)}(w) = \left(\frac{\partial w}{\partial z}\right)^{-2}(T_{(z)}(z)-\frac{c}{12}S(w,z)) \tag{730}$$

where

$$S(w,z):=\frac{\partial_z^3 w}{\partial_z w}-\frac{3}{2}\left(\frac{\partial_z^2 w}{\partial_z w}\right)^2 \tag{731}$$

is the so-called *Schwarzian derivative* of the holomorphic map $f\colon z\mapsto w(z)$ (we will also use the notation $S(f)$ for the Schwarzian derivative).

Here are some properties of the Schwarzian derivative:

(a) $S$ vanishes on Möbius transformations,

(b) $S$ satisfies a chain-like rule

$$S(f\circ g)=(S(f)\circ g)\cdot(g')^2+S(g). \tag{732}$$

In particular, combining with (a), we have that for $f$ a Möbius transformation and $g$ any holomorphic map, $S(f\circ g)=S(g)$.

(c) $S$ can be restricted to smooth maps $S^1\to S^1$. This restriction can be understood as a degree 1 group cocycle of diffeomorphisms of the circle with coefficients in the module of densities of weight 2:

$$S\in H^1(\mathrm{Diff}(S^1),\mathrm{Dens}^2(S^1)). \tag{733}$$

This is ultimately a consequence of the "chain rule" (732).

As in Section (5.5.1), the transformation law (730) can either be understood in "active way" (moving points on the surface $\Sigma$) or "passive way" (action of a coordinate transformation).

**Example 5.39.** Consider $w=\log(z)$ as a holomorphic map from the punctured plane to the cylinder

$$\begin{array}{ccc} \mathbb{C}\backslash\{0\} & \to & \mathbb{C}/2\pi i\mathbb{Z} \\ z & \mapsto & w=\log(z) \end{array} \tag{734}$$

From (731) one finds

$$S(w,z)=\frac{1}{2z^2}. \tag{735}$$

In particular, (730) becomes

$$T_{(z)}(z)\mapsto T_{(w)}(w)=z^2T_{(z)}(z)-\frac{c}{24}. \tag{736}$$

In particular, on $\mathbb{C}$ one has $\langle T(z)\rangle_{\text{plane}}=0$ (this is a consequence of e.g. Lemma 5.25). Thus, on the cylinder one has

$$\langle T(w)\rangle_{\text{cylinder}}=-\frac{c}{24}. \tag{737}$$

This is just throwing in a physics buzzword

Thus, the vacuum energy on the cylinder should be $-\frac{c+\bar{c}}{24}$ instead of zero. In physics this mysterious effect has the name "Casimir energy associated with periodic boundary conditions."

## 6. More free CFTs

6.1. **Free scalar field with values in $S^1$.** An important deficiency of the free scalar field, our main (and only) example of a CFT so far, is that the evolution operator it assigns to a cylinder (or annulus) is not trace-class, which leads to the genus one partition function being ill-defined. This is remedied if we consider free scalar field with values in a circle (instead of values in $\mathbb{R}$). This model is also known as "free boson compactified on $S^1$" (compactification refers to the target) or "compactified free boson."

6.1.1. *Classical theory.* We will introduce the model and quickly retrace our steps in Sections 4.2, 4.3.1, pointing out where the change of target from $\mathbb{R}$ to $S^1$ changes the story.

Classically, the model on a Minkowski cylinder $\Sigma = \mathbb{R} \times S^1$ is defined by the action functional

$$S(\phi) = \int dt \underbrace{\int d\sigma \, \frac{\kappa}{2}((\partial_t \phi)^2 - (\partial_\sigma \phi)^2)}_{\mathsf{L}} \tag{738}$$

(the same formula as (432)) where $\phi$ now is a smooth map $\Sigma \to S^1_{\text{target}}$ where $S^1_{\text{target}} = \mathbb{R}/2\pi r\mathbb{Z}$ is a circle of a fixed radius $r$. Such a maps $\phi$ fall into homotopy classes, classified by a winding number $\mathsf{m} \in \mathbb{Z}$: a map with winding $\mathsf{m}$ satisfies $\phi(t, \sigma + 2\pi) = \phi(t, \sigma) + 2\pi r\mathsf{m}$. We included the conventional normalization $\kappa = \frac{1}{4\pi}$ in (738).

Thus, the space of fields splits as a disjoint union of spaces of maps to $S^1_{\text{target}}$ with a given winding number:

$$\text{Fields} = \text{Map}(\Sigma, S^1_{\text{target}}) = \bigsqcup_{\mathsf{m} \in \mathbb{Z}} \underbrace{\text{Map}_{\mathsf{m}}(\Sigma, S^1_{\text{target}})}_{\text{maps with winding number } \mathsf{m}} \tag{739}$$

One can then consider this model as classical mechanics with target

$$X = \bigsqcup_{\mathsf{m} \in \mathbb{Z}} \text{Map}_{\mathsf{m}}(S^1, S^1_{\text{target}}) \tag{740}$$

with Lagrangian $\mathsf{L}$ as in (738). A field $\phi \in X_{\mathsf{m}}$ with winding $\mathsf{m}$ can be expanded in Fourier modes, plus a shift linear in $\Sigma$, accounting for the winding:

$$\phi(\sigma) = \mathsf{m} r\sigma + \sum_{n \in \mathbb{Z}} \phi_n e^{in\sigma} \tag{741}$$

Transitioning to the Hamiltonian formalism (by Legendre transform), we have the phase space

$$\Phi = T^*X = \bigsqcup_{\mathsf{m} \in \mathbb{Z}} \underbrace{T^*X_{\mathsf{m}}}_{\Phi_{\mathsf{m}}} \tag{742}$$

parameterized in $\mathsf{m}$-th sector by the field $\phi(\sigma)$ and the Darboux-conjugate "momentum" $\pi(\sigma) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} \pi_n e^{in\sigma}$. The modes satisfy the standard Poisson brackets (449). The Hamiltonian on $\Phi_{\mathsf{m}}$ in terms of Fourier modes is

$$H = \pi_0^2 + \left(\frac{\mathsf{m}r}{2}\right)^2 + \sum_{n \neq 0} (\pi_n \pi_{-n} + \frac{1}{4} n^2 \phi_n \phi_{-n}). \tag{743}$$

Note that this differs from the Hamiltonian (452) by a shift $\left(\frac{\mathsf{m}r}{2}\right)^2$ which arises from the $\sigma$-linear term in (741).

6.1.2. *Canonical quantization.* We proceed with canonical quantization of the theory. The splitting (742) of the phase space means that the space of states splits as a direct sum

$$\mathcal{H} = \bigoplus_{\mathsf{m}\in\mathbb{Z}} \mathcal{H}_\mathsf{m}, \tag{744}$$

where $\mathcal{H}_\mathsf{m}$ consists of states with winding number $\mathsf{m}$.

Let $\widehat{\mathsf{m}}$ be the operator on $\mathcal{H}$ which has eigenvalue $\mathsf{m}$ on $\mathcal{H}_\mathsf{m}$. The quantum hamiltonian is

$$\widehat{H} = \widehat{\pi}_0^2 + \left(\frac{\widehat{\mathsf{m}}r}{2}\right)^2 + \sum_{n\neq 0}(\widehat{\pi}_n\widehat{\pi}_{-n} + \frac{1}{4}n^2\widehat{\phi}_n\widehat{\phi}_{-n}). \tag{745}$$

Similarly to Section 4.2.4, the Hamiltonian splits into

- A collection of harmonic oscillators (one for each $n\neq 0$, with frequency $\omega_n = |n|$). For the oscillators we introduce creation/annihilation operators $\widehat{a}_n$, $\widehat{\bar{a}}_n$, $n\neq 0$, exactly as before (471); they satisfy the usual commutation relations (472).
- A free particle of mass $\mu=\frac{1}{2}$ $\underline{\text{with values in } S^1_{\text{target}}}$ (described by $\widehat{\phi}_0$, $\widehat{\pi}_0$).
- A shift by a constant depending on winding, $\left(\frac{\widehat{\mathsf{m}}r}{2}\right)^2$.

For a free quantum particle on $S^1_{\text{target}}$ the space of states in Schrödinger representation is $L^2(S^1_{\text{target}})$ (the space of $2\pi r$-periodic $L^2$ functions $\psi(\phi_0)$) with $\widehat{\phi}_0$ acting by multiplication $\psi(\phi_0)\mapsto\phi_0\psi(\phi_0)$ and $\widehat{\pi}_0 = -i\frac{\partial}{\partial\phi_0}$ the derivation. Two important points here (in comparison with Section 4.2.3):

- The eigenvectors of $\widehat{\pi}_0$ are functions $\psi_\mathsf{e}(\phi_0) = e^{\frac{i\mathsf{e}}{r}\phi_0}$ with $\mathsf{e}\in\mathbb{Z}$, the corresponding eigenvalue is $\frac{\mathsf{e}}{r}$. In particular, the eigenvalue spectrum of $\widehat{\pi}_0$ is *discrete*:
$$\left\{\frac{\mathsf{e}}{r}\right\}_{\mathsf{e}\in\mathbb{Z}} = \frac{1}{r}\mathbb{Z}, \tag{746}$$
unlike the case of a free particle on $\mathbb{R}$ where the spectrum of momentum operator is $\mathbb{R}$.
- "Operator" $\widehat{\phi}_0$ is multi-valued (defined modulo $2\pi r\mathbb{Z}\cdot\mathrm{Id}$). In particular, it is not a well-defined operator in the usual sense, though exponentials $\widehat{v}^n :\, = e^{i\frac{n}{r}\widehat{\phi}_0}$ are well-defined operators for $n\in\mathbb{Z}$.[106] They satisfy the commutation relation
$$[\widehat{\pi}_0, \widehat{v}^n] = \frac{n}{r}\widehat{v}^n. \tag{747}$$

Retracing our steps with the scalar field, we proceed with the canonical quantization, construct the Heisenberg field operator, switch to Euclidean cylinder by Wick rotation and map to $\mathbb{C}\backslash\{0\}$ by the exponential map, arriving at the Heisenberg field operator

$$\widehat{\phi}(z) = \widehat{\phi}_0 - i\frac{\widehat{\mathsf{m}}r}{2}\log\frac{z}{\bar{z}} - i\widehat{\pi}_0\log(z\bar{z}) + \sum_{n\neq 0}\frac{i}{n}(\widehat{a}_n z^{-n} + \widehat{\bar{a}}_n\bar{z}^{-n}) \tag{748}$$

[106] In $\widehat{v}^n$, the superscript can be read either as index or as a power (of the operator $\widehat{v}=\widehat{v}^1$).

As we discussed above, $\widehat{\pi}_0$ has eigenvalue spectrum $\frac{1}{r}\mathbb{Z}$. So, we introduce the operator $\widehat{\mathsf{e}} := r\widehat{\pi}_0$ which has integer eigenvalues. In terms of this new notation, the field operator (748) is

$$\widehat{\phi}(z) = \widehat{\phi}_0 - i\frac{\widehat{\mathsf{m}}r}{2}\log\frac{z}{\bar{z}} - i\frac{\widehat{\mathsf{e}}}{r}\log(z\bar{z}) + \sum_{n\neq 0}\frac{i}{n}(\widehat{a}_n z^{-n} + \widehat{\bar{a}}_n\bar{z}^{-n}) \tag{749}$$

Lecture 28, 11/2/2022

The derivatives of the field operator are

$$i\partial\widehat{\phi}(z) = \sum_{n\in\mathbb{Z}}\widehat{a}_n z^{-n-1}, \quad i\bar{\partial}\widehat{\phi}(z) = \sum_{n\in\mathbb{Z}}\widehat{\bar{a}}_n\bar{z}^{-n-1} \tag{750}$$

(same formulae as (535)), where we defined

$$\widehat{a}_0 := \frac{\widehat{\mathsf{e}}}{r} + \frac{\widehat{\mathsf{m}}r}{2}, \quad \widehat{\bar{a}}_0 := \frac{\widehat{\mathsf{e}}}{r} - \frac{\widehat{\mathsf{m}}r}{2}. \tag{751}$$

The stress-energy tensor is given by the same formula as for the scalar field valued in $\mathbb{R}$, $T =: -\frac{1}{2}\partial\phi\partial\phi :$ (the normal ordering is defined as usual, putting the operator $\widehat{a}_{\geq 0}$, $\widehat{\bar{a}}_{\geq 0}$ to the right). Thus, the Virasoro generators are again given by (613), (614) and the Hamiltonian and total momentum operators are given by (616):

$$\begin{aligned} \widehat{H} &= \widehat{L}_0 + \widehat{\bar{L}}_0 = \frac{1}{2}\sum_{n\in\mathbb{Z}} :\widehat{a}_n\widehat{a}_{-n} + \widehat{\bar{a}}_n\widehat{\bar{a}}_{-n}:, \\ \widehat{P} &= \widehat{L}_0 - \widehat{\bar{L}}_0 = \frac{1}{2}\sum_{n\in\mathbb{Z}} :\widehat{a}_n\widehat{a}_{-n} - \widehat{\bar{a}}_n\widehat{\bar{a}}_{-n}: \end{aligned} \tag{752}$$

6.1.3. *Space of states.* The space of states of the scalar field with values in $S^1_{\mathrm{target}}$ (the Fock space) is

$$\mathcal{H} = \mathrm{Span}_{\mathbb{C}}\left\{\widehat{a}_{-n_r}\cdots\widehat{a}_{-n_1}\widehat{\bar{a}}_{-\bar{n}_s}\cdots\widehat{\bar{a}}_{-\bar{n}_1}|\mathsf{e},\mathsf{m}\rangle \,\middle|\, \begin{array}{c} 1\leq n_1\leq\cdots\leq n_r, \\ 1\leq\bar{n}_1\leq\cdots\leq\bar{n}_s, \\ (\mathsf{e},\mathsf{m})\in\mathbb{Z}^2 \end{array}\right\} \tag{753}$$

The vector $|\mathsf{e},\mathsf{m}\rangle\in\mathcal{H}$ ("pseudovacuum") is annihilated by the annihilation operators $\widehat{a}_{>0}$, $\widehat{\bar{a}}_{>0}$ and is an eigenvector of $\widehat{a}_0, \widehat{\bar{a}}_0$:

$$\widehat{a}_0|\mathsf{e},\mathsf{m}\rangle = \underbrace{\left(\frac{\mathsf{e}}{r} + \frac{\mathsf{m}r}{2}\right)}_{\alpha_{\mathsf{e},\mathsf{m}}}|\mathsf{e},\mathsf{m}\rangle, \qquad \widehat{\bar{a}}_0|\mathsf{e},\mathsf{m}\rangle = \underbrace{\left(\frac{\mathsf{e}}{r} - \frac{\mathsf{m}r}{2}\right)}_{\bar{\alpha}_{\mathsf{e},\mathsf{m}}}|\mathsf{e},\mathsf{m}\rangle \tag{754}$$

where we introduced the notations $\alpha_{\mathsf{e},\mathsf{m}}$, $\bar{\alpha}_{\mathsf{e},\mathsf{m}}$ for the respective eigenvalues.

Another way to express the the space of states is as a direct sum of Verma modules of the Lie algebra $\mathrm{Heis}\oplus\overline{\mathrm{Heis}}$ (the direct sum of two Heisenberg Lie algebras (476)) with highest weights (eigenvalues of $\widehat{a}_0, \widehat{\bar{a}}_0$) given by pairs $(\alpha_{\mathsf{e},\mathsf{m}}, \bar{\alpha}_{\mathsf{e},\mathsf{m}})$:

$$\mathcal{H} = \bigoplus_{(\mathsf{e},\mathsf{m})\in\mathbb{Z}^2}\underbrace{\mathbb{V}^{\mathrm{Heis}\oplus\overline{\mathrm{Heis}}}_{(\alpha_{\mathsf{e},\mathsf{m}},\bar{\alpha}_{\mathsf{e},\mathsf{m}})}}_{\mathcal{H}_{\mathsf{e},\mathsf{m}}} \tag{755}$$

Note that the main distinction from the case of the usual free scalar theory (485) is the structure of pseudovacua: previously we had a *continuum family* of pseudovacua $|\pi_0\rangle$ characterized by the value of the zero-mode momentum $\pi_0\in\mathbb{R}$, whereas now we have a *lattice* of pseudovacua $|\mathsf{e},\mathsf{m}\rangle$ characterized by the (integer) zero-mode momentum $\mathsf{e}$ and the winding number $\mathsf{m}$.[107]

The energy and total momentum of pseudovacua are found from (752):

$$\begin{aligned} \widehat{H}|\mathsf{e},\mathsf{m}\rangle &= \frac{1}{2}(\alpha^2_{\mathsf{e},\mathsf{m}}+\bar\alpha^2_{\mathsf{e},\mathsf{m}})|\mathsf{e},\mathsf{m}\rangle = \left(\left(\frac{\mathsf{e}}{r}\right)^2+\left(\frac{\mathsf{m}r}{2}\right)^2\right)|\mathsf{e},\mathsf{m}\rangle,\\ \widehat{P}|\mathsf{e},\mathsf{m}\rangle &= \frac{1}{2}(\alpha^2_{\mathsf{e},\mathsf{m}}-\bar\alpha^2_{\mathsf{e},\mathsf{m}})|\mathsf{e},\mathsf{m}\rangle = \mathsf{em}|\mathsf{e},\mathsf{m}\rangle \end{aligned} \tag{756}$$

Note that while the eigenvalue of $\widehat{H}$ is a non-negative real number, the eigenvalue of $\widehat{P}$ is always an integer. Also note that the only pseudovacuum with zero energy (eigenvalue of $\widehat{H}$) is $|\mathsf{e}=0,\mathsf{m}=0\rangle$. It also has zero total momentum and we identify this particular state as the "true" (as opposed to "pseudo-") vacuum, $|\mathrm{vac}\rangle :=|0,0\rangle$

As in the ordinary free scalar theory, we have that

- applying $\widehat{a}_{-n}$ to a state changes energy-momentum by $(n,n)$ (creates a "left-mover"),
- applying $\widehat{\bar{a}}_{-n}$ to a state changes energy-momentum by $(n,-n)$ (creates a "right-mover"),

where we assume $n>0$.

The pseudovacuum $|\mathsf{e},\mathsf{m}\rangle$ is also an eigenvector of the Virasoro generators $\widehat{L}_0$, $\widehat{\bar{L}}_0$ with

$$\widehat{L}_0|\mathsf{e},\mathsf{m}\rangle = \underbrace{\frac{1}{2}\alpha^2_{\mathsf{e},\mathsf{m}}}_{h_{\mathsf{e},\mathsf{m}}}|\mathsf{e},\mathsf{m}\rangle,\quad \widehat{\bar{L}}_0|\mathsf{e},\mathsf{m}\rangle = \underbrace{\frac{1}{2}\bar\alpha^2_{\mathsf{e},\mathsf{m}}}_{\bar h_{\mathsf{e},\mathsf{m}}}|\mathsf{e},\mathsf{m}\rangle. \tag{757}$$

6.1.4. *Vertex operators.* The counterpart of pseudovacua $|\mathsf{e},\mathsf{m}\rangle$ via the field-state correspondence are the vertex operators $V_{\mathsf{e},\mathsf{m}}\in V$, constructed somewhat differently than in the non-compactified scalar field theory.

Let us introduce an "operator" $\widehat{\mu}$ on $\mathcal{H}$, defined modulo $2\pi\mathbb{Z}\cdot\mathrm{Id}$ (similarly to the operator $\widehat{\phi}_0$) satisfying $[\widehat{\mu},\widehat{m}]=i$ and commuting with $\widehat{a}_{\neq 0},\widehat{\bar{a}}_{\neq 0},\widehat{e},\widehat{\phi}_0$. Then for $k\in\mathbb{Z}$ the exponential $e^{ik\widehat{\mu}}$ is a well-defined operator on $\mathcal{H}$ satisfying

$$[\widehat{m},e^{ik\widehat{\mu}}]=ke^{ik\widehat{\mu}}, \tag{758}$$

cf. (747), i.e., the operator

$$e^{ik\widehat{\mu}}\colon\quad \begin{array}{ccc} \mathcal{H}_{\mathsf{e},\mathsf{m}} & \to & \mathcal{H}_{\mathsf{e},\mathsf{m}+k} \\ \widehat{a}_{-n_r}\cdots\widehat{a}_{-n_1}\widehat{\bar{a}}_{-\bar{n}_s}\cdots\widehat{\bar{a}}_{-\bar{n}_1}|\mathsf{e},\mathsf{m}\rangle & \mapsto & \widehat{a}_{-n_r}\cdots\widehat{a}_{-n_1}\widehat{\bar{a}}_{-\bar{n}_s}\cdots\widehat{\bar{a}}_{-\bar{n}_1}|\mathsf{e},\mathsf{m}+k\rangle \end{array} \tag{759}$$

shifts the magnetic (or winding) number $\mathsf{m}$ by $k$.[108] Similarly, due to (747), the operator $e^{il\widehat{\phi}_0/r}$ shifts the electric number $\mathsf{e}$ by $l$:

$$e^{il\widehat{\phi}_0/r}\colon\quad \begin{array}{ccc} \mathcal{H}_{\mathsf{e},\mathsf{m}} & \to & \mathcal{H}_{\mathsf{e}+l,\mathsf{m}} \\ \widehat{a}_{-n_r}\cdots\widehat{a}_{-n_1}\widehat{\bar{a}}_{-\bar{n}_s}\cdots\widehat{\bar{a}}_{-\bar{n}_1}|\mathsf{e},\mathsf{m}\rangle & \mapsto & \widehat{a}_{-n_r}\cdots\widehat{a}_{-n_1}\widehat{\bar{a}}_{-\bar{n}_s}\cdots\widehat{\bar{a}}_{-\bar{n}_1}|\mathsf{e}+l,\mathsf{m}\rangle \end{array}. \tag{760}$$

[107]The notations $\mathsf{e},\mathsf{m}$ correspond to "electric" and "magnetic" number.

[108]Instead of introducing the operator $\widehat{\mu}$, one can treat (759) as the definition of a family of operators on $\mathcal{H}$, formally denoted $e^{ik\widehat{\mu}}$. From this viewpoint, $\widehat{\mu}$ is a purely notational device, only meaningful in the combination $e^{ik\widehat{\mu}}$.

Further, let us introduce the following two multivalued operators (the "holomorphic/antiholomorphic parts of $\widehat{\phi}$"):

(761)

$$\widehat{\chi}(z) = \frac{1}{2}\widehat{\phi}_0 + \frac{\widehat{\mu}}{r} - i\widehat{a}_0 \log z + \sum_{n\neq 0} \frac{i}{n}\widehat{a}_n z^{-n}, \quad \widehat{\bar{\chi}}(z) = \frac{1}{2}\widehat{\phi}_0 - \frac{\widehat{\mu}}{r} - i\widehat{\bar{a}}_0 \log \bar{z} + \sum_{n\neq 0} \frac{i}{n}\widehat{\bar{a}}_n \bar{z}^{-n}.$$

In particular, one has $\widehat{\phi}(z) = \widehat{\chi}(z) + \widehat{\bar{\chi}}(z)$.

**Definition 6.1.** The vertex operator $V_{\mathsf{e},\mathsf{m}}$ in the compactified free scalar field CFT is defined by

$$\widehat{V}_{\mathsf{e},\mathsf{m}}(z) := \; : e^{i\alpha_{\mathsf{e},\mathsf{m}}\widehat{\chi}(z)} e^{i\bar{\alpha}_{\mathsf{e},\mathsf{m}}\widehat{\bar{\chi}}(z)} : \tag{762}$$

Here the parameters $\mathsf{e}, \mathsf{m}$ are integers and $\alpha_{\mathsf{e},\mathsf{m}}$, $\bar{\alpha}_{\mathsf{e},\mathsf{m}}$ are as in (754).

The normal ordering puts operators $\widehat{a}_{\geq 0}, \widehat{\bar{a}}_{\geq 0}$ to the right and operators $\widehat{a}_{<0}, \widehat{\bar{a}}_{<0}, \widehat{\phi}_0, \widehat{\mu}$ to the left. Written more explicitly, the vertex operator is

$$\begin{aligned}\widehat{V}_{\mathsf{e},\mathsf{m}}(z) = {} & e^{i\mathsf{e}\widehat{\phi}_0/r} e^{i\mathsf{m}\widehat{\mu}} e^{-\sum_{n<0}\frac{1}{n}(\alpha_{\mathsf{e},\mathsf{m}}\widehat{a}_n z^{-n} + \bar{\alpha}_{\mathsf{e},\mathsf{m}}\widehat{\bar{a}}_n \bar{z}^{-n})} \cdot \\ & \cdot e^{-\sum_{n>0}\frac{1}{n}(\alpha_{\mathsf{e},\mathsf{m}}\widehat{a}_n z^{-n} + \bar{\alpha}_{\mathsf{e},\mathsf{m}}\widehat{\bar{a}}_n \bar{z}^{-n})} e^{\alpha_{\mathsf{e},\mathsf{m}}\widehat{a}_0 \log z + \bar{\alpha}_{\mathsf{e},\mathsf{m}}\widehat{\bar{a}}_0 \log \bar{z}}.\end{aligned} \tag{763}$$

Somewhat non-obviously, this is a single-valued operator: the multi-valued operators $\widehat{\phi}_0$, $\widehat{\mu}$ are only present in single-valued exponential expressions; the last exponential is single valued when acting on $\mathcal{H}_{\mathsf{e}',\mathsf{m}'}$ since one has

$$\alpha_{\mathsf{e},\mathsf{m}}\alpha_{\mathsf{e}',\mathsf{m}'} - \bar{\alpha}_{\mathsf{e},\mathsf{m}}\bar{\alpha}_{\mathsf{e}',\mathsf{m}'} = \mathsf{e}\mathsf{m}' + \mathsf{m}\mathsf{e}' \in \mathbb{Z}. \tag{764}$$

Performing computations similar to those of Section 5.3.1, 5.5.2, 5.5.3, one proves the following properties of vertex operators:

- $V_{\mathsf{e},\mathsf{m}}$ is a primary field of conformal weight
$$h_{\mathsf{e},\mathsf{m}} = \frac{1}{2}\left(\frac{\mathsf{e}}{r} + \frac{\mathsf{m}r}{2}\right)^2, \quad \bar{h}_{\mathsf{e},\mathsf{m}} = \frac{1}{2}\left(\frac{\mathsf{e}}{r} - \frac{\mathsf{m}r}{2}\right)^2 \tag{765}$$
– same $h_{\mathsf{e},\mathsf{m}}, \bar{h}_{\mathsf{e},\mathsf{m}}$ as in (757).
- One has
$$\lim_{z\to 0} \widehat{V}_{\mathsf{e},\mathsf{m}}(z)|\mathrm{vac}\rangle = |\mathsf{e},\mathsf{m}\rangle, \tag{766}$$
i.e., as claimed in the beginning of this section, the state corresponding to the vertex operator $V_{\mathsf{e},\mathsf{m}}$ by the field-state correspondence is the pseudovacuum $|\mathsf{e},\mathsf{m}\rangle$. More generally, one has
(767)
$$\lim_{z\to 0} : \prod_{j=1}^{r} \frac{i\partial^{n_j}\widehat{\phi}(z)}{(n_j-1)!} \prod_{k=1}^{s} \frac{i\bar{\partial}^{\bar{n}_k}\widehat{\phi}(z)}{(\bar{n}_k-1)!} \widehat{V}_{\mathsf{e},\mathsf{m}}(z) : |\mathrm{vac}\rangle = \widehat{a}_{-n_r}\cdots\widehat{a}_{-n_1}\widehat{\bar{a}}_{-\bar{n}_s}\cdots\widehat{\bar{a}}_{\bar{n}_1}|\mathsf{e},\mathsf{m}\rangle,$$
i.e., the fields corresponding to basis states of $\mathcal{H}$ are the vertex operators multiplied by differential polynomials in $\phi$.
- The correlator of $n$ vertex operators is

(768)

$$\left\langle \prod_{k=1}^{n} V_{\mathsf{e}_k,\mathsf{m}_k}(z_k) \right\rangle = \begin{cases} \prod_{1\leq i<j\leq n} (z_i - z_j)^{\alpha_{\mathsf{e}_i,\mathsf{m}_i}\alpha_{\mathsf{e}_j,\mathsf{m}_j}} (\bar{z}_i - \bar{z}_j)^{\bar{\alpha}_{\mathsf{e}_i,\mathsf{m}_i}\bar{\alpha}_{\mathsf{e}_j,\mathsf{m}_j}}, & \text{if } \sum_{i=1}^n \mathsf{e}_i = \sum_{i=1}^n \mathsf{m}_i = 0, \\ 0, & \text{otherwise} \end{cases}$$

Despite the real exponents appearing here, the entire expression on the right is in fact a single-valued function on $C_n(\mathbb{CP}^1)$, due to (764). For instance, for $n=2$ one has

$$\langle V_{\mathsf{e},\mathsf{m}}(w)V_{-\mathsf{e},-\mathsf{m}}(z)\rangle = |w-z|^{-2\left(\left(\frac{\mathsf{e}}{r}\right)^2+\left(\frac{\mathsf{m}r}{2}\right)^2\right)}\left(\frac{w-z}{\bar{w}-\bar{z}}\right)^{-\mathsf{em}} \tag{769}$$

– note that the first exponent on the right is real while the second is an integer, making the expression single-valued.

6.1.5. *Torus partition function in a general CFT.* Consider the torus $\mathbb{T}$ obtained from the annulus $\{z\in\mathbb{C}\mid r_{\mathrm{in}}\leq |z|\leq r_{\mathrm{out}}\}$ by identifying the inner and outer circles via the identification $r_{\mathrm{in}}e^{i\sigma}\sim r_{\mathrm{out}}e^{i\sigma}$. Equivalently, we map the annulus by the map $z\mapsto \zeta=\log z$ to the cylinder

$$\mathrm{cyl}=\{\zeta=t+i\sigma\in\mathbb{C}/2\pi i\mathbb{Z}\mid\ \log r_{\mathrm{in}}\leq t\leq \log r_{\mathrm{out}}\} \tag{770}$$

and identify the boundary circles by $\log r_{\mathrm{in}}+i\sigma\sim\log r_{\mathrm{out}}+i\sigma$. This yields a complex torus with modular parameter

$$\tau=\frac{i}{2\pi}T \tag{771}$$

with $T=\log\frac{r_{\mathrm{out}}}{r_{\mathrm{in}}}$.

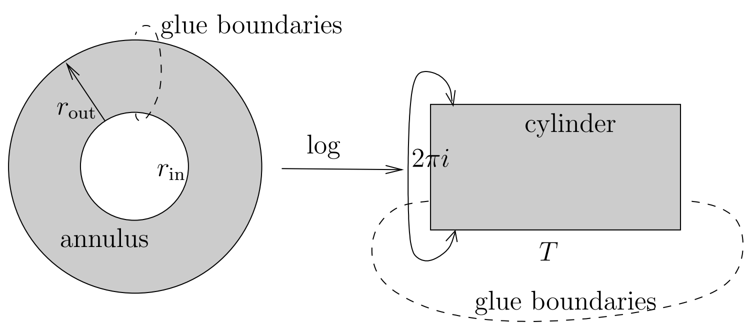



FIGURE 31. Torus obtained from annulus or cylinder by identifying the boundary circles.

The evolution operator for the cylinder of Euclidean length $T$ is

$$Z(\mathrm{cyl}_T)=e^{-T\widehat{H}}=e^{-T(\widehat{L}_0+\widehat{\bar{L}}_0)} \tag{772}$$

The partition function for the torus is the trace of this evolution operator over the space of states,

$$Z(\mathbb{T}_\tau)=\mathrm{tr}_{\mathcal{H}}e^{-T\widehat{H}}=\mathrm{tr}_{\mathcal{H}}e^{2\pi i\tau(\widehat{L}_0+\widehat{\bar{L}}_0)} \tag{773}$$

Lecture 29, 11/4/2022

with $\tau$ the modular parameter (771).

Gluing with a twist by angle $\theta$. More generally, one can glue the inner and outer boundary circles of the annulus with a twist by angle $\theta$: $r_{\mathrm{in}}e^{i\sigma}\sim r_{\mathrm{out}}e^{i\sigma+\theta}$, or equivalently identify the boundary circles of the cylinder as $\log r_{\mathrm{in}}+i\sigma\sim\log r_{\mathrm{out}}+i(\sigma+\theta)$. Denote $\mathrm{cyl}_{T,\theta}$ the mapping cylinder (6) of length $T$ (understood as a cobordism $S^1 \to S^1$), associated with mapping $\rho_r\colon S^1 \to S^1$ rotating the circle by angle $\theta$. Then one has

$$Z(\mathrm{cyl}_{T,\theta}) = e^{-T\widehat{H}-i\theta\widehat{P}} = e^{2\pi i\tau\widehat{L}_0 - 2\pi i\bar{\tau}\widehat{\overline{L}}_0} = q^{\widehat{L}_0}\bar{q}^{\widehat{\overline{L}}_0} \tag{774}$$

with $\widehat{P}$ the total momentum operator. In (774) we denoted

$$q = e^{2\pi i\tau} \tag{775}$$

and $\bar{q}$ is its complex conjugate; note that since $\mathrm{Im}(\tau) > 0$, one has $|q| < 1$. We used the expressions (616) for the total energy/momentum as $\widehat{L}_0 \pm \widehat{\overline{L}}_0$.

This yields a complex torus with modular parameter $\tau = \frac{i}{2\pi}(T + i\theta)$ and the corresponding partition function is

$$Z(\mathbb{T}_\tau) = \mathrm{tr}_{\mathcal{H}} Z(\mathrm{cyl}_{T,\theta}) = \mathrm{tr}_{\mathcal{H}} q^{\widehat{L}_0}\bar{q}^{\widehat{\overline{L}}_0} \tag{776}$$

<u>Correction due to central charge.</u> In fact, one needs to introduce a correction in (776):

$$Z(\mathbb{T}_\tau) = \mathrm{tr}_{\mathcal{H}} Z(\mathrm{cyl}_{T,\theta}) = \mathrm{tr}_{\mathcal{H}} q^{\widehat{L}_0 - \frac{c}{24}}\bar{q}^{\widehat{\overline{L}}_0 - \frac{\bar{c}}{24}}, \tag{777}$$

with $(c, \bar{c})$ the holomorphic/antiholomorphic central charge of the CFT (one also needs similar correction in (774)). The reason for this correction can be explained in several ways:

(i) The correction in (777) arises from the Schwarzian derivative correction in the transformation law of the stress-energy tensor (730), (736), which implies

$$\widehat{H}_{\mathrm{cyl}} = \frac{1}{2\pi}\oint_{t=\mathrm{const}} \iota_{\frac{\partial}{\partial t}}(T_{\mathrm{cyl}}(d\zeta)^2 + \overline{T}_{\mathrm{cyl}}(d\bar{\zeta})^2) = \widehat{H}_{\mathrm{plane}} - \frac{c+\bar{c}}{24} = \widehat{L}_0 + \widehat{\overline{L}}_0 - \frac{c+\bar{c}}{24}. \tag{778}$$

and similarly for the total momentum operator; Virasoro generators $\widehat{L}_0$, $\widehat{\overline{L}}$ are understood as pertaining to the plane and to the radial quantization picture (thus, when mapping to the cylinder by the map $z \mapsto \zeta = \log(z)$ they receive the Schwarzian correction).

(ii) Expression (777) is the partition function for a torus with *flat metric* (obtained from the flat metric on the cylinder), whereas (776) is the partition function for the torus with a singular metric obtained by taking the flat annulus and identifying the two boundary circles (the glued surface has a metric which is flat almost everywhere, except at the circle where the gluing was performed – there the metric is singular, since e.g. the identified circles had different lengths). Conformal anomaly means that the partition function has a dependence on the metric within the conformal class (32). Thus, the factor $q^{-\frac{c}{24}}\bar{q}^{-\frac{\bar{c}}{24}}$ in (777) is the explonential of the Liouville action in (32) corresponding to the change from the singular metric on $\mathbb{T}$ coming from the annulus to the flat metric.

(iii) Pragmatic viewpoint: the partition function for the torus is expected to be modular invariant, in particular, it should be invariant under $\tau \mapsto -\frac{1}{\tau}$. As we will see in the example of the free scalar field with values in $S^1$, expression (777) has this property, while (776) does not. This is connected with item (ii) above: flat tori with modular parameters $\tau$ and $-1/\tau$ are connected by a constant Weyl transformation, for which the Liouville action in (32) is zero. For the singular metric coming from the annulus, this is not true: the metric tori $\mathbb{T}_\tau$, $\mathbb{T}_{-1/\tau}$ have "scars" – singular loci of the metric – and they are not intertwined by the conformal map $\mathbb{T}_\tau \to \mathbb{T}_{-1/\tau}$.

6.1.6. *Torus partition function for the free scalar field with values in $S^1$.* In our case the central charge is $c = \bar{c} = 1$ and the formula (777) becomes

$$Z(\tau) = \mathrm{tr}_\mathcal{H} q^{\widehat{L}_0 - \frac{1}{24}} \bar{q}^{\widehat{\bar{L}}_0 - \frac{1}{24}} = \\ = \sum_{(\mathsf{e},\mathsf{m})\in\mathbb{Z}^2} \sum_{1\leq n_1 \leq \cdots \leq n_r,\ 1\leq \bar{n}_1 \leq \cdots \leq \bar{n}_s} q^{h_{\mathsf{e},\mathsf{m}} + \sum_{i=1}^r n_i - \frac{1}{24}} \bar{q}^{\bar{h}_{\mathsf{e},\mathsf{m}} + \sum_{j=1}^s \bar{n}_j - \frac{1}{24}} \tag{779}$$

For brevity we denote the partition function of the torus with modular parameter $\tau$ simply as $Z(\tau)$. Here we used that the operators $\widehat{L}_0$, $\widehat{\bar{L}}_0$ are diagonal in the basis (753); the exponents in the r.h.s. of (779) are the corresponding eigenvalues shifted by $-\frac{1}{24}$; $h_{\mathsf{e},\mathsf{m}}$, $\bar{h}_{\mathsf{e},\mathsf{m}}$ are the conformal weights of the pseudovacua (757), (765). Continuing the computation, we have

$$Z(\tau) = \sum_{(\mathsf{e},\mathsf{m})\in\mathbb{Z}^2} q^{h_{\mathsf{e},\mathsf{m}}} \bar{q}^{\bar{h}_{\mathsf{e},\mathsf{m}}} (q\bar{q})^{\frac{1}{24}} \sum_{k,l\geq 0} P(k)P(l) q^k \bar{q}^l, \tag{780}$$

where $P(k)$ is the number of partitions of $k$, i.e., the number of nondecreasing sequences $1 \leq n_1 \leq \cdots n_r$ such that $k = n_1 + \cdots + n_r$, for some $r \geq 1$. For instance, one has

$$\begin{aligned} 4 &= 1+1+1+1 \\ &= 1+1+2 \\ &= 2+2 \\ &= 1+3 \\ &= 4, \end{aligned} \tag{781}$$

thus, $P(4) = 5$. In (780), the left factor is the sum over pseudovacua, the middle factor is the central charge correction, and the right factor accounts for the contributions of $\mathrm{Heis} \oplus \overline{\mathrm{Heis}}$-descendants of the pseudovacuum (and $P(k)P(l)$ is the count of descendants of conformal weight $(h_{\mathsf{e},\mathsf{m}} + k, \bar{h}_{\mathsf{e},\mathsf{m}} + l)$).

The generating function for the numbers of partitions is a well-studied object of combinatorics,

$$\sum_{k\geq 0} P(k) q^k = \frac{1}{\prod_{n\geq 1}(1-q^n)} = \frac{q^{\frac{1}{24}}}{\eta(\tau)} \tag{782}$$

where

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n\geq 1} (1-q^n) \tag{783}$$

is the Dedekind eta-function which satisfies the modular equivariance properties[109]

$$\eta(\tau+1) = e^{i\pi/12}\eta(\tau), \tag{784}$$

$$\eta(-1/\tau) = (-i\tau)^{\frac{1}{2}}\eta(\tau) \tag{785}$$

[109] Property (784) is obvious from the definition (783). Property (785) follows from the Euler's identity $\prod_{n\geq 1}(1-q^n) = \sum_{j\in\mathbb{Z}} (-1)^j q^{\frac{3j^2-j}{2}}$ by applying Poisson summation formula (cf. footnote 10).

Finally, the partition function (780) can be written in the form

$$Z(\tau)=\frac{1}{\eta(\tau)\eta(\bar\tau)}\sum_{(\mathsf{e},\mathsf{m})\in\mathbb{Z}^2}q^{\frac12\left(\frac{\mathsf{e}}{r}+\frac{\mathsf{m}r}{2}\right)^2}\bar q^{\frac12\left(\frac{\mathsf{e}}{r}-\frac{\mathsf{m}r}{2}\right)^2} \tag{786}$$

When we are interested in the dependence of the partition function on the radius of the target circle, we will write it as a function of two arguments $Z(\tau,r)$.

**Lemma 6.2** (Properties of $Z(\tau)$)**.** *The torus partition function (786) satisfies the following properties.*

*(a) Modular invariance:*

$$Z(\tau+1) = Z(\tau), \tag{787}$$

$$Z(-1/\tau) = Z(\tau). \tag{788}$$

*(b) "T-duality":*

$$Z(\tau,r)=Z(\tau,2/r) \tag{789}$$

*(c) Large-radius asymptotics*

$$Z(\tau,r)\underset{r\to\infty}{\sim} r\frac{1}{\sqrt{\mathrm{Im}(\tau)}\,\eta(\tau)\eta(\bar\tau)} \tag{790}$$

Modular invariance (787), (788) means that the genus one partition function belongs to $C^\infty(\Pi_+)^{PSL_2(\mathbb{Z})}$, i.e., descends to a smooth function on the moduli space of complex tori $\mathcal{M}_{1,0}$ – which is the general feature expected in any CFT, cf. Section 1.4.1.

"T-duality" (or "target-space duality") is a term originating in string theory. T-duality means that there is an equivalence of sigma-models with target a circle of radius $r$ and target a circle of radius $2/r$.

Property (790) means in particular that if we think of the scalar field with target $\mathbb{R}$ as a limit of the scalar field with target $S^1$ of radius $r$, as $r\to\infty$, we are seeing explicitly how the partition function diverges (as the volume of the target). This gives us a better understanding of the claim made in the very beginning of Section 6.1 that the genus one partition function of the $\mathbb{R}$-valued free scalar theory diverges.

*Proof.* Item (a) is proven by Poisson summation in $\mathsf{e},\mathsf{m}$. add the detailed computation?

For the item (b), we notice that the exponents in (786) satisfy

$$h_{\mathsf{e},\mathsf{m}}(r)=h_{\mathsf{m},\mathsf{e}}(2/r),\ \bar h_{\mathsf{e},\mathsf{m}}(r)=\bar h_{\mathsf{m},\mathsf{e}}(2/r) \tag{791}$$

where we indicate explicitly the dependence of the exponents (conformal weights of the pseudovacuum $|\mathsf{e},\mathsf{m}\rangle$) on $r$. From this observation, the equality (789) is obvious. (Interestingly, the inversion of the target radius $r\mapsto 2/r$ is compensated by the interchange of the electric and magnetic numbers $(\mathsf{e},\mathsf{m})\mapsto(\mathsf{m},\mathsf{e})$.)

For the item (c) one applies Poisson summation just in the variable $\mathsf{e}$ to (786): one has

$$Z(\tau,r)=\frac{1}{\eta(\tau)\eta(\bar\tau)}\sum_{(p,\mathsf{m})\in\mathbb{Z}^2}\frac{r}{\sqrt{\mathrm{Im}(\tau)}}e^{-\frac{\pi^2}{2}r^2\left(\frac{(p+\mathsf{m}\mathrm{Re}(\tau))^2}{\mathrm{Im}(\tau)}+\mathsf{m}^2\right)}, \tag{792}$$

where we denoted $p$ the dual variable to $\mathsf{e}$ (w.r.t. Poisson summation). In the sum (792), the asymptotics as $r\to\infty$ is given by the term $p=\mathsf{m}=0$ (and it is the r.h.s. of (790)), while the sum of all other terms is exponentially suppressed – it behaves as $O(re^{-Ar^2})$ with some constant $A>0$.

□

6.1.7. *Path integral approach to the torus partition of the free scalar field with values in $S^1$.* In this part we follow K. Gawedzki [14], we refer the reader to this source for more details.

In the path integral approach, the partition function of the torus $\Sigma = \mathbb{T}_\tau$ is represented by the integral over smooth maps $\phi\colon \Sigma \to S^1_{\mathrm{target}}$:

$$Z^{\mathrm{PI}}(\Sigma) = \int_{\mathrm{Map}(\Sigma,S^1)} \mathcal{D}\phi\, e^{-S(\phi)} \tag{793}$$

with $S(\phi)$ the classical (Euclidean) action of the model,

$$S(\phi) = \frac{1}{8\pi}\int_\Sigma d\phi\wedge *d\phi = \frac{1}{8\pi}\int_\Sigma dt d\sigma((\partial_t\phi)^2 + (\partial_\sigma\phi)^2). \tag{794}$$

Note that we have $\pi_0\mathrm{Map}(\Sigma, S^1_{\mathrm{target}}) \simeq \mathbb{Z}^2$. More specifically, maps $\phi$ fall into classes of homotopy equivalent maps, according to the pair of winding numbers $(n_1, n_2) \in \mathbb{Z}^2$ of $\phi$ around two closed curves $\gamma_{1,2} \subset \Sigma$ – the generators of $\pi_1(\Sigma)$.

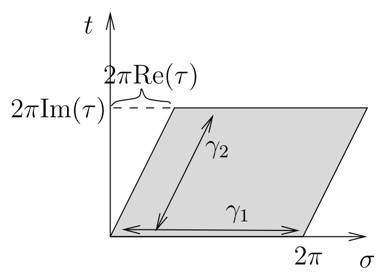



FIGURE 32. Torus with modular parameter $\tau$ with two generators of $\pi_1$.

Thus, the mapping space breaks into connected components

$$\mathrm{Map}(\Sigma, S^1_{\mathrm{target}}) = \bigsqcup_{(n_1,n_2)\in\mathbb{Z}^2} \mathrm{Map}_{n_1,n_2}(\Sigma, S^1_{\mathrm{target}}) \tag{795}$$

where $\mathrm{Map}_{n_1,n_2}$ consists of maps with prescribed winding numbers $n_1, n_2$. Therefore, we can rewrite (793) as

$$Z^{\mathrm{PI}}(\Sigma) = \sum_{(n_1,n_2)\in\mathbb{Z}^2} \int_{\mathrm{Map}_{n_1,n_2}(\Sigma,S^1)} \mathcal{D}\phi\, e^{-S(\phi)} \tag{796}$$

Notice that for each pair $(n_1, n_2) \in \mathbb{Z}$ there exists a unique (up to a constant shift) solution of the Euler-Lagrange equation $\Delta\phi = 0$ with winding numbers $(n_1, n_2)$. Explicitly it can be represented by the function

$$\phi^{\mathrm{cl}}_{n_1,n_2}(\sigma, t) = r\cdot\left(n_1\sigma + \frac{n_2 - n_1\mathrm{Re}(\tau)}{\mathrm{Im}(\tau)}t\right). \tag{797}$$

Note that it is a linear function in coordinates $\sigma, t$ on the torus. The classical action evaluated on the classical solution (797) is

$$S(\phi^{\mathrm{cl}}_{n_1,n_2}) = \frac{\pi r^2}{2}\frac{|n_2 - \tau n_1|^2}{\mathrm{Im}(\tau)} \tag{798}$$

A general smooth map $\phi \in \mathrm{Map}_{n_1,n_2}(\Sigma, S^1_{\mathrm{target}})$ can be uniquely decomposed as

$$\phi = \phi_0 + \phi^{\mathrm{cl}}_{n_1,n_2} + \widetilde{\phi}, \tag{799}$$

where

- $\phi_0$ is a constant function valued in $S^1_{\mathrm{target}}$ (the constant shift of a classical solution),
- $\phi^{\mathrm{cl}}_{n_1,n_2}$ is the "standard" classical solution with given winding numbers (797),
- the "fluctuation" $\widetilde{\phi}$ is a smooth function with no winding (i.e. lifting to a function $\Sigma \to \mathbb{R}$) and satisfying the condition

$$\int_\Sigma dt\, d\sigma\, \widetilde{\phi} = 0 \tag{800}$$

(this condition is imposed to have uniqueness of the decomposition (799)). We denote the space of maps $\phi\colon \Sigma \to \mathbb{R}$ satisfying (800) by $\mathrm{Map}'(\Sigma, \mathbb{R})$ (it is the orthogonal complement of constant maps).

Note that the first two terms in (799) together give the general classical solution with given winding numbers. Substituting the decomposition (799) into the action (794), we obtain

$$S(\phi) = S(\phi^{\mathrm{cl}}_{n_1,n_2}) + S(\widetilde{\phi}). \tag{801}$$

Thus, the path integral (796) is

$$Z^{\mathrm{PI}}(\Sigma) = \sum_{(n_1,n_2)\in\mathbb{Z}^2} \underbrace{\oint_{S^1_{\mathrm{target}}} d\phi_0}_{2\pi r} \underbrace{\int_{\mathrm{Map}'(\Sigma,\mathbb{R})} \mathcal{D}\widetilde{\phi}\, e^{-S(\widetilde{\phi})}}_{(\det{}' \Delta_\Sigma)^{-\frac{1}{2}}} \cdot e^{-S(\phi^{\mathrm{cl}}_{n_1,n_2})} \tag{802}$$

The integral over $\phi_0 \in S^1_{\mathrm{target}}$ here is the integral over the space of classical solutions. The Gaussian functional integral in the middle is formally evaluated to the determinant-prime (i.e. excluding the zero eigenvalue) of the Laplacian on $\Sigma$ raised to the power $-\frac{1}{2}$, cf. Section 4.5.3. This determinant can be calculated explicitly in the sense of zeta-function regularization (this is a rather nontrivial computation for which we refer the reader again to Gawedzki [14]), yielding

$$\det{}'\Delta_\Sigma = (2\pi)^2 \mathrm{Im}(\tau)\, |\eta(\tau)|^4, \tag{803}$$

where the Dedekind eta-function makes an appearance. Thus, continuing (802), we have

$$Z^{\mathrm{PI}}(\Sigma) = \sum_{(n_1,n_2)\in\mathbb{Z}^2} \cancel{2\pi} r \frac{1}{\cancel{2\pi}\sqrt{\mathrm{Im}(\tau)}|\eta(\tau)|^2} e^{-\frac{\pi r^2}{2}\frac{|n_2-\tau n_1|^2}{\mathrm{Im}(\tau)}} \tag{804}$$

This expression coincides with result of the operator formalism in the form (792)!

To see this coincidence, we identify $n_1$ with $\mathsf{m}$ (which is not surprising, since $\mathsf{m}$ was the winding number along the fixed-time circle) and $n_2$ with $p$ (i.e., the second winding number gets identified with the Poisson-dual variable to $\mathsf{e}$ – the zero-mode momentum).

Ultimately, we obtained a check that the operator formalism of CFT (relying on the study of the space of states) and the path integral formalism yield the same answer for the genus one partition function.

We remark that in the path integral formalism, the modular invariance of the torus partition function is manifest (unlike the operator formalism where it is a nontrivial consequence of Poisson summation). Indeed, the values of the action evaluated on classical solutions (798) on the tori $\Sigma = \mathbb{T}_\tau$ and $\Sigma' = \mathbb{T}_{-1/\tau}$ are the same (if one identifies the winding numbers as $(n_1, n_2) \leftrightarrow (n_2, -n_1)$). Likewise, the eigenvalue spectra of Laplacians on on $\Sigma, \Sigma'$ are the same, and hence the determinants are the same. Put another way, in the path integral formalism modular invariance is manifest, because the classical (Lagrangian) theory is conformally invariant.[110]

Lecture 30, 11/7/2022

6.2. **Aside: conformal blocks.** In a general CFT on a surface $\Sigma$ (e.g. $\Sigma = \mathbb{C}$ or $\mathbb{CP}^1$), for a collection of fields $\Phi_1, \ldots, \Phi_n \in V$ one is interested in writing the correlator as a sum of products of holomorphic and antiholomorphic functions

$$\langle \Phi_1(z_1) \cdots \Phi_n(z_n) \rangle = \sum_{\rho \in I(\Phi_1, \ldots, \Phi_n)} F_\rho(z_1, \ldots, z_n) F'_\rho(\bar{z}_1, \ldots, \bar{z}_n). \tag{805}$$

Here

- The correlator in the l.h.s. is a smooth single-valued function on the open configuration space $C_n(\Sigma)$.
- In the r.h.s. the index $\rho$ ranges over some set $I(\Phi_1, \ldots, \Phi_n)$ depending on the input fields (in nice cases it is a finite set, but generally does not have to be).
- $F_\rho$, $F'_\rho$ are respectively holomorphic and antiholomorphic (possibly multivalued[111]) functions on $C_n(\Sigma)$; they are called the "conformal blocks" for the correlator in the l.h.s. of (805).

Similarly, the genus one partition function can be written as

$$Z(\tau) = \sum_{\rho \in I_{1,0}} \chi_\rho(\tau) \chi'_\rho(\bar{\tau}) \tag{806}$$

with $\chi_\rho, \chi'_\rho$ – the "conformal blocks for the torus partition function" – respectively holomorphic and antiholomorphic multivalued functions on the moduli space $\mathcal{M}_{1,0}$.

6.2.1. *Chiral (holomorphic) free boson with values in $S^1$.* Consider the version of the compactified free boson theory where one only considers one copy of the Heisenberg algebra (generated by $\widehat{a}_n$, but not $\widehat{\bar{a}}_n$), and the space of states is the sum of Verma modules for this single Heisenberg algebra:

(807)
$$\mathcal{H}^{\mathrm{chiral}} = \bigoplus_{(\mathsf{e},\mathsf{m}) \in \mathbb{Z}^2} \mathbb{V}^{\mathrm{Heis}}_{\mathsf{e},\mathsf{m}} = \mathrm{Span}\left\{ \widehat{a}_{-k_r} \cdots \widehat{a}_{-k_1} |\mathsf{e},\mathsf{m}\rangle \;\middle|\; (\mathsf{e},\mathsf{m}) \in \mathbb{Z}^2,\ 1 \le k_1 \le \cdots \le k_r \right\}$$

In this model, one can consider the chiral vertex operator

$$\widehat{V}^{\mathrm{chiral}}_{\mathsf{e},\mathsf{m}}(z) =: e^{i\alpha_{\mathsf{e},\mathsf{m}} \widehat{\chi}(z)} :, \tag{808}$$

[110] In this form this argument is a bit formal and implicitly assumes conformal invariance of the path integral measure.

[111] In particular, $F_\rho, F'_\rho$ are allowed to have monodromy as one puncture goes around another one. Put another way, $F_\rho, F'_\rho$ are single-valued holomorphic/antiholomorphic functions on some covering space of $C_n(\Sigma)$.

with $\widehat{\chi}(z)$ as in (761) – the "holomorphic part" of the field operator $\widehat{\phi}(z)$. The expression (808) should be thought of as the "holomorphic half" of the vertex operator (762) of the full (non-chiral) theory.

From Wick's lemma one obtains correlators

$$(809)\quad \langle\prod_{i=1}^n V^{\mathrm{chiral}}_{\mathsf{e}_i,\mathsf{m}_i}(z_i)\rangle = \begin{cases} \prod_{1\leq i<j\leq n}(z_i-z_j)^{\alpha_{\mathsf{e}_i\mathsf{m}_i}\alpha_{\mathsf{e}_j\mathsf{m}_j}}, & \text{if } \sum \mathsf{e}_i = \sum \mathsf{m}_i = 0, \\ 0, & \text{otherwise} \end{cases}$$

This expression is holomorphic and multivalued (has monodromies) on $C_n(\mathbb{C})$. If the radius of the target circle satisfies $r^2\in\mathbb{Q}$, then the monodromies are rational and the correlator lifts to as a single-valued function on a finite-degree covering space of $C_n(\mathbb{C})$.

We remark that multivaluedness of correlators is linked to the fact that the conformal weights of chiral vertex operators $(h,\bar{h}) = (\frac12\alpha^2_{\mathsf{e},\mathsf{m}},0)$ fail the assumption (652). In the chiral theory the antiholomorphic stress-energy tensor vanishes identically $\overline{T}=0$ and any field has $\bar{h}=0$, so $V$ and $\mathcal{H}$ are graded just by the holomorphic conformal weight $h$.

The correlator (768) of vertex operators in the full compactified free boson theory factorizes as the correlator of holomorphic chiral vertex operators (809) times the correlator of (analogous) antiholomorphic chiral vertex operators:[112]

$$(810)\qquad \langle\prod_{i=1}^n V^{\mathrm{non-chiral}}_{\mathsf{e}_i,\mathsf{m}_i}(z_i)\rangle = \langle\prod_{i=1}^n V^{\mathrm{chiral}}_{\mathsf{e}_i,\mathsf{m}_i}(z_i)\rangle\cdot\langle\prod_{i=1}^n \overline{V}^{\mathrm{chiral}}_{\mathsf{e}_i,\mathsf{m}_i}(\bar{z}_i)\rangle$$

Comparing with (805), we can say that correlators of holomorphic/antiholmorphic chiral vertex operators in the respective chiral compactified free boson theories yield the conformal blocks for the correlator of vertex operators in the full (non-chiral) compactified free boson theory. In particular, in this example the indexing set $I$ of (805) is a single-element set.

The genus one partition function (786) of the compactified free boson admits the representation (806) with $I_{1,0}$ a finite set if and only if the target radius satisfies $r^2\in\mathbb{Q}$.

For example, for $r=\sqrt{2}$ (the so-called self-dual radius, since it is a stationary point of T-duality (789)), one has

$$(811)\quad Z(\tau) = \Big(\frac{1}{\eta(\tau)}\sum_{k\in\mathbb{Z}} q^{k^2}\Big)\Big(\frac{1}{\eta(\bar\tau)}\sum_{l\in\mathbb{Z}}\bar{q}^{l^2}\Big) + \Big(\frac{1}{\eta(\tau)}\sum_{k\in\mathbb{Z}+\frac12} q^{k^2}\Big)\Big(\frac{1}{\eta(\bar\tau)}\sum_{l\in\mathbb{Z}+\frac12}\bar{q}^{l^2}\Big)$$

I.e., here $I_{1,0}$ is a 2-element set: one has two holomorphic and two antiholomorphic conformal blocks.

### 6.3. **Free fermion.**

6.3.1. *Classical Lagrangian theory on a surface.* As a Lagrangian field theory, 2d free fermion on a Riemannian surface $\Sigma$ is defined by the classical action

$$(812)\qquad S = \frac{i}{4\pi}\int_\Sigma \boldsymbol{\psi}\bar{\boldsymbol{\partial}}\boldsymbol{\psi} - \bar{\boldsymbol{\psi}}\boldsymbol{\partial}\bar{\boldsymbol{\psi}} = \frac{1}{2\pi}\int_\Sigma d^2z(\psi\bar\partial\psi + \bar\psi\partial\bar\psi)$$

[112]In the antiholomorphic chiral theory, one only retains the creation/annihilation operators $\widehat{\bar{a}}_n$, all fields have conformal weight of the form $(0,\bar{h})$ and $T=0$. Correlators are antiholomorphic and multivalued.

Here $\boldsymbol{\partial} = dz\,\partial$, $\bar{\boldsymbol{\partial}} = d\bar{z}\,\bar{\partial}$ are the holomorphic/antiholomorphic Dolbeault differentials, $z, \bar{z}$ refers to a local complex coordinate on $\Sigma$ and $d^2z = \frac{i}{2}dz \wedge d\bar{z}$ is the coordinate area element. The fields of the model are fermions (spinors)[113]

$$\boldsymbol{\psi} = \psi(dz)^{1/2} \in \Gamma(\Sigma, K^{\otimes \frac{1}{2}}), \qquad \bar{\boldsymbol{\psi}} = \bar{\psi}(d\bar{z})^{1/2} \in \Gamma(\Sigma, \overline{K}^{\otimes \frac{1}{2}}). \tag{813}$$

Here $K, \overline{K}$ are the line bundles $(T^{1,0})^*\Sigma, (T^{0,1})^*\Sigma$. Two important points:

- To define the square root of these line bundles, one needs to choose the sign of the root of the transition function. This choice of sign is known as the spin structure on $\Sigma$.[114]
- One treats the values of the fields $\psi, \bar{\psi}$ as *anticommuting* (or "odd" or "Grassmann") variables.

Thus, the space of fields of the model is (purely odd) vector superspace

$$\mathrm{Fields}_\Sigma = \bigoplus_s \Gamma(\Sigma, \Pi K_s^{\otimes \frac{1}{2}} \oplus \Pi \overline{K}_s^{\otimes \frac{1}{2}})) \tag{814}$$

refer to Cimasoni-Reshetikhin?

where the sum is over the spin structures $s$ on $\Sigma$;[115] $\Pi$ is the parity reversal symbol, implying that $\mathrm{Fields}_\Sigma$ is the space of sections of a supervector bundle with purely odd fiber.

The Euler-Lagrange equation for the action reads

$$\bar{\boldsymbol{\partial}}\boldsymbol{\psi} = 0, \quad \boldsymbol{\partial}\bar{\boldsymbol{\psi}} = 0, \tag{815}$$

or equivalently, in a local complex coordinate,

$$\bar{\partial}\psi = 0, \quad \partial\bar{\psi} = 0. \tag{816}$$

*Remark* 6.3. The system described by the action functional (812), with fields $\psi, \bar{\psi}$ is called the free Majorana fermion.[116] One can also consider the system with only field $\psi$ (or only $\bar{\psi}$), with the action $S_{\mathrm{chiral}} = \frac{1}{2\pi}\int_\Sigma d^2z\, \psi\bar{\partial}\psi$ (respectively, $\frac{1}{2\pi}\int_\Sigma d^2z\, \bar{\psi}\partial\bar{\psi}$) – it is called the chiral or Weyl fermion. When one wants to distinguish between the chiral fermion $\psi$ and the chiral fermion $\bar{\psi}$, they are called respectively left- and right-chiral fermions.

6.3.2. *Hamiltonian picture.* As a Hamiltonian theory on a cylinder, the model has phase space – the purely odd vector superspace

$$\Phi = \bigsqcup_{s \in \{P,A\}} C_s^\infty(S^1) \otimes \mathbb{C}^{0|2} \tag{817}$$

where $\mathbb{C}^{0|2}$ is another notation for the odd two-dimensional complex space $\Pi\mathbb{C}^2$; $s \in \{P, A\}$ is a choice of spin structure on the cylinder – a choice of either periodic (P) or antiperiodic (A) boundary conditon. Elements of $\Phi$ are pairs $(\psi, \bar{\psi})$ of functions on $S^1$ satisfying simultaneously either P or A condition,

$$\psi(\sigma + 2\pi) = \epsilon\psi(\sigma), \quad \bar{\psi}(\sigma + 2\pi) = \epsilon\bar{\psi}(\sigma) \tag{818}$$

[113]One understands $\psi, \bar{\psi}$ as two independent fields.

[114]Put another way, it is a choice of a consistent set of periodicity/antiperiodicity conditions for the fermion field $\psi, \bar{\psi}$, as one traverses a closed curve $\gamma$ on $\Sigma$.

[115]Generally, spin structures form a torsor over $H^1(\Sigma, \mathbb{Z}_2)$, thus there are $2^{B_1}$ spin structures on a surface with first Betti number $B_1$.

[116]Majorana fermion is "uncharged" as opposed to Dirac fermion, which is "charged" – possesses an extra $U(1)$-symmetry $\psi \to e^{i\theta}\psi$. Majorana and Dirac fermions are also referred to as "real" and "complex" fermions, respectively.

with $\epsilon = +1$ is $s = P$ and $\epsilon = -1$ if $s = A$.

One has the symplectic form on the phase space,

$$\omega = \frac{i}{4\pi}\oint_{S^1} d\sigma\Big(\delta\psi\wedge\delta\psi + \delta\bar\psi\wedge\delta\bar\psi\Big) \qquad \in \Omega^2(\Phi). \tag{819}$$

The corresponding Poisson (anti-)brackets[117] are

$$\{\psi(\sigma),\psi(\sigma')\} = 2\pi i\delta(\sigma-\sigma'),\ \{\bar\psi(\sigma),\bar\psi(\sigma')\} = 2\pi i\delta(\sigma-\sigma'),\ \{\psi(\sigma),\bar\psi(\sigma')\} = 0. \tag{820}$$

The Hamiltonian of the model is

$$H = \frac{i}{4\pi}\oint_{S^1} d\sigma\,(\psi\partial_\sigma\psi - \bar\psi\partial_\sigma\bar\psi). \tag{821}$$

It is obtained by writing the action functional on the Minkowski cylinder

$$S_{\mathrm{Mink}} = \int dt\, \underbrace{\frac{i}{4\pi}\oint d\sigma(\psi\partial_t\psi - \psi\partial_\sigma\psi + \bar\psi\partial_t\bar\psi + \bar\psi\partial_\sigma\bar\psi)}_{\mathsf{L}} \tag{822}$$

and performing the Legendre transform. Since the Lagrangian $\mathsf{L}$ is linear in velocities $\dot\psi(\sigma)$, the corresponding momenta $\pi(\sigma) = \frac{\delta}{\delta\dot\psi(\sigma)}\mathsf{L} = -\frac{i}{4\pi}\psi(\sigma)$ are not independent and drop out of the Legendre transform.

One can expand the fields in Fourier modes,

$$\psi(\sigma) = \sum_n e^{-in\sigma} b_n, \quad \bar\psi(\sigma) = \sum_n e^{-in\sigma}\bar b_n \tag{823}$$

where $n$ ranges over integers if $s = P$ and over half-integers ($n \in \mathbb{Z} + \frac12$) if $s = A$. Poisson brackets (820) imply to following Poisson brackets for the Fourier modes:

$$\{b_n, b_m\} = i\delta_{n,-m}, \quad \{\bar b_n, \bar b_m\} = i\delta_{n,-m}, \quad \{b_n, \bar b_m\} = 0. \tag{824}$$

6.3.3. *Canonical quantization.* Proceeding to canonical quantization, one replaces coordinates $b_n$, $\bar b_n$ on the phase space with operators $\widehat{b}_n, \widehat{\bar b}_n$ acting on some space of states $\mathcal{H}$ (to be described), subject to the following anticommutation relations (obtained from (824) by the canonical quantization prescription):

Expand on canonical quantization prescription?

$$[\widehat{b}_n, \widehat{b}_m]_+ = \delta_{n,-m}\widehat{\mathbb{1}}, \quad [\widehat{\bar b}_n, \widehat{\bar b}_m]_+ = \delta_{n,-m}\widehat{\mathbb{1}}, \quad [\widehat{b}_n, \widehat{\bar b}_m]_+ = 0, \tag{825}$$

where $[A,B]_+ := AB + BA$ is the anticommutator.

*Remark* 6.4. Generally, given a vector space $W$ with an inner product $g$, one can form the Clifford algebra $\mathrm{Cl}(W,g)$ – the associative unital algebra generated by the elements of $W$ subject to the relation

$$uv + vu = g(u,v)\mathbb{1} \tag{826}$$

for any $u, v \in W$. Then, the algebra spanned by the operators $\widehat{b}_n$ above (with $n \in \mathbb{Z}$ for $s = P$ and $n \in \mathbb{Z} + \frac12$ for $s = A$) is the Clifford algebra for the vector space $W = C^\infty_s(S^1)$ with inner product $g(u,v) = \oint d\sigma u(\sigma)v(\sigma)$.[118] Thus, the Clifford algebra for $W$ plays a similar role in the free fermion theory to the role of the Weyl

[117] Instead of being skew-symmetric, they are symmetric

[118] Or, more invariantly, one should set $W = \Gamma(S^1, (T^*S^1)_s^{\otimes\frac12})$ – the space of half-densities on $S^1$ with periodicity condition $s \in \{P, A\}$. Then one has $g(u,v) = \oint \mathbf{uv}$ for $\mathbf{u}, \mathbf{v} \in W$ two half-densities.

algebra in the free boson theory. We will denote these two Clifford algebras $\mathrm{Cl}_s$ with $s \in \{P, A\}$:

$$\begin{aligned} \mathrm{Cl}_P &= \mathbb{C}\langle \ldots, \widehat{b}_{-1}, \widehat{b}_0, \widehat{b}_1, \ldots \rangle \Big/ (\widehat{b}_n \widehat{b}_m + \widehat{b}_m \widehat{b}_n = \delta_{n,-m} \widehat{\mathbb{1}}), \\ \mathrm{Cl}_A &= \mathbb{C}\langle \ldots, \widehat{b}_{-3/2}, \widehat{b}_{-1/2}, \widehat{b}_{1/2}, \widehat{b}_{3/2} \ldots \rangle \Big/ (\widehat{b}_n \widehat{b}_m + \widehat{b}_m \widehat{b}_n = \delta_{n,-m} \widehat{\mathbb{1}}) \end{aligned} \tag{827}$$

The Heisenberg field operator on the cylinder is

$$\widehat{\psi}(\zeta) = \sum_{n \in \mathbb{Z}_s} e^{-n\zeta} \widehat{b}_n, \quad \widehat{\bar{\psi}}(\zeta) = \sum_{n \in \mathbb{Z}_s} e^{-n\bar{\zeta}} \widehat{\bar{b}}_n, \tag{828}$$

where $\zeta = t + i\sigma$, with $t$ the Euclidean time, and where we denoted $\mathbb{Z}_P \colon = \mathbb{Z}$, $\mathbb{Z}_A \colon = \mathbb{Z} + \frac{1}{2}$.

Mapping from the cylinder to the punctured plane by $\exp\colon \mathbb{C}/2\pi i\mathbb{Z} \to \mathbb{C}\backslash\{0\}$, $\zeta \mapsto z = e^\zeta$, we have $\psi_{\mathrm{plane}}(z)(dz)^{\frac{1}{2}} = \psi_{\mathrm{cyl}}(\zeta)(d\zeta)^{\frac{1}{2}}$ and thus

$$\psi_{\mathrm{plane}}(z) = \underbrace{z^{-\frac{1}{2}}}_{\left(\frac{dz}{d\zeta}\right)^{-\frac{1}{2}}} \psi_{\mathrm{cyl}}(\zeta) \tag{829}$$

where the power of derivative is minus the power of $K$ in (813), cf. also (659). Similarly, one has

$$\bar{\psi}_{\mathrm{plane}}(z) = \bar{z}^{-\frac{1}{2}} \bar{\psi}_{\mathrm{cyl}}(\zeta). \tag{830}$$

By this reasoning, Heisenberg field operators on the cylinder (828) mapped to the punctured plane become

$$\widehat{\psi}(z) = \sum_{n \in \mathbb{Z}_s} \widehat{b}_n z^{-n-\frac{1}{2}}, \quad \widehat{\bar{\psi}}(z) = \sum_{n \in \mathbb{Z}_s} \widehat{\bar{b}}_n \bar{z}^{-n-\frac{1}{2}} \tag{831}$$

where the $-\frac{1}{2}$ shift in the exponent comes from (829), (830).

Periodic boundary condition (P) on the cylinder ($\psi_{\mathrm{cyl}}(\sigma + 2\pi) = \psi_{\mathrm{cyl}}(\sigma)$) maps to the antiperiodic condition on the plane,

$$\psi_{\mathrm{plane}}(e^{2\pi i} z) = e^{-\frac{1}{2} 2\pi i} \psi_{\mathrm{plane}}(z) = -\psi_{\mathrm{plane}}(z), \tag{832}$$

i.e., when travelling along a closed simple contour around zero, the field $\psi_{\mathrm{plane}}(z)$ changes sign. This spin structure on $\mathbb{C}\backslash\{0\}$ (or "sector" of the phase space/space of states) is called "Ramond sector." Thus, in P or Ramond sector one has $\widehat{\psi}(z) = \sum_{n \in \mathbb{Z}} \widehat{b}_n z^{-n-\frac{1}{2}}$ and similarly for $\widehat{\bar{\psi}}(z)$.

Similarly, antiperiodic condition (A) on the cylinder becomes periodic condition on the plane, $\psi_{\mathrm{plane}}(e^{2\pi i} z) = +\psi_{\mathrm{plane}}(z)$. This is the so-called "Neveu-Schwarz spin structure/sector." Thus, in A or Neveu-Schwarz sector one has $\widehat{\psi}(z) = \sum_{n \in \mathbb{Z} + \frac{1}{2}} \widehat{b}_n z^{-n-\frac{1}{2}}$ and similarly for $\widehat{\bar{\psi}}(z)$.

Lecture 31, 11/9/2022

6.3.4. *Space of states for the chiral fermion.* Let us restrict our attention to the chiral fermion $\psi$, cf. Remark 6.3.

The space of states splits into P- and A-sectors:

$$\mathcal{H} = \mathcal{H}_P \oplus \mathcal{H}_A \tag{833}$$

with $\mathcal{H}_P$ a highest weight $\mathrm{Cl}_P$-module (cf. (827)) generated by the highest vector $|\mathrm{vac}_P\rangle$ satisfying $\widehat{b}_{>0} |\mathrm{vac}_P\rangle = 0$. Similarly, $\mathcal{H}_A$ a highest weight $\mathrm{Cl}_A$-module generated by the highest vector $|\mathrm{vac}_A\rangle$ satisfying $\widehat{b}_{>0}|\mathrm{vac}_A\rangle = 0$. Thus, one has

$$\mathcal{H}_P = \mathrm{Span}\left\{ \cdots \widehat{b}_{-2}^{p_2}\widehat{b}_{-1}^{p_1}\widehat{b}_0^{p_0}|\mathrm{vac}_P\rangle \;\middle|\; \begin{array}{c} p_0, p_1, p_2, \ldots \in \{0,1\}, \\ \text{finitely many } p_n \text{ are nonzero} \end{array} \right\} \tag{834}$$

Fermionic occupation numbers $p_0, p_1, \ldots$ are in $\{0,1\}$ since from the anticommutation relations (825) one has $(\widehat{b}_n)^2 = 0$ for $n \neq 0$ and $(\widehat{b}_0)^2 = \frac{1}{2}\widehat{\mathbb{1}}$. Similarly, one has

$$\mathcal{H}_A = \mathrm{Span}\left\{ \cdots \widehat{b}_{-5/2}^{p_{5/2}}\widehat{b}_{-3/2}^{p_{3/2}}\widehat{b}_{1/2}^{p_{1/2}}|\mathrm{vac}_A\rangle \;\middle|\; \begin{array}{c} p_{1/2}, p_{3/2}, p_{5/2}, \ldots \in \{0,1\}, \\ \text{finitely many } p_n \text{ are nonzero} \end{array} \right\} \tag{835}$$

6.3.5. *2-point function* $\langle\psi\psi\rangle$. Tu understand which of the Clifford highest vectors $|\mathrm{vac}_P\rangle$, $|\mathrm{vac}_A\rangle$ is the true vacuum of the system, let us calculate the correlation function $\langle\psi(w)\psi(z)\rangle$ in the operator formalism. Assume for simplicity $|w| > |z| > 0$.

In the P-sector we have

(836)
$$\langle\psi(w)\psi(z)\rangle_P :=\langle\mathrm{vac}_P|\widehat{\psi}(w)\widehat{\psi}(z)|\mathrm{vac}_P\rangle = \sum_{n,m\in\mathbb{Z}} \langle\mathrm{vac}_P|\widehat{b}_n\widehat{b}_m|\mathrm{vac}_P\rangle w^{-n-\frac{1}{2}}z^{-m-\frac{1}{2}}$$

From the fact that $\widehat{b}_{>0}|\mathrm{vac}_P\rangle = 0$, $\langle\mathrm{vac}_P|\widehat{b}_{<0} = 0$ and from the anticommutation relation (825) we see that the only surviving terms are $n = m = 0$ and $n = -m > 0$, i.e., one has

(837)
$$\langle\psi(w)\psi(z)\rangle_P = \underbrace{\langle\mathrm{vac}_P|\widehat{b}_0\widehat{b}_0|\mathrm{vac}_P\rangle}_{\frac{1}{2}} w^{-\frac{1}{2}}z^{-\frac{1}{2}} + \sum_{n=1}^{\infty}\underbrace{\langle\mathrm{vac}_P|\widehat{b}_n\widehat{b}_{-n}|\mathrm{vac}_P\rangle}_{1} w^{-n-\frac{1}{2}}z^{n-\frac{1}{2}} =$$
$$= (wz)^{-\frac{1}{2}}\Big(\frac{1}{2} + \sum_{n=1}^{\infty}\Big(\frac{z}{w}\Big)^n\Big) = \frac{1}{2}\frac{\left(\frac{w}{z}\right)^{\frac{1}{2}} + \left(\frac{z}{w}\right)^{\frac{1}{2}}}{w-z}$$

By a similar computation, in the A-sector we have

$$\langle\psi(w)\psi(z)\rangle_A = \sum_{n\in\mathbb{Z}+\frac{1}{2},\, n>0} \underbrace{\langle\mathrm{vac}_A|\widehat{b}_n\widehat{b}_{-n}|\mathrm{vac}_A\rangle}_{1} w^{-n-\frac{1}{2}}z^{n-\frac{1}{2}} = \tag{838}$$
$$= \frac{1}{w} + \frac{z}{w^2} + \frac{z^2}{w^3} + \cdots = \frac{1}{w-z}$$

Note that the expression (838) is translation-invariant as expected of a 2-point correlator in any CFT (cf. Lemma 5.26), while (837) is not translation-invariant. This suggests that we should identify $|\mathrm{vac}\rangle := |\mathrm{vac}_A\rangle$ as the true vacuum vector in $\mathcal{H}$, while $|\mathrm{vac}_P\rangle$ is a pseudovacuum (similar to the states $|\pi_0\rangle$ with $\pi_0 \neq 0$ in the scalar field theory). In particular, (838) should be understood as the actual 2-point correlator

$$\langle\psi(w)\psi(z)\rangle = \frac{1}{w-z} \tag{839}$$

On the other hand, the computation (837) should be understood as a 4-point correlator on $\mathbb{CP}^1$,[119]

$$\langle \sigma(\infty)\psi(w)\psi(z)\sigma(0)\rangle_{\mathbb{CP}^1}, \tag{840}$$

with a certain field $\sigma$ (so-called "twist field," to be discussed later), corresponding by field-state correspondence to $|\mathrm{vac}_P\rangle$, inserted at the points 0 and $\infty$. This explains why we don't see translation invariance in (837) – because "secretly" it is a 4-point function and a translation would displace the field $\sigma$ away from the origin.

In the free fermion model, the space of states $\mathcal{H}$ and the space of fields $V$ are $\mathbb{Z}_2$-graded and we understand that when the radial ordering is applied, we have a sign when we have to permute field operators:

$$\mathcal{R}\widehat{\Phi}_1(w)\widehat{\Phi}_2(z) = \begin{cases} \widehat{\Phi}_1(w)\widehat{\Phi}_2(z), & \text{if } |w| \geq |z| \\ (-1)^{|\Phi_1|\cdot|\Phi_2|}\widehat{\Phi}_2(z)\widehat{\Phi}_2(w), & \text{if } |z| \geq |w| \end{cases} \tag{841}$$

Here $|\Phi| \in \mathbb{Z}_2$ is the parity of the field. With this prescription, for instance, the computation (838) extends to the case $|w| \leq |z|$, yielding the same formula:

$$\langle \phi(w)\psi(z)\rangle :=\langle \mathrm{vac}_A|\mathcal{R}\widehat{\psi}(w)\widehat{\psi}(z)|\mathrm{vac}_A\rangle = \frac{1}{w-z}. \tag{842}$$

with any $w \neq z \in \mathbb{C}\backslash\{0\}$.

Note that the 2-point function (842) satisfies

$$\langle \psi(w)\psi(z)\rangle = -\langle \psi(z)\psi(w)\rangle \tag{843}$$

– the correlation function is antisymmetric under swapping the positions of fermions (as expected in Fermi statistics).

6.3.6. *Stress-energy tensor.* Classically, the stress-energy tensor (computed as a variation of the action w.r.t. metric) for the chiral fermion is

$$T(z) = -\frac{1}{2}\psi(z)\partial\psi(z) \tag{844}$$

for the holomorphic component and $\overline{T}(z) = 0$ for the antiholomorphic component.

For the corresponding quantum object – an operator on $\mathcal{H}$, we consider separately $\widehat{T}(z)$ as an operator on $\mathcal{H}_A$ and on $\mathcal{H}_P$.

<u>A-sector.</u> Set

$$\widehat{T}(z) := -\frac{1}{2} :\widehat{\psi}(z)\partial\widehat{\psi}(z): \tag{845}$$

where the normal ordering puts fermion annihilation operators $\widehat{b}_{>0}$ to the right and fermion creation operators $\widehat{b}_{<0}$ to the left; we understand that when we interchange two $\widehat{b}$'s, the sign of the expression is flipped.

From Wick's lemma (or rather its obvious adaptation to the Clifford algebra) we find the standard OPE

$$\mathcal{R}\widehat{T}(w)\widehat{T}(z) = \frac{\frac{c}{2}\widehat{\mathbb{1}}}{(w-z)^4} + \frac{2\widehat{T}(z)}{(w-z)^2} + \frac{\partial\widehat{T}(z)}{w-z} + \mathrm{reg.} \tag{846}$$

[119] The field $\sigma(\infty)$ here is with respect to the coordinate chart at $\infty \in \mathbb{CP}^1$. Writing this correlator in terms of $\mathbb{C}$, and using the result from further along this section that $\sigma$ has conformal weight $(\frac{1}{16}, 0)$, one should write $\lim_{y\to\infty} y^{\frac{1}{8}}\langle \sigma(y)\psi(w)\psi(z)\sigma(0)\rangle_{\mathbb{C}}$.

(cf. (601)) with holomorphic central charge $c = \frac{1}{2}$. Since $\overline{T} = 0$, the $T\overline{T}$ and $\overline{TT}$ OPEs are satisfied trivially, with antiholomorphic central charge $\bar{c} = 0$.

P-sector. Set

$$\widehat{T}^{\mathrm{naive}}(z) := -\frac{1}{2} : \widehat{\psi}(z)\partial\widehat{\psi}(z) : \tag{847}$$

with the same definition of normal ordering as above. Interestingly, it does not satisfy the expected OPE (601), thus it fails a basic axiom of a CFT (in particular its modes do not satisfy the Virasoro algebra relations). It turns out that a good definition is as follows:

$$\widehat{T}(z) := \lim_{w\to z}\left(-\frac{1}{2}\mathcal{R}\widehat{\psi}(w)\partial\widehat{\psi}(z) + \frac{1}{2}\frac{\widehat{\mathbb{1}}}{(w-z)^2}\right) \tag{848}$$

– we split the two points in the definition of the stress-energy tensor (844) and subtract the (translation-invariant) singular part of OPE, $-\frac{1}{2}\psi(w)\partial\psi(z) - [-\frac{1}{2}\psi(w)\partial\psi(z)]_{\mathrm{sing}}$. Then one has[120]

$$\widehat{T}(z) = \widehat{T}^{\mathrm{naive}}(z) + \frac{\widehat{\mathbb{1}}}{16z^2} \tag{849}$$

– with this $\frac{\widehat{\mathbb{1}}}{16z^2}$ shift included, $\widehat{T}$ does satisfy the desired OPE (846), again with $c = \frac{1}{2}$.

In particular, we have nonzero expectation value of the stress-energy tensor in P-sector

$$\langle T(z)\rangle_P := \langle \mathrm{vac}_P|\widehat{T}(z)|\mathrm{vac}_P\rangle = \frac{1}{16z^2}. \tag{850}$$

We remark that in A-sector, prescription (848) is compatible with the construction via normal ordering (845). Thus, (848) can be taken as a universal recipe for the fermion stress-energy tensor (applies to both A- and P-sector).

Virasoro generators. Virasoro generators can be obtained from the stress-energy tensor $\widehat{T}(z) = \sum_{n\in\mathbb{Z}} z^{-n-2}\widehat{L}_n$. Thus, from (845) and (849) one obtains:

$$\begin{aligned}
&\text{A-sector:} \quad \widehat{L}_n = \sum_{m\in\mathbb{Z}+\frac{1}{2}}\left(\frac{m}{2}+\frac{1}{4}\right) : \widehat{b}_{n-m}\widehat{b}_m :, \\
&\text{P-sector:} \quad \widehat{L}_n = \sum_{m\in\mathbb{Z}}\left(\frac{m}{2}+\frac{1}{4}\right) : \widehat{b}_{n-m}\widehat{b}_m : + \delta_{n,0}\frac{\widehat{\mathbb{1}}}{16}.
\end{aligned} \tag{851}$$

All operators $\overline{L}_n$ vanish identically.

In particular, one has

$$\widehat{L}_0|\mathrm{vac}_A\rangle = 0, \qquad \widehat{L}_0|\mathrm{vac}_P\rangle = \frac{1}{16}|\mathrm{vac}_P\rangle \tag{852}$$

In particular, the true vacuum $|\mathrm{vac}_A\rangle$ has zero energy and total momentum, while $|\mathrm{vac}_P\rangle$ has both energy and total momentum $\frac{1}{16}$.

---

[120] Indeed, repeating the computation (836), (837), without pairing to $|\mathrm{vac}_P\rangle$, we have $\mathcal{R}\widehat{\psi}(w)\widehat{\psi}(z) - : \widehat{\psi}(w)\widehat{\psi}(z) : = \frac{1}{2}\frac{(\frac{w}{z})^{\frac{1}{2}} + (\frac{z}{w})^{\frac{1}{2}}}{w-z}\widehat{\mathbb{1}} = (\frac{1}{w-z} + \frac{1}{8z^2}(w-z) + O((w-z)^2))\widehat{\mathbb{1}}$. Hence, $-\frac{1}{2}\mathcal{R}\widehat{\psi}(w)\partial\widehat{\psi}(z) = -\frac{1}{2} : \widehat{\psi}(w)\partial\widehat{\psi}(z) : + (-\frac{1}{2}\frac{1}{(w-z)^2} + \frac{1}{16z^2} + O(w-z))\widehat{\mathbb{1}}$, or equivalently $-\frac{1}{2}\mathcal{R}\widehat{\psi}(w)\partial\widehat{\psi}(z) + \frac{1}{2}\frac{\widehat{\mathbb{1}}}{(w-z)^2} = -\frac{1}{2} : \widehat{\psi}(w)\partial\widehat{\psi}(z) : + \frac{\widehat{\mathbb{1}}}{16z^2} + O(w-z)$. Taking the limit $w \to z$, we obtain (849).

One also has

$$[\widehat{L}_0, \widehat{b}_{-n}] = n\widehat{b}_n \tag{853}$$

in both A- and P-sectors. I.e., applying $\widehat{b}_{-n}$, one increases the $\widehat{L}_0$-eigenvalue (conformal weight) by $n$.

6.3.7. *Back to the space of states.* Let us list the states in A- and P-sectors with small conformal weights $h$ (i.e., $\widehat{L}_0$-eigenvalues).

| $h$ | state |
|---|---|
| $0$ | $\|\mathrm{vac}_A\rangle$ |
| $\frac{1}{2}$ | $\widehat{b}_{-\frac{1}{2}}\|\mathrm{vac}_A\rangle$ |
| $1$ | $\varnothing$ |
| $\frac{3}{2}$ | $\widehat{b}_{-\frac{3}{2}}\|\mathrm{vac}_A\rangle$ |
| $2$ | $\widehat{b}_{-\frac{3}{2}}\widehat{b}_{-\frac{1}{2}}\|\mathrm{vac}_A\rangle$ |
| $\frac{5}{2}$ | $\widehat{b}_{-\frac{5}{2}}\|\mathrm{vac}_A\rangle$ |
| $3$ | $\widehat{b}_{-\frac{5}{2}}\widehat{b}_{-\frac{1}{2}}\|\mathrm{vac}_A\rangle$ |
| ... | ... |

| $h$ | state |
|---|---|
| $\frac{1}{16}$ | $\|\mathrm{vac}_P\rangle$, $\widehat{b}_0\|\mathrm{vac}_P\rangle$ |
| $1+\frac{1}{16}$ | $\widehat{b}_{-1}\|\mathrm{vac}_P\rangle$, $\widehat{b}_{-1}\widehat{b}_0\|\mathrm{vac}_P\rangle$ |
| $2+\frac{1}{16}$ | $\widehat{b}_{-2}\|\mathrm{vac}_P\rangle$, $\widehat{b}_{-2}\widehat{b}_0\|\mathrm{vac}_P\rangle$ |
| $3+\frac{1}{16}$ | $\widehat{b}_{-3}\|\mathrm{vac}_P\rangle$, $\widehat{b}_{-3}\widehat{b}_0\|\mathrm{vac}_P\rangle$, $\widehat{b}_{-2}\widehat{b}_{-1}\|\mathrm{vac}_P\rangle$, $\widehat{b}_{-2}\widehat{b}_{-1}\widehat{b}_0\|\mathrm{vac}_P\rangle$ |
| ... | ... |

Here we have states in A-sector on the left and states in P-sector on the right.
States

$$|\mathrm{vac}_A\rangle, \quad \widehat{b}_{-\frac{1}{2}}|\mathrm{vac}_A\rangle, \quad |\mathrm{vac}_P\rangle, \quad \widehat{b}_0|\mathrm{vac}_P\rangle \tag{854}$$

are Virasoro-primary (annihilated by $\widehat{L}_{>0}$) – and they are the only Virasoro-primary states in $\mathcal{H}$. We will also denote these four states according to their conformal weight by $|0\rangle$, $|\frac{1}{2}\rangle$, $|\frac{1}{16}\rangle_+$, $|\frac{1}{16}\rangle_-$. Their $\mathbb{Z}_2$-grading is, respectively, even, odd, even, odd.[121]

Thus, the space of states of the chiral fermion splits into four conformal families (irreducible representations of Virasoro algebra):

$$\mathcal{H} = \underbrace{M_0 \oplus \Pi M_{\frac{1}{2}}}_{\mathcal{H}_A} \oplus \underbrace{M_{\frac{1}{16}} \oplus \Pi M_{\frac{1}{16}}}_{\mathcal{H}_P} \tag{855}$$

where $M_h$ is the irreducible Virasoro highest weight module with central charge $\frac{1}{2}$ and highest weight $h$; $\Pi$ is the parity reversal symbol (i.e. $M_h$ is an even vector space and $\Pi M_h$ is an odd (super)vector space).

By the field-state correspondence, the four primary states (854) correspond to four primary fields

$$\mathbb{1}, \quad \psi(z), \quad \sigma(z), \quad \mu(z) \tag{856}$$

with conformal weight $h$ being $0, \frac{1}{2}, \frac{1}{16}, \frac{1}{16}$, respectively (and $\bar{h} = 0$ for all fields in the chiral theory). Fields $\sigma, \mu$ are the so-called "twist fields." One has for instance the OPE

$$\psi(w)\sigma(z) \sim (w-z)^{-\frac{1}{2}}\mu(z) + \mathrm{reg.} \tag{857}$$

Is it phrased ok?

In particular, the insertion of the twist field $\sigma(z)$ creates a monodromy $-1$ around $z$ for the fermion $\psi(w)$.

Lecture 32, 11/11/2022

[121] The logic with $\mathbb{Z}_2$ grading is that vectors $|\mathrm{vac}_A\rangle$, $|\mathrm{vac}_P\rangle$ are even, while action by any single Clifford generator $\widehat{b}$ changes the parity of the vector.

6.3.8. *Non-chiral (Majorana) fermion.* We pair the left- and right- (or holomorphic/antiholomorphic) chiral fermion CFTs, with the following conventions:

- We require that the P/A boundary condition is the same for $\psi$ and $\bar{\psi}$.
- We impose $\widehat{b}_0 = \widehat{\bar{b}}_0$ (cf. (474)).

The space of states splits as a sum of irreducible highest weight modules of $\mathrm{Vir} \oplus \overline{\mathrm{Vir}}$ with central charge $c = \bar{c} = \frac{1}{2}$:

$$\mathcal{H}_{\mathrm{non-chiral}} = M_{0,0} \oplus \Pi M_{\frac{1}{2},0} \oplus \Pi M_{0,\frac{1}{2}} \oplus M_{\frac{1}{2},\frac{1}{2}} \oplus M_{\frac{1}{16},\frac{1}{16}} \oplus \Pi M_{\frac{1}{16},\frac{1}{16}} \tag{858}$$

whre the two indices of $M$ are the highest weight (conformal weight) $(h, \bar{h})$ of the highest vector. The highest weight vectors themselves and the corresponding primary fields are, respectively:

| highest vector | $\lvert\mathrm{vac}_A\rangle$ | $\widehat{b}_{-\frac{1}{2}}\lvert\mathrm{vac}_A\rangle$ | $\widehat{\bar{b}}_{-\frac{1}{2}}\lvert\mathrm{vac}_A\rangle$ | $\widehat{b}_{-\frac{1}{2}}\widehat{\bar{b}}_{-\frac{1}{2}}\lvert\mathrm{vac}_A\rangle$ | $\lvert\mathrm{vac}_P\rangle$ | $\widehat{b}_0\lvert\mathrm{vac}_P\rangle$ |
|---|---|---|---|---|---|---|
| primary field | $\mathbb{1}$ | $\psi(z)$ | $\bar{\psi}(z)$ | $\epsilon(z) = \psi(z)\bar{\psi}(z)$ | $\sigma(z)$ | $\mu(z)$ |
| $(h, \bar{h})$ | $(0,0)$ | $(\frac{1}{2}, 0)$ | $(0, \frac{1}{2})$ | $(\frac{1}{2}, \frac{1}{2})$ | $(\frac{1}{16})$ | $(\frac{1}{16}, \frac{1}{16})$ |
| $\mathbb{Z}_2$-parity | even | odd | odd | even | even | odd |

*Remark* 6.5. Free Majorana fermion is the CFT model corresponding to the Ising model at critical temperature, see [6] and [8] for a detailed discussion. In particular, correlation functions of the spin field in Ising model can be recovered as correlation functions of the field $\sigma$ in the free fermion CFT.

6.3.9. *Examples of correlators.* From the computation (838) we know the 2-point correlator

$$\langle \psi(w)\psi(z)\rangle = \frac{1}{w-z}. \tag{859}$$

The correlator of any number of fields $\psi$, $\bar{\psi}$ can be computed by Wick's lemma, as a sum over perfect matchings (where one needs to be careful with signs incurred when moving $\hat{\psi}$ over other $\hat{\psi}$'s.) For the correlator of several $\psi$ fields, this sum over perfect matchings can written as a Pfaffian formula

$$\langle \psi(z_1)\cdots\psi(z_n)\rangle = \begin{cases} \mathrm{Pf}\left(\frac{1}{z_i - z_j}\right) & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd} \end{cases} \tag{860}$$

For example, for $n = 4$ one has

$$\langle \psi(z_1)\psi(z_2)\psi(z_3)\psi(z_4)\rangle = \frac{1}{z_{12}z_{34}} - \frac{1}{z_{13}z_{24}} + \frac{1}{z_{14}z_{23}}, \tag{861}$$

where $z_{ij} = z_i - z_j$.

The 2-point correlator $\langle\sigma(w)\sigma(z)\rangle$ cannot be found from Wick's lemma (we don't have an explicit description of the field $\sigma$ in terms of Clifford generators $\widehat{b}_n, \widehat{\bar{b}}_n$ at our disposal), however we have an ansatz for it from global conformal symmetry, cf. Lemma 5.26:

$$\langle\sigma(w)\sigma(z)\rangle = C\frac{1}{(w-z)^{\frac{1}{16}+\frac{1}{16}}} \cdot \frac{1}{(\bar{w}-\bar{z})^{\frac{1}{16}+\frac{1}{16}}} = C\frac{1}{|w-z|^{\frac{1}{4}}} \tag{862}$$

with $C$ some constant. By choosing a convenient normalization for the field $\sigma$, we can assume $C = 1$.[122]

---

[122] This normalization agrees with the convention that the state corresponding to $\sigma$, $\lvert\mathrm{vac}_P\rangle$, has unit norm $\langle\mathrm{vac}_P|\mathrm{vac}_P\rangle = 1$, cf. (699).

The exponent $\frac{1}{4}$ in (862) is exactly the one appearing in the spin-spin correlator in Ising model at critical temperature (as known from the explicit solution of 2d Ising model), thus corroborating the free fermion-Ising correspondence.

4-point correlator of $\sigma$ fields. As the next example, consider the 4-point function of $\sigma$ fields. From global conformal invariance (cf. Lemma 5.31) one has

$$\langle\sigma(z_1)\sigma(z_2)\sigma(z_3)\sigma(z_4)\rangle = \left|\frac{z_{13}z_{24}}{z_{12}z_{23}z_{34}z_{41}}\right|^{\frac{1}{4}} F(\lambda), \tag{863}$$

where $F(\lambda)$ is some smooth function of the cross-ratio $\lambda = \frac{z_{12}z_{34}}{z_{13}z_{24}} \in \mathbb{CP}^1\backslash\{0,1,\infty\}$. To fix the function $F$, we need some other idea than just global conformal invariance.

In the free fermion theory one has a vanishing descendant of the state $|\mathrm{vac}_P\rangle$ at level 2:

$$(\widehat{L}_{-2} - \frac{4}{3}\widehat{L}_{-1}^2)|\mathrm{vac}_P\rangle = 0 \tag{864}$$

– this can be verified by using the expressions (851) for Virasoro generators in terms of Clifford generators.[123] Thus, the corresponding primary field also has a vanishing descendant:

$$(L_{-2} - \frac{4}{3}L_{-1}^2)\sigma(z) = 0. \tag{865}$$

Thus, by Ward identity (cf. Example 5.22) one has

$$0 = \langle (L_{-2} - \frac{4}{3}L_{-1}^2)\sigma(z_1)\ \sigma(z_2)\sigma(z_3)\sigma(z_4)\rangle = \mathcal{D}\langle\sigma(z_1)\sigma(z_2)\sigma(z_3)\sigma(z_4)\rangle \tag{866}$$

with $\mathcal{D}$ some differential operator in $z_i$'s. Substituting the ansatz (863), we obtain a differential equation on the function $F(\lambda)$ – the hypergeometric equation

$$\left(\lambda(1-\lambda)\frac{\partial^2}{\partial\lambda^2} + (\frac{1}{2}-\lambda)\frac{\partial}{\partial\lambda} + \frac{1}{16}\right) F(\lambda) = 0. \tag{867}$$

This equation has two independent solutions

$$f_{1,2}(\lambda) = (1 \pm \sqrt{1-\lambda})^{\frac{1}{2}} \tag{868}$$

and the general solution has the form $f_1(\lambda)g_1(\bar\lambda) + f_2(\lambda)g_2(\bar\lambda)$ with $g_{1,2}$ some anti-holomorphic functions. Using the conditions that $F$ should be a real, single valued function, fixes the solution to the form

$$F(\lambda) = a(f_1(\lambda)f_1(\bar\lambda) + f_2(\lambda)f_2(\bar\lambda)) \tag{869}$$

with $a$ a constant. Using additionally the OPE $\sigma(w)\sigma(z) \sim \frac{\mathbb{1}}{|w-z|^{\frac{1}{4}}} + \cdots$ (where the normalization follows from $C = 1$ in (862)), one obtains $a = \frac{1}{2}$. Thus, putting everything together, one has

$$\langle\sigma(z_1)\sigma(z_2)\sigma(z_3)\sigma(z_4)\rangle = \frac{1}{2}\left|\frac{z_{13}z_{24}}{z_{12}z_{23}z_{34}z_{41}}\right|^{\frac{1}{4}} (|1+\sqrt{1-\lambda}| + |1-\sqrt{1-\lambda}|). \tag{870}$$

[123] In fact, it is true generally that in the Verma module $\mathbb{V}_{c,h}$ for the Virasoro algebra at central charge $c$ with highest weight $h$ one has a singular vector at level 2 (cf. Remark 5.16), of the form $|\chi\rangle = (L_{-2} + \alpha L_{-1}^2)|h\rangle$, if and only if one has $\begin{vmatrix} 3 & 4h+2 \\ \frac{c}{2}+4h & 6h \end{vmatrix} = 0$ and then $|\chi\rangle$ is a singular vector if $\alpha = -\frac{3}{4h+2}$. In particular, the pair $c = \frac{1}{2}$, $h = \frac{1}{16}$ satisfies the determinant condition and gives $\alpha = -\frac{4}{3}$, i.e., $(L_{-2} - \frac{4}{3}L_{-1}^2)|\frac{1}{16}\rangle$ is a singular vector in the Verma module. Thus, in the irreducible Virasoro module it has to be set to zero.

6.4. $bc$ **system.** The $bc$ system (or "reparametrization ghost system") is a CFT classically defined on a Riemannian surface $\Sigma$ by the action functional signs? normalization?

$$S_{bc} = \frac{i}{2\pi}\int_\Sigma \mathbf{b}\bar{\boldsymbol{\partial}}\mathbf{c} + \bar{\mathbf{b}}\boldsymbol{\partial}\bar{\mathbf{c}} = \frac{1}{\pi}\int_\Sigma d^2 z\,(b\bar\partial c + \bar b \partial \bar c), \tag{871}$$

where the fields are a $(1,0)$-vector field and a quadratic differential

$$\mathbf{c} = c\partial_z \in \Gamma(\Sigma, \underbrace{K^{-1}}_{T^{1,0}}), \quad \mathbf{b} = b(dz)^2 \in \Gamma(\Sigma, K^{\otimes 2}) \tag{872}$$

and their antiholomorphic counterparts

$$\bar{\mathbf{c}} = \bar c\partial_{\bar z} \in \Gamma(\Sigma, \underbrace{\overline{K}^{-1}}_{T^{0,1}}), \quad \bar{\mathbf{b}} = \bar b(d\bar z)^2 \in \Gamma(\Sigma, \overline{K}^{\otimes 2}). \tag{873}$$

Fields $b, c, \bar b, \bar c$ are understood as odd (anticommuting). Since no fractional powers of $K$ appear in the definition of the fields, there is no choice of a spin structure/boundary condition involved.

It is easier to analyze the model in the path integral formalism. One finds the 2-point function

$$\langle b(w)c(z)\rangle = \frac{1}{w-z} \tag{874}$$

as the Green's function for the operator $\frac{1}{\pi}\bar\partial$. Similarly, by the method of Section 4.5.3 one finds the OPE

$$b(w)c(z) \sim \frac{\mathbb{1}}{w-z} + \text{reg.} \tag{875}$$

The stress-energy tensor is

$$T(z) =: 2\partial c(z)\, b(z) + c(z)\partial b(z) : . \tag{876}$$

and similarly for $\overline{T}$. The normal ordering here means that inside a correlator Wick contractions of fields inside $:\cdots:$ are prohibited. Using Wick's lemma as in Section 4.5.3, one computes the OPEs of $b(z)$, $c(z)$, $T(z)$ with $T(w)$ or $\overline{T}(w)$ and finds that:

- $c$ is a primary field of conformal weight $(-1,0)$ (similarly, $\bar c$ is $(0,-1)$-primary),
- $b$ is a primary field of conformal weight $(2,0)$ (similarly, $\bar b$ is $(0,2)$-primary),
- one has the standard OPE of the stress-energy with itself (601), (603), (602) with central charge

$$c = \bar c = -26. \tag{877}$$

*Remark* 6.6. One can consider a modified ghost system, with fields as above and with modified stress-energy tensor

$$T(z) =: \partial c(z) b(z) + j\partial\big(c(z)b(z)\big) : \tag{878}$$

with $j \in \mathbb{R}$ a parameter of the system (the case of reparametrization ghosts corresponds to $j=1$). Then one obtains by similar computations to the above that $c$ is $(-j,0)$-primary, $b$ is $(j+1,0)$-primary and the central charge is $c = -12j^2 - 12j - 2$.

6.4.1. *Correlators on $\mathbb{CP}^1$, soaking field, ghost number anomaly.* Note that the correlator (874) seems to contradict Lemma 5.26 (b): we have a nonvanishing correlator of two primary fields of *different* conformal weight (2 and $-1$). The answer to this seeming paradox is that the field $c$ on $\mathbb{CP}^1$ has zero-modes: there is a 3-dimensional space of holomorphic vector fields on $\mathbb{CP}^1$. When we wrote the Green's function (874), we implicitly imposed the condition that the vector field $c(z)\partial_z$ vanishes together with its first and second derivatives at the point $\infty \in \mathbb{CP}^1$. This is tantamount to inserting a certain field $s$ ("zero-mode soaking field") of conformal weight $(h,\bar{h}) = (0,0)$ at $z=\infty$. So, the correlator (874) is "secretly" a 3-point function

$$\langle s(\infty)b(w)c(z)\rangle_{\mathbb{CP}^1}. \tag{879}$$

From this standpoint, there is no contradiction in the fact that the correlator is nonzero. For three arbitrary points on $\mathbb{CP}^1$, the correlator (879) becomes a Möbius-invariant expression

$$\langle b(z_1)(dz_1)^2\, c(z_2)\partial_{z_2}\, s(z_3)\rangle = \nu_{12}\nu_{13}^3\nu_{23}^{-3}, \tag{880}$$

where

$$\nu_{ij} :\, = \frac{d^{\frac{1}{2}}z_i d^{\frac{1}{2}}z_j}{z_i - z_j} \tag{881}$$

is (the square root of) the Szegö kernel (708). The soaking field $s$ can be written as

$$s = \frac{1}{4}(c\,\partial c\,\partial^2 c)(\bar{c}\,\bar{\partial}\bar{c}\,\bar{\partial}^2\bar{c}). \tag{882}$$

We refer to [43, Section 10] and [29, Section 2.4] for details on soaking fields.

The presence of zero-modes also means that for instance one has

$$\langle 1\rangle_{\mathbb{CP}^1} = \langle \mathrm{vac}|\mathrm{vac}\rangle = 0 \tag{883}$$

(which means that the theory does not satisfy the usual BPZ axiomatics). On the other hand,

$$\langle s(\infty)\rangle_{\mathbb{CP}^1} = \langle s|\mathrm{vac}\rangle = 1. \tag{884}$$

One can assign the "left ghost number" $+1$ to the field $c$ and $-1$ to $b$ and likewise "right ghost number" $+1$ to $\bar{c}$ and $-1$ to $\bar{b}$. Then for a correlator on $\mathbb{CP}^1$ of some collection of differential monomials inserted at points $z_1,\ldots,z_n \in \mathbb{CP}^1$ to be possibly nonzero, one needs the following selection rule to hold: the total left ghost number and the total right ghost number (of the entire expression under the correlator) should both be $+3$:

$$\#c - \#b = 3, \quad \#\bar{c} - \#\bar{b} = 3 \tag{885}$$

This phenomenon is known as the "ghost number anomaly." For example, one has

$$\langle c(z_1)c(z_2)c(z_3)\rangle^{\mathrm{chiral}}_{\mathbb{CP}^1} = z_{12}z_{13}z_{23} \tag{886}$$

Here for brevity we wrote the correlator in the chiral $bc$ system (ignoring the fields $\bar{b},\bar{c}$). Taking the limit $\lim_{z_2\to z_1}\frac{1}{z_{12}}(\cdots)$ in (886), replacing $c(z_2)$ with its Taylor expansion around $z_1$, we have

$$\langle (c\,\partial c)(z_1)c(z_3)\rangle^{\mathrm{chiral}}_{\mathbb{CP}^1} = -z_{13}^2. \tag{887}$$

Taking here the limit $\lim_{z_3\to z_1} \frac{1}{z_{13}^2}\cdots$, replacing $c(z_3)$ with its expansion around $z_1$, we obtain

$$\langle (\frac{1}{2} c\,\partial c\,\partial^2 c)(z_1)\rangle^{\mathrm{chiral}}_{\mathbb{CP}^1} = 1, \tag{888}$$

which is the chiral counterpart of (884).

For a surface $\Sigma$ of genus $g$, the ghost number anomaly (885) is given by Riemann-Roch theorem, as the dimension of the space of holomorphic vector fields minus the dimension of the space of holomorphic quadratic differentials:

$$\dim H^0_{\bar\partial}(\Sigma, K^{-1}) - \dim H^0_{\bar\partial}(\Sigma, K^{\otimes 2}) = 3 - 3g. \tag{889}$$

6.4.2. *Operator formalism for the bc system.* One can develop the canonical quantization picture for the $bc$ system, similarly to how we did it for the other free field models before. Then one obtains the Heisenberg fields on $\mathbb{C}\backslash\{0\}$,

$$\widehat{c}(z) = \sum_{n\in\mathbb{Z}} \widehat{c}_n z^{-n+1}, \quad \widehat{b}(z) = \sum_{n\in\mathbb{Z}} \widehat{b}_n z^{-n-2} \tag{890}$$

with operators $\widehat{c}_n, \widehat{b}_n$ subject to the anticommutation relations

$$[\widehat{b}_n, \widehat{c}_m]_+ = \delta_{n,-m}\widehat{\mathbb{1}}, \quad [\widehat{b}_n, \widehat{b}_m]_+ = 0, \quad [\widehat{c}_n, \widehat{c}_m]_+ = 0. \tag{891}$$

One has similar mode expansions and anticommutation relations for $\bar{b}, \bar{c}$. Here the the splitting of the mode operators into creation and annihilation operators is as follows:

$$\underbrace{\ldots, \widehat{c}_{-1}, \widehat{c}_0, \widehat{c}_1}_{\text{creation}}, \underbrace{\widehat{c}_2, \widehat{c}_3, \ldots}_{\text{annihilation}}, \qquad \underbrace{\ldots, \widehat{b}_{-3}, \widehat{b}_{-2}}_{\text{creation}}, \underbrace{\widehat{b}_{-1}, \widehat{b}_0, \widehat{b}_1, \ldots}_{\text{annihilation}} \tag{892}$$

and similarly for $\widehat{\bar{b}}_n, \widehat{\bar{c}}_n$.[124] The vacuum vector $|\mathrm{vac}\rangle$ is killed by annihilation operators, while creation operators produce nonzero vectors out of $|\mathrm{vac}\rangle$. The hermitian conjugates are $(\widehat{b}_n)^+ = \widehat{b}_{-n}$, $(\widehat{c}_n)^+ = \widehat{c}_{-n}$. The special vector $|s\rangle$ corresponding to the soaking field (882) is

$$|s\rangle = \widehat{c}_{-1}\widehat{c}_0\widehat{c}_1\widehat{\bar{c}}_{-1}\widehat{\bar{c}}_0\widehat{\bar{c}}_0|\mathrm{vac}\rangle. \tag{893}$$

The space of states $\mathcal{H}$ is generated freely by acting on the vector $|\mathrm{vac}\rangle$ repeatedly with creation operators (i.e., $\mathcal{H}$ is a Verma module for the Clifford algebra (891), tensored with the conjugate one).

Reproducing the 2-point correlation function (874) in the language of operator quantization, we have (assuming $|w| > |z|$ for simplicity):

$$\langle b(w)c(z)\rangle = \langle s|\widehat{b}(w)\widehat{c}(z)|\mathrm{vac}\rangle = \sum_{m,n\in\mathbb{Z}} \langle s|\widehat{b}_n\widehat{c}_m|\mathrm{vac}\rangle w^{-n-2}z^{-m+1}. \tag{894}$$

Here we notice that the expression $\langle s|\widehat{b}_n\widehat{c}_m|\mathrm{vac}\rangle$ has the following properties:

- Vanishes for $\widehat{c}_m$ an annihilation operator (since then $\widehat{c}_n|\mathrm{vac}\rangle = 0$), i.e., for $m \geq 2$.
- Vanishes for $n \neq -m$ and $\widehat{b}_n$ an annihilation operator (since $\widehat{b}_n$ commutes past $\widehat{c}_m$ and acts on $|\mathrm{vac}\rangle$), i.e., for $n \neq -m$, $n \geq -1$.
- Vanishes for $\widehat{b}_n$ a creation operator (in this case $\langle s|\widehat{b}_n = 0$), i.e., for $n \leq -2$.

[124] This nontrivial splitting of modes into creation and annihilation operators is forced by the field-state correspondence: one wants limits $\lim_{z\to 0}\widehat{\Phi}(z)|\mathrm{vac}\rangle$ to be well-defined and nonzero for $\Phi = b, c, \bar{b}, \bar{c}$.

- Vanishes for $\widehat{c}_m$ a creation operator if $n \neq -m$ (in this case, $\widehat{c}_m$ commutes to the left past $\widehat{b}_n$ and annihilates $\langle h|$), i.e., for $n \neq -m$, $m \leq 1$.

Thus, the only surviving terms in (894) are $n = -m \geq -1$, i.e., we have

(895) $$\langle b(w)c(z)\rangle = \sum_{n\geq -1} \langle s|\underbrace{\widehat{b}_n\widehat{c}_{-n}}_{\widehat{\mathbb{1}}-\widehat{c}_{-n}\widehat{b}_n}|\mathrm{vac}\rangle w^{-n-2}z^{n+1} = \sum_{n\geq -1} w^{-n-2}z^{n+1} = \\ = \frac{1}{w}\left(1+\frac{z}{w}+\left(\frac{z}{w}\right)^2+\cdots\right) = \frac{1}{w-z}$$

6.5. **Bosonic string.** We start with outlining the heuristic idea of bosonic string theory. One wants to integrate over maps $\phi$ of a smooth surface $\Sigma$ (worldsheet) to the target $\mathbb{R}^D$ (for some dimension $D \geq 1$):

(896) $$Z_{\mathrm{string}}(\Sigma,\mathbb{R}^D) = \int_{\mathrm{Met}(\Sigma)} \mathcal{D}g \int_{\mathrm{Map}(\Sigma,\mathbb{R}^D)} \mathcal{D}\phi\, e^{-S_{\mathrm{Polyakov}}(g,\phi)}$$

where

(897) $$S_{\mathrm{Polyakov}}(g,\phi) = \sum_{k=1}^{D} \frac{1}{2}\int_\Sigma \mathrm{dvol}_g d\phi^k \wedge *d\phi^k$$

is the action for $D$ non-interacting free bosons $\phi^1,\ldots,\phi^D$ on $\Sigma$; the action depends on a choice of Riemannian metric $g$ on the surface, and this choice is averaged over in (896). The integrand in (896) is invariant under diffeomorphisms of $\Sigma$, and one wants to switch to integration over the quotient $\mathrm{Met}(\Sigma)\times\mathrm{Map}(\Sigma,\mathbb{R}^D)/\mathrm{Diff}(\Sigma)$.[125] Next, one writes the metric as

(898) $$g = e^{2\sigma} g_0$$

where $g_0$ is the canonical "uniformization" metric of constant scalar curvature $K \in \{0,\pm 1\}$ representing the conformal class of $g$ – the metric arising from uniformization theorem; $\Omega = e^{2\sigma}$ with $\sigma \in C^\infty(\Sigma)$ is the Weyl factor, transforming $g_0$ into $g$; one calls $\sigma$ the Liouville field. With this in mind, the path integral (896) becomes the integral over

(899)
$$\{\text{conformal structures on }\Sigma\}\times\{\text{Weyl factors }\Omega = e^{2\sigma}\}\times \mathrm{Map}(\Sigma,\mathbb{R}^D)/\mathrm{Diff}(\Sigma) \simeq \\ \simeq \frac{\{\text{conformal structures on }\Sigma\}}{\mathrm{Diff}(\Sigma)}\times\{\text{Weyl factors }\Omega = e^{2\sigma}\}\times\mathrm{Map}(\Sigma,\mathbb{R}^D)/\mathrm{Diff}(\Sigma)$$

where in the first factor on the r.h.s. we recognize the moduli space of conformal structures $\mathcal{M}_\Sigma$. For the integral over the quotient by diffeomorphisms, one employs the Faddeev-Popov gauge-fixing mechanism, which results in the path integral
(900)
$$\int_{\mathcal{M}_\Sigma}\mathcal{D}\xi \int_{C^\infty(\Sigma)}\mathcal{D}\sigma \int_{\Pi\mathfrak{X}(\Sigma)\times\Pi\Gamma(\Sigma,K^{\otimes 2}\oplus\overline{K}^{\otimes 2})} \mathcal{D}c\mathcal{D}\bar{c}\mathcal{D}b\mathcal{D}\bar{b}e^{-S_{bc}} \int_{\mathrm{Map}(\Sigma,\mathbb{R}^D)} e^{-S_{\mathrm{Polyakov}}}$$

[125] Heuristically, transitioning to integration over the quotient rescales the result by an "infinite constant" – the volume of $\mathrm{Diff}(\Sigma)$.

where the auxiliary fields $c\partial_z+\bar{c}\partial_{\bar{z}}$ (an odd vector field) and $b(dz)^2+\bar{b}(d\bar{z})^2$ (an odd quadratic differential) appear as Faddeev-Popov ghosts corresponding to the quotient by diffeomorphisms (or "reparametrizations," hence the name "reparametrization ghosts"); the action $S_{bc}$ is as in (871). The Gaussian integral over ghosts is an integral representation of a Jacobian, canceling the dependence of the integral over a section of the quotient {conf. structures}/Diff$(\Sigma)$ on the choice of the section.

Exploiting the result (32), we have that the bosonic string path integral is

$$\int_{\mathcal{M}_\Sigma} \mathcal{D}\xi \int_{C^\infty(\Sigma)} \mathcal{D}\sigma\, Z_{\mathrm{CFT}}\left( \begin{array}{c} D \text{ free bosons} \\ +bc \text{ system} \end{array}, \xi \right) e^{icS_{\mathrm{Liouville}}(\sigma)} \tag{901}$$

where

$$c = D - 26 \tag{902}$$

is the central charge of the CFT comprised of $D$ free bosons and a single $bc$ system. The case $D=26$ is special and corresponds to the so-called "critical" bosonic string – in this case the central charge vanishes and the integrand is independent of the Liouville field $\sigma$.

In summary, bosonic string is the conformal field theory comprised of $D$ free bosons and a $bc$ system, with classical action fix normalizations

$$S_{\mathrm{string}} = \frac{1}{\pi}\int_\Sigma d^2z \Big( \underbrace{\sum_{k=1}^D \partial\phi^k \bar\partial\phi^k}_{D \text{ free bosons}} + \underbrace{b\bar\partial c + \bar{b}\partial\bar{c}}_{bc \text{ system}} \Big) \tag{903}$$

where to get the full string path integral one needs to integrate the CFT partition function (or correlator) over the moduli space $\mathcal{M}_\Sigma$ (and if $D\neq 26$, also factor in the Liouville path integral).[126] Lecture 33, 11/14/2022

6.5.1. *The BRST differential $Q$ in bosonic string.* Fix $D=26$. Consider the fields

$$\begin{aligned} J &= :cT_{\mathrm{bosons}} + \frac{1}{2}cT_{bc} + \frac{3}{2}\partial^2 c: = :\sum_{k=1}^D -\frac{1}{2}c\partial\phi^k\partial\phi^k + c\,\partial c\, b + \frac{3}{2}\partial^2 c:, \\ \bar{J} &= :\bar{c}\overline{T}_{\mathrm{bosons}} + \frac{1}{2}\bar{c}\overline{T}_{bc} + \frac{3}{2}\bar\partial^2 \bar{c}: = :\sum_{k=1}^D -\frac{1}{2}\bar{c}\bar\partial\phi^k\bar\partial\phi^k + \bar{c}\,\bar\partial\bar{c}\,\bar{b} + \frac{3}{2}\bar\partial^2 \bar{c}: \end{aligned} \tag{904}$$

They satisfy the following properties.

- $J$ is a holomorphic $(1,0)$-primary field, $\bar{J}$ is an antiholomorphic $(0,1)$-primary field.
- The OPE $J(w)J(z)$ does not contain a first-order pole[127] (but contains second and third-order poles) and similarly for $\bar{J}(w)\bar{J}(z)$. The mixed OPE $J(w)\bar{J}(z)$ is regular.

[126] In a jargon, one couples the CFT (903) on $\Sigma$ with "2d gravity on $\Sigma$."

[127] This property relies on $D=26$. More explicitly, if one defines $J_\alpha =: cT_{\mathrm{bosons}} + \frac{1}{2}cT_{bc} + \alpha\partial^2 c:$, then one has the OPE

$$J_\alpha(w)J_\alpha(z) \sim \frac{(3-\frac{D}{2}+4\alpha)\,c\partial c}{(w-z)^3} + \frac{(\frac{3}{2})-\frac{D}{4}+2\alpha)\,c\partial^2 c}{(w-z)^2} + \frac{(\frac{2}{3}-\frac{D}{12}+\alpha)c\partial^3 c + (-\frac{3}{2}+\alpha)\partial c\partial^2 c}{w-z} + \mathrm{reg.}$$

Here all fields on the right are at the point $z$. In particular, for $D=26$ and $\alpha=\frac{3}{2}$ one has $J(w)J(z)\sim -\frac{4c\partial c}{(w-z)^3} - \frac{2c\partial^2 c}{(w-z)^2} + \mathrm{reg.}$

- One can introduce an operator $Q\colon V_z \to V_z$ given by

$$Q\colon \Phi(z) \mapsto \frac{1}{2\pi i}\oint_{\gamma_z} (dw J(w) + d\bar{w}\bar{J}(w))\Phi(z), \tag{905}$$

with $\gamma_z$ a contour around $z$. This operator satisfies

$$Q^2 = 0, \tag{906}$$

as a consequence of (905) (proven by the contour integration technique of Section 5.2.2). One can equip $V$ with $\mathbb{Z}$-grading by *(total) ghost number*, by prescribing the ghost numbers to elementary fields as follows:

| field | $c$ | $b$ | $\bar{c}$ | $\bar{b}$ | $\phi^k$ |
|---|---|---|---|---|---|
| ghost number | 1 | $-1$ | 1 | $-1$ | 0 |

– This is the sum of the left and right ghost numbers of Section 6.4.1. According to this $\mathbb{Z}$-grading, $V$ is a cochain complex, with differential $Q$ (known as the "BRST operator"), increasing the ghost number by $+1$.
- The stress-energy tensor satisfies

$$T = Q(b), \quad \overline{T} = Q(\bar{b}) \tag{907}$$

– the stress-energy tensor is $Q$-exact.

*Remark* 6.7. If one omits the $\frac{3}{2}\partial^2 c$ term in $J$ and likewise in $\bar{J}$ then the residue of the first-order pole in $JJ$ OPE will be nonzero, but it will be exact, so the operator $Q$ would not change. Also, with this modification $J, \bar{J}$ would not be primary.

*Remark* 6.8. Fields $J, \bar{J}$ are also $Q$-exact:

$$J = Q(:bc:), \quad \bar{J} = Q(:\bar{b}\bar{c}:). \tag{908}$$

We also note that in the computation of the r.h.s. it is the double Wick contractions that result in $\frac{3}{2}\partial^2 c$ term in $J$ in the l.h.s.; in this sense, the term $\frac{3}{2}\partial^2 c$ should be regarded as a quantum ("1-loop" in the language of Feynman diagrams) correction to $J$.[128]

### 6.6. Topological conformal field theories.

**Definition 6.9.** A CFT is called *topological* (or TCFT) is the space of fields $V$ (and the space of states $\mathcal{H}$) is endowed with the structure a cochain complex with differential $Q$ of degree $+1$, such that the stress-energy tensor is $Q$-exact,

$$T = Q(G), \quad \overline{T} = Q(\overline{G}), \tag{909}$$

with $G, \overline{G}$ some fields of cohomological degree $-1$, such that

(a) One has regular OPEs

$$G(w)G(z), \quad G(w)\overline{G}(z), \quad \overline{G}(w)\overline{G}(z). \tag{910}$$

(b) $G$ is a holomorphic $(2,0)$-primary field, $\overline{G}$ is an antiholomorphic $(0,2)$-primary field.

[128]In a bit more detail: in the classical field theory defined by the action functional (903) one has an odd symmetry $Q_{cl} \in \mathfrak{X}(\mathrm{Fields}_\Sigma)$ acting on the space of classical fields and squaring to zero. For this symmetry one has an associated Noether current $dz J_{cl} + d\bar{z}\bar{J}_{cl}$, where $J_{cl}, \bar{J}_{cl}$ are given by the formulae (904) without the $\frac{3}{2}\partial^2 c$, $\frac{3}{2}\bar{\partial}^2\bar{c}$ terms and without normal ordering. Thus, the quantum fields (904) are the "naive" quantization of $J_{cl}, \bar{J}_{cl}$ (replacing a differential polynomial in free classical fields by a normally ordered expression), plus a "quantum correction" $\frac{3}{2}\partial^2 c$, $\frac{3}{2}\bar{\partial}^2\bar{c}$.

(c) There exist fields $J, \bar{J} \in V$ of degree $+1$ and conformal weights $(1,0)$ for $J$ and $(0,1)$ for $\bar{J}$, such that:
- The 1-form-valued field $\mathbb{J}(z) = dz\, J + d\bar{z}\, \bar{J} \in V_z \otimes T_z^*\Sigma$ is $d$-closed under the correlator (or equivalently $\bar\partial J - \partial \bar{J} = 0$).[129]
- The differential $Q$ is given by

sign?

$$QΦ(z) = \frac{1}{2\pi i} \oint_{\gamma_z} \mathbb{J}(w)\Phi(z). \tag{911}$$

- $\mathbb{J}$ satisfies

$$\oint_{\gamma_z} \mathbb{J}(w)\mathbb{J}(z) = 0. \tag{912}$$

This property implies $Q^2 = 0$.

Do we need an axiom that $\mathbb{1}$ is not $Q$-exact? Do we need to ask that $\mathbb{J}$ is primary?

In particular, bosonic string with $D = 26$ is an example of a TCFT.

One can introduce mode operators for $G, \overline{G}$, defined similarly to (638):

$$G_n\Phi(z) = \frac{1}{2\pi i}\oint_{\gamma_z} dw\,(w-z)^{n+1}G(w)\Phi(z), \quad \overline{G}_n\Phi(z) = \frac{1}{2\pi i}\oint_{\gamma_z} d\bar{w}(\bar{w}-\bar{z})^{n+1}\overline{G}(w)\Phi(z) \tag{913}$$

Then the property (909) implies that one has[130]

$$L_n = [Q, G_n], \quad \overline{L}_n = [Q, \overline{G}_n] \tag{914}$$

for $n \in \mathbb{Z}$, i.e., Virasoro generators are $Q$-exact. In turn this implies that the central charge of the CFT must vanish (because the coefficient of the fourth-order pole in $TT$ OPE must be $Q$-exact; since it is proportional to identity, it must vanish[131]):

$$c = \bar{c} = 0. \tag{915}$$

Property (910) implies

$$[G_n, G_m] = 0, \quad [G_n, \overline{G}_m] = 0, \quad [\overline{G}_n, \overline{G}_m] = 0. \tag{916}$$

From the OPEs between $T, \overline{T}$ and $G, \overline{G}$, which are encoded in the axiom (b) above:

$$\begin{gathered} T(w)G(z) \sim \frac{2G(z)}{(w-z)^2} + \frac{\partial G(z)}{w-z} + \text{reg.}, \\ \overline{T}(w)\overline{G}(z) \sim \frac{2\overline{G}(z)}{(\bar{w}-\bar{z})^2} + \frac{\bar\partial\, \overline{G}(z)}{\bar{w}-\bar{z}} + \text{reg.}, \\ T(w)\overline{G}(z) \sim \text{reg.},\ \overline{T}(w)G(z) \sim \text{reg.}, \end{gathered} \tag{917}$$

one has the commutation relations

$$[L_n, G_m] = (n-m)G_{n+m}, \quad [\overline{L}_n, \overline{G}_m] = (n-m)\overline{G}_{n+m}, \quad [L_n, \overline{G}_m] = [\overline{L}_n, G_m] = 0. \tag{918}$$

---

[129] In some TCFTs one has that $\bar\partial J$ and $\partial\bar{J}$ vanish separately. This means that $Q$ splits into two commuting differentials $Q = Q_L + Q_R$ which square to zero separately. This extra symmetry is present, e.g., in bosonic string theory, but fails in some other examples, see e.g. [29].

[130] We write $[A, B] = AB - (-1)^{|A|\cdot|B|}BA$ for the supercommutator of two operators $A, B$. It is the usual commutator if either $A$ or $B$ (or both) are even and it is the anticommutator if $A$ and $B$ are odd.

[131] An equivalent argument: the commutator $[L_n, L_m] = [Q, [G_n, [Q, G_m]]]$ has the form $[Q, -]$, so it cannot contain a nonzero term proportional to identity/central element (with is not of the form $[Q, -]$).

**Lemma 6.10.** *In a TCFT, assume that $\Phi_1, \ldots, \Phi_n \in V$ are $Q$-closed elements. Then:*

*(i) The correlator on $\mathbb{CP}^1$*

$$\langle \Phi_1(z_1) \cdots \Phi_n(z_n) \rangle \tag{919}$$

*is a constant function on the configuration space $C_n(\mathbb{CP}^1)$.*

*(ii) For any $\Psi \in V$, one has*

$$\langle Q\Psi(z_0)\ \Phi_1(z_1) \cdots \Phi_n(z_n) \rangle = 0 \tag{920}$$

*– the correlator of a $Q$-exact field with several $Q$-closed fields vanishes.*

*Proof.* For, (i) consider the derivative of the correlator (919) in $z_i$, $i = 1, \ldots, n$. We have

$$\begin{aligned} \partial_{z_i} \langle \Phi_1(z_1) \cdots \Phi_n(z_n) \rangle &= \langle \Phi_1(z_1) \cdots \underbrace{L_{-1}\Phi_i}_{QG_{-1}\Phi_i}(z_i) \cdots \Phi_n(z_n) \rangle = \\ &= \pm \frac{1}{2\pi i} \oint_{\gamma_{z_i}} \langle \mathbb{J}(w) \Phi_1(z_1) \cdots G_{-1}\Phi_i(z_i) \cdots \Phi_n(z_n) \rangle \end{aligned} \tag{921}$$

where $\gamma_{z_i}$ is a contour going around $z_i$ and not enclosing any other $z_j$'s. One then deforms $\gamma_{z_i}$ into a collection of contours going around $z_j$'s for $j \neq i$: $\gamma_{z_i} \sim \sqcup_{i \neq j} - \gamma_{z_j}$ (cf. Section 5.6). Thus, one has

$$\partial_{z_i} \langle \Phi_1(z_1) \cdots \Phi_n(z_n) \rangle = \sum_{j \neq i} \langle \Phi_1(z_1) \cdots \underbrace{Q\Phi_j}_{=0}(z_j) \cdots G_{-1}\Phi_i \cdots \Phi_n(z_n) \rangle = 0. \tag{922}$$

So, we obtain that all holomorphic derivatives of the correlator vanish; by a similar argument, the antiholomorphic derivatives vanish too. Hence, the correlator is constant.

The proof of (ii) is similar: one represents $Q$ acting on $\Psi$ by a contour integral around $z_0$ and then deform the contour to a collection of contours going around $z_j$, $j \neq 0$; those give correlators containing $Q\Phi_j = 0$. $\square$

edit? (or maybe it's ok already..)

*Remark* 6.11. The statement and proof of Lemma 6.10 actually extends to correlators on Riemannian surfaces $\Sigma$ of any genus $g$, since on any $\Sigma$ the 1-cycle $\gamma_{z_i}$ is *homologous* (though not homotopic for $g > 0$) to $\sqcup_{i \neq j} - \gamma_{z_j}$. Since one has $d\mathbb{J} = 0$ (under a correlator, away from the punctures $z_j$), this homology statement is sufficient to justify the switch of contours in (921).

**Lemma 6.12.** *If a field $\Phi \in V$ is $Q$-closed and has conformal weight $h \neq 0$, then $\Phi$ is $Q$-exact.*

*Proof.* Since $\Phi$ has conformal weight $h$, we have

$$h\Phi = L_0 \Phi = (QG_0 + \cancel{G_0 Q})\Phi = QG_0\Phi. \tag{923}$$

Thus, we have

$$\Phi = Q(\frac{1}{h} G_0 \Phi). \tag{924}$$

$\square$

Similarly, one shows that if a $Q$-closed field has $\bar{h} \neq 0$, then it is $Q$-exact. Therefore, a nontrivial $Q$-cocycle (homogeneous w.r.t. grading by conformal weight) must have $(h, \bar{h}) = 0$. By "nontrivial" we mean "not $Q$-exact" or equivalently defining a nonzero element in the cohomology of $Q$, $H_Q(V)$.

**Example 6.13.** Here is an example of a $Q$-cocycle in bosonic string: fix a unit "momentum" vector $p \in (\mathbb{R}^D)^*$, $||p|| = 1$. Then the field

$$\Phi =: c\bar{c}\, e^{i\sqrt{2}\sum_{k=1}^{D} p_k \phi^k} : \tag{925}$$

is a nontrivial $Q$-cocycle. (In string theory, in Lorentzian signature on $\mathbb{R}^D$ it is called the "tachyon field.") Note that the condition $||p|| = 1$ guarantees that $\Phi$ has confromal weight $(0, 0)$.

**Example 6.14.** In any TCFT one has $QJ = Q\bar{J} = 0$, as a consequence of (912). Thus, using homotopy (924) one has

$$J = Q(G_0(J)), \quad \bar{J} = Q(\overline{G}_0(\bar{J})), \tag{926}$$

i.e., fields $J, \bar{J}$ are always $Q$-exact. This generalizes the observation (908) in bosonic string theory.

6.6.1. *Witten's descent equation.* Witten's descent equation is a sequence of equations on a tower of $p$-form valued fields $\Phi^{(0)}$, $\Phi^{(1)}$, $\Phi^{(2)}$, where[132]

$$\Phi^{(p)}(z) \in V_z^{(p)} = V_z \otimes \wedge^p T_z^* \Sigma \tag{927}$$

(we denoted $V_z^{(p)}$ the space of $p$-form-valued fields at $z$). Descent equation reads

$$d\Phi^{(p-1)} = Q\Phi^{(p)}, \qquad p = 0, 1, 2. \tag{928}$$

Here we understand that $\Phi^{(-1)} :\, = 0$. Thus, explicitly, the equations are:

$$Q\Phi^{(0)} = 0 \tag{929}$$

$$Q\Phi^{(1)} = d\Phi^{(0)} \tag{930}$$

$$Q\Phi^{(2)} = d\Phi^{(1)} \tag{931}$$

One can think of this sequence as follows: one fixed a $Q$-cocycle $\Phi^{(0)}$ – a "0-observable," then one wants to solve (930) for the "1-observable" $\Phi^{(1)}$ and subsequently solve (931) for the 2-observable $\Phi^{(2)}$.

*Remark* 6.15. Descent equations (928) are meaningful not just in dimension 2 (then $p$ goes up to the dimension of the manifold). Originally, they appeared in the work of Witten on 4-dimensional Donaldson theory [39].

From Lemma 6.10, correlators of $Q$-closed 0-observables $\langle \Phi_1^{(0)}(z_1) \cdots \Phi_n^{(0)}(z_n) \rangle$ are constant functions of positions $z_1, \ldots, z_n$ (as long as points are distinct).

Equation (930) implies that one can construct an "extended observable" (localized on a 1-cycle rather than at a point)

$$\oint_\gamma \Phi^{(1)} \tag{932}$$

[132] Recall that we already encountered a situation where it is convenient to consider form-valued observables, – transformation of primary fields and Ward identity for primary fields, cf. (662), (691).

with $\gamma$ some closed contour. Then (930) implies by Stokes' theorem that (932) is $Q$-closed:[133]

$$Q\oint_\gamma \Phi^{(1)}=0 \tag{933}$$

By repeating the argument of Lemma 6.10, we have that, given $Q$-cocycles $\Phi^{(0)},\Phi_1^{(0)},\ldots,\Phi_n^{(0)}$, the correlator

$$\langle \oint_\gamma \Phi^{(1)}\ \ \Phi_1^{(0)}(z_1)\cdots\Phi_n^{(n)}(z_n)\rangle \tag{934}$$

does not change when one moves points $z_i$ or deforms the contour $\gamma$ (as long as the points and the contour keep disjoint), however it can change when some point $z_i$ crosses $\gamma$.

The correlator (934) is an example of a "topological correlator" – one invariant under small deformations insertion points of fields (and the contour over which the 1-observable is integrated).

Edit, think through more

A 2-observable $\Phi^{(2)}$ gives rise to a $Q$-closed extended observable

$$\int_\Sigma \Phi^{(2)} \tag{935}$$

and can be understood as defining an infinitesimal deformation of a TCFT, deforming the correlators as

$$\begin{aligned}\langle\Phi_1(z_1)\cdots\Phi_n(z_n)\rangle &\mapsto \\ \mapsto \langle\Phi_1(z_1)\cdots\Phi_n(z_n)\rangle &+\epsilon\Big\langle\left(\int_{\Sigma-\sqcup_{i=1}^n D_i}\Phi^{(2)}\right)\Phi_1(z_1)\cdots\Phi_n(z_n)\Big\rangle\end{aligned} \tag{936}$$

Here $D_i$ is a small disk centered at $z_i$; $\epsilon$ is the infinitesimal deformation parameter. This deformation should be accompanied by a deformation of the rest of TCFT data, $Q,\mathbb{J},G,\overline{G},T,\overline{T}$, so that the relations of TCFT hold (up to $O(\epsilon^2)$) for the deformed package.

?

The deformation (936) in the path integral language can be interpreted as the deformation of the action functional,

$$S\mapsto S-\epsilon\int_\Sigma \Phi^{(2)}. \tag{937}$$

6.6.2. *Canonical solution of descent equations using the G-field.* In a TCFT, one can find a canonical solution of the equation (928) starting from any $Q$-cocycle $\Phi^{(0)}$.

Consider the operator

$$\Gamma=-dz\,G_{-1}-d\bar{z}\,\overline{G}_{-1}\colon\ V_z^{(p)}\to V_z^{(p+1)}, \tag{938}$$

where $G_{-1},\overline{G}_{-1}$ are particular mode operators of the fields $G,\overline{G}$, cf. (913). Note that the commutator of $\Gamma$ with $Q$ is the de Rham operator:

$$[Q,\Gamma]=dz[Q,G_{-1}]+d\bar{z}[Q,\overline{G}_{-1}]=dzL_{-1}+d\bar{z}\overline{L}_{-1}=dz\partial_z+d\bar{z}\partial_{\bar{z}}=d. \tag{939}$$

[133] We understand (932) as an element of $V$ – in that sense it is clear what acting by $Q$ means. Equivalently, the action of $Q$ on (932) can be understood as $\frac{1}{2\pi i}\int_{\partial U_\gamma\ni w}\mathbb{J}(w)\oint_{\gamma\ni z}\Phi^{(1)}(z)$, with the integral being over the boundary of a thickening $U_\gamma$ of the contour $\gamma$.

Note also that one has

$$[d, \Gamma] = 0, \tag{940}$$

since operators $L_{-1}, \overline{L}_{-1}$ commute with $G_{-1}, \overline{G}_{-1}$, cf. (918).

**Lemma 6.16.** *Given a $Q$-cocycle $\Phi^{(0)}$, the sequence*

$$\Phi^{(0)}, \quad \Phi^{(1)} := \Gamma\Phi^{(0)}, \quad \Phi^{(2)} := \frac{1}{2}\Gamma^2\Phi^{(0)} \tag{941}$$

*solves the descent equation (928).*

*Proof.* The equation (929) is given, since $\Phi^{(0)}$ is a $Q$-cocycle. For (930) we have

$$Q\Gamma\Phi^{(1)} \underset{(939)}{=} (d + \Gamma Q)\Phi^{(0)} = d\Phi^{(0)}. \tag{942}$$

For (931) we have

$$Q\frac{1}{2}\Gamma\Gamma\Phi^{(0)} = \frac{1}{2}(d + \Gamma Q)\Phi^{(1)} \underset{(930)}{=} \frac{1}{2}(d\Phi^{(1)} + \Gamma d\Phi^{(0)}) = \tag{943}$$

$$\underset{(940)}{=} \frac{1}{2}(d\Phi^{(1)} + d\Gamma\Phi^{(0)}) = d\Phi^{(1)}.$$

□

**Example 6.17.** In bosonic string, starting with the $Q$-cocycle (925) and applying the canonical descent construction (941), we obtain the descent sequence

$$\Phi^{(0)} =: c\bar{c}V_p :, \quad \Phi^{(1)} =: (-dz\, \bar{c} + dz\, c)V_p :, \quad \Phi^{(2)} =: dz\, d\bar{z}V_p :, \tag{944}$$

where we denoted $V_p = e^{i\sqrt{2}\sum_k p_k\phi^k}$, with the momentum satisfying $||p|| = 1$.

For $\Phi = \Phi^{(0)}$ a $Q$-cocycle, one can assemble the descendants into the field valued in nonhomogeneous forms

$$\widetilde{\Phi} := e^{\Gamma}\Phi = \Phi + \Gamma\Phi + \frac{1}{2}\Gamma^2\Phi \tag{945}$$

– the "total descendant." Then the descent equation (928) can be written as

$$(d - Q)\widetilde{\Phi} = 0. \tag{946}$$

More generally, for $\Phi$ not necessarily $Q$-closed, one has an easily proven identity

$$(d - Q)e^{\Gamma}\Phi = -e^{\Gamma}(Q\Phi). \tag{947}$$

*Remark* 6.18. Given a collection of $Q$-cocycles $\Phi_1^{(0)}, \ldots, \Phi_n^{(0)}$, the correlator of their total descendants (945) is a *closed* form on the open configuration space:

$$\langle\widetilde{\Phi}_1(z_1)\cdots\widetilde{\Phi}_n(z_n)\rangle \in \Omega_{\mathrm{closed}}(C_n(\mathbb{CP}^1)), \tag{948}$$

since one has

$$d\langle\widetilde{\Phi}_1(z_1)\cdots\widetilde{\Phi}_n(z_n)\rangle = \sum_{j=1}^{n}\langle\widetilde{\Phi}_1(z_1)\cdots\underbrace{(d-Q)\widetilde{\Phi}_j(z_j)}_{0}\cdots\widetilde{\Phi}_n(z_n)\rangle = 0. \tag{949}$$

In fact, the form (948) is $PSL_2(\mathbb{C})$-basic and thus descends to a closed form on the moduli space $\mathcal{M}_{0,n} \simeq C_n(\mathbb{CP}^1)/PSL_2(\mathbb{C})$.

In the example of bosonic string theory the degree of the form (948) is

$$\sum_{j=1}^{n} \mathrm{gh}(\Phi_j^{(0)}) - 6 \tag{950}$$

– the sum of the ghost numbers of the fields $\Phi_j^{(0)}$ minus the total (left plus right) ghost number anomaly.

By Remark 6.11, the correlator (948) can be considered on a surface $\Sigma$ of any genus, yielding again a closed form on the configuration space.

6.6.3. *BV algebra structure on Q-cohomology.*

**Definition 6.19.** A Batalin-Vilkovisky algebra (or "BV algebra") is a $\mathbb{Z}$-graded supercommutative unital algebra $(W, \cdot, \mathbb{1})$ equipped additionally with:

- A degree $-1$ Poisson bracket[134] (or "BV bracket," or "antibracket")

$$(,)\colon W \otimes W \to W \tag{951}$$

  which is a derivation in both slots and satisfies (graded) Jacobi identity.
- A degree $-1$ operator $\Delta\colon W \to W$ (the "BV Laplacian") satisfying the Leibniz identity for a second-order differential operator

$$\Delta(xyz) \pm \Delta(xy)z \pm \Delta(xz)y \pm \Delta(yz)x \pm xy\Delta(z) \pm xz\Delta(y) \pm yz\Delta(x) = 0 \tag{952}$$

  and the properties

$$\Delta(\mathbb{1}) = 0, \tag{953}$$

$$\Delta(xy) = \Delta(x)y + (-1)^{|x|}x\Delta(y) + (-1)^{|x|}(x, y). \tag{954}$$

  In particular, the BV bracket arises as the defect of the first order Leibniz identity for $\Delta$.

In the setting of TCFT, we consider the graded vector space $W = H_Q(V)$ – the cohomology of $Q$ (with grading by the "ghost number"), the unit element is $\mathbb{1}$ – the cohomology class of the identity field.

The supercommutative product on $W$ is given by OPEs. Notice that if $\Phi_1, \Phi_2$ are two nontrivial $Q$-cocycles, the OPE has the form

$$\Phi_1(w)\Phi_2(z) \sim \sum_{\Phi} (w - z)^{h(\Phi)}(\bar{w} - \bar{z})^{\bar{h}(\Phi)}\Phi(z) \tag{955}$$

where we used that $\Phi_{1,2}$ must have conformal weight $(0, 0)$ and used Lemma 5.24. Terms in the right hand side of the OPE must also be $Q$-closed, and the ones containing nontrivial $Q$-cocycles $\Phi$ have to contribute with exponents $h(\Phi) = \bar{h}(\Phi) = 0$. Therefore, for $\Phi_1, \Phi_2$ two $Q$-cocycles one has an OPE of the form

$$\Phi_1(w)\Phi_2(z) \sim (\Phi_1 \cdot \Phi_2)(z) \quad \text{modulo } Q\text{-exact terms.} \tag{956}$$

with $\Phi_1 \cdot \Phi_2$ some $Q$-cocycle. Thus, in $Q$-cohomology OPE, is always constant and induces a supercommutative product.

---

[134]The grading convention that we use here, with $(,)$ and $\Delta$ of degree $-1$, is adapted to BV algebras arising from 2d TCFT. In the setting where BV algebras originally appeared – Batalin-Vilkovisky quantization of gauge theories – the natural convention is to assign degree $+1$ to $(,)$ and $\Delta$ (the same degree as the operator $Q$, whereas in TCFT the degrees are opposite to the degree of $Q$).

The BV bracket is given by the following construction: for $\Phi_1 = \Phi_1^{(0)}$, $\Phi_2 = \Phi_2^{(0)}$ two $Q$-cocycles, we set

$$(\Phi_1, \Phi_2)(z) = \frac{1}{2\pi i} \oint_{\gamma_z} \Phi_1^{(1)}(w) \Phi_2^{(0)}(z), \tag{957}$$

where $\gamma_z$ is a contour around $z$ and $\Phi_1^{(1)} = \Gamma \Phi_1^{(0)}$ is the first descent of $\Phi_1$.

The BV Laplacian is constructed as the operator

$$\Delta \colon = G_0 - \overline{G}_0, \tag{958}$$

also denoted $G_{0,-}$, where $G_0, \overline{G}_0$ are particular mode operators of $G$, cf. (913).

We refer to [28] for an example of a TCFT with explicitly computed BV algebra structure on $Q$-cohomology.

6.6.4. *Action of the operad of framed little disks on $V$.* The BV algebra structure on $Q$-cohomology $H_Q(V)$ has a "lift" to the full space of fields $V$, as an "algebra over the operad of framed little 2-disks."

**Definition 6.20.** The operad of framed little 2-disks $E_2^{\mathrm{fr}}$ is a sequence of manifolds $(E_2^{\mathrm{fr}})_n$, where $(E_2^{\mathrm{fr}})_n$ is the space of configurations of $n \geq 0$ disjoint disks inside a unit disk in $\mathbb{R}^2 \simeq \mathbb{C}$, each disk is equipped with a "framing" – a marked point on the boundary circle.[135] The marked point on the unit circle is fixed at $(1, 0)$.

One has composition maps

$$\circ_i \colon (E_2^{\mathrm{fr}})_n \times (E_2^{\mathrm{fr}})_m \to (E_2^{\mathrm{fr}})_{n+m-1} \tag{959}$$

for $i = 1, \ldots, n$, defined as follows. A configuration of disks $o_2 \in (E_2^{\mathrm{fr}})_m$ is scaled and rotated so that its outer disk fits with $i$-th disk in the configuration $o_1 \in (E_2^{\mathrm{fr}})_n$ (and the marked points should coincide). Then the new configuration $o_1 \circ_i o_2$ consists of the rescaled/rotated configuration $o_2$ and all disks of $o_1$ except the $i$-th disk.

It is convenient to think of a configuration of disks as "holes" in the unit disk. Then the composition map fits one $m$-holed inside the $i$-th hole of another $n$-holed disk.

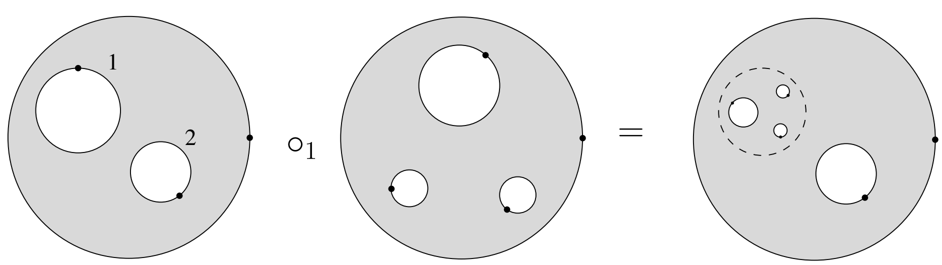



FIGURE 33. Composition in the operad of framed little 2-disks.

[135] In particular, $(E_2^{\mathrm{fr}})_n$ is a manifold of real dimension $4n$, parameterized by positions of centers of the $n$ disks, $n$ radii and $n$ angles (of the marked point).

Given a TCFT, one can construct a sequence of differential forms $\omega_n$ on $(E_2^{\mathrm{fr}})_n$ valued in $\mathrm{Hom}(V^{\otimes n}, V)$, for $n \geq 1$, defined by

$$\omega_n(\Phi_1, \ldots, \Phi_n) = \prod_{k=1}^{n} e^{\zeta_k L_0 + \bar{\zeta}_k \overline{L}_0 + d\zeta_k G_0 + d\bar{\zeta}_k \overline{G}_0} e^{\Gamma} \Phi_k(z_k). \tag{960}$$

Here $z_k$ are positions of the centers of disks, $\zeta_k = \log r_k + i\theta_k$, with $r_k$ the radii and $\theta_k$ the angles; $\Gamma$ is the descent operator (938). The expression in the r.h.s. of (960) is to be understood under a correlator with an arbitrary collection of test fields inserted outside the unit disk. Thus, the r.h.s. of (960) is a "multi-OPE."

One has the property

$$(d - \mathrm{ad}_Q)\omega_n = 0 \tag{961}$$

where $\mathrm{ad}_Q$ means the sum of terms where $Q$ on an input or the output field of $\omega_n$. More explicitly,

$$\begin{aligned} &(d - \mathrm{ad}_Q)\omega_n(\Phi_1, \ldots, \Phi_n) \colon = \\ &= d\omega_n(\Phi_1, \ldots, \Phi_n) - Q\omega_n(\Phi_1, \ldots, \Phi_n) + \sum_{k=1}^{n} \pm\omega_n(\Phi_1, \ldots, Q\Phi_k, \ldots, \Phi_n) = 0 \end{aligned} \tag{962}$$

This property is a consequence of (947).

The property (961) implies that one has a map of cochain complexes, from singular chains of the framed little disk operad to multilinear operators on $V$:

$$\begin{array}{ccc} C_{-\bullet}((E_2^{\mathrm{fr}})_n) & \to & \mathrm{Hom}(V^{\otimes n}, V) \\ \mathrm{chain} & \mapsto & \Big(\Phi_1 \otimes \cdots \otimes \Phi_n \mapsto \int_{\mathrm{chain}} \omega_n(\Phi_1, \ldots, \Phi_n)\Big) \end{array} \tag{963}$$

Note that we put the reverse grading on chains, so that singular chains are seen as a cochain complex. This map is a representation of the operad, i.e., is compatible with compositions.

In particular, passing to cohomology, we obtain a map from homology of $(E_2^{\mathrm{fr}})_n$ to $\mathrm{Hom}(W^{\otimes n}, W)$, where $W = H_Q(V)$.

What is the original reference?

Here is a known fact (see [16]): homology of the operad $E_2^{\mathrm{fr}}$ is the operad of BV algebras, with generators $\mathbb{1}, \cdot, (,), \Delta$ (subject to the relations as in Definition 6.19).[136]

More explicitly, the homology of $E_2^{\mathrm{fr}}$ is generated (using compositions $\circ_i$) by four homology classes:

(i) The tautological 0-class in $H_0((E_2^{\mathrm{fr}})_0)$.
(ii) The 0-class in $H_0((E_2^{\mathrm{fr}})_2)$, represented by any configuration of two disjoint disks in the unit disk.
(iii) The 1-class in $H_1((E_2^{\mathrm{fr}})_2)$, represented by one disk moving a full circle around the other disk.
(iv) The 1-class in $H_1((E_2^{\mathrm{fr}})_1)$, represented by rotating the disk (or equivalently rotating the marked point on the boundary) a full circle.

[136] We think of the unit as an operation of "arity zero," $\mathbb{1} \in \mathrm{Hom}(V^{\otimes 0}, V) \simeq V$.

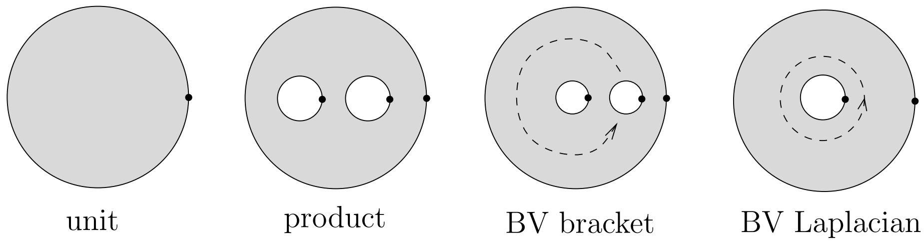



FIGURE 34. Generators of homology of $E_2^{\mathrm{fr}}$.

These classes correspond to the elements $\mathbb{1}, \cdot, (,), \Delta$ of the BV operad and are represented on $W$ by the corresponding operations: cohomology class of the unit field, (956), (957), (958), and one can see that those formulae follow from integrating the form (960) over the respective cycles. Thus, indeed, the BV algebra structure on $Q$-cohomology that we constructed in Section 6.6.3 is induced from the $E_2^{\mathrm{fr}}$-algebra structure on $V$ via passing to cohomology.

For details on the operadic viewpoint on TCFTs, we refer the reader to [16]. For an explicit example, we refer to [28]. Lecture 34, 11/16/2022

## 7. Bits of representation theory of Virasoro algebra

7.1. **Verma modules of Virasoro algebra, null vectors.** Let $V_{c,h}$ be the Verma module[137] of Virasoro algebra with central charge $c \in \mathbb{C}$ and highest weight $h \in \mathbb{C}$, i.e., it is generated by the highest weight vector which we denote $|h\rangle$ which satisfies

$$L_{>0}|h\rangle = 0, \quad L_0|h\rangle = h|h\rangle \tag{964}$$

– is killed by the positive part of the Virasoro algebra and is an eigenvector for $L_0$ with eigenvalue $h$. The Verma module is then

$$V_{c,h} = \mathrm{Span}_{\mathbb{C}}\{L_{-n_r} \cdots L_{-n_1}|h\rangle \mid 1 \le n_1 \le \cdots \le n_r,\ r \ge 0\} \tag{965}$$

The descendant

$$L_{-n_r} \cdots L_{-n_1}|h\rangle \tag{966}$$

has conformal weight ($L_0$-eigenvalue) $h + N$ where $N = n_1 + \cdots + n_r$. One says that (966) is a "level-$N$" vector in $V_{c,h}$. One has a splitting of $V_{c,h}$ by level:

$$V_{c,h} = \bigoplus_{N \ge 0} V_{c,h}^N, \tag{967}$$

where $V_{c,h}^N$ is the subspace of the Verma module spanned by level-$N$ vectors (i.e., it is the $(h+N)$-eigenspace of $L_0$). Note that the dimension of $V_{c,h}^N$ is

$$\dim V_{c,h}^N = P(N) \tag{968}$$

– the number of partitions of $N$ (cf. Section 6.1.6).

There is a unique sesquilinear form $\langle , \rangle$ on $V_{c,h}$ characterized by the properties

$$\langle h|h\rangle \quad = \quad 1 \tag{969}$$

[137] Given any $\mathbb{Z}$-graded Lie algebra $A = A_\bullet$, one defines the Verma module as follows. Let $W$ be a module over $A_{\ge 0}$ where $A_{>0}$ acts by zero. Then the Verma module is the $U(A)$-module induced from $W$, i.e., $U(A) \otimes_{U(A_{\ge 0})} W$, where $U(\cdots)$ is the enveloping algebra.

$$(L_n)^+ = L_{-n}, \quad n \in \mathbb{Z} \tag{970}$$

Generally, $\langle , \rangle$ is not positive-definite and may be degenerate.

**Definition 7.1.** A vector $|\chi\rangle \neq |h\rangle$ in $V_{c,h}$ is called a "singular vector" or "null vector" if it satisfies

$$L_{>0}|\chi\rangle = 0. \tag{971}$$

Note that a null vector is automatically orthogonal to the entire $V_{c,h}$, since one has

$$\Big\langle L_{-n_r} \cdots L_{-n_1}|h\rangle, |\chi\rangle\Big\rangle = \langle h|L_{n_1} \cdots \underbrace{L_{n_r}|\chi\rangle}_{=0 \text{ since } n_r>0} = 0. \tag{972}$$

In particular, a null vector has zero norm:

$$\langle \chi|\chi\rangle = 0. \tag{973}$$

Assume that there exists a null vector $|\chi\rangle$ at level $N$ in $V_{c,h}$. Then Virasoro descendants of $|\chi\rangle$ form a submodule of $V_{c,h}$ isomorphic to the Verma module $V_{c,h+N}$:

$$\underbrace{\mathrm{Span}\{L_{-n_r} \cdots L_{-n_1}|\chi\rangle\}}_{\simeq V_{c,h+N}} \subset V_{c,h} \tag{974}$$

In fact, this entire submodule is orthogonal to $V_{c,h}$, by an argument similar to (972).

Let us consider when null vectors can appear at small levels $N$ (the full answer for general $N$ is given by Kac determinant formula in Section 7.2 below).

**Example 7.2.** Assume that $V_{c,h}$ contains a null vector at level $N = 1$. That means $|\chi\rangle = L_{-1}|h\rangle$ (ignoring a possible normalization factor). Note that $L_{\geq 2}|\chi\rangle$ is a vector at level $-1$, so it automatically vanishes. The only case of (971) that needs checking is $L_1|\chi\rangle$ :

$$L_1|\chi\rangle = L_1L_{-1}|h\rangle = (2L_0 - L_{-1}\underbrace{L_1)|h\rangle}_{0} = 2h|h\rangle. \tag{975}$$

Thus, $|\chi\rangle = L_{-1}|h\rangle$ is a null vector if and only if $h = 0$.

**Example 7.3.** Assume that $V_{c,h}$ contains a null vector at level $N = 2$. This means

$$|\chi\rangle = (\alpha L_{-2} + \beta L_{-1}^2)|h\rangle \tag{976}$$

with $\alpha, \beta \in \mathbb{C}$ not simultaneously zero. By the same argument as above, $L_{\geq 3}|\chi\rangle$ vanishes automatically, so we only need to check $L_1|\chi\rangle$ and $L_2|\chi\rangle$. We have

(977)
$$L_1(\alpha L_{-2} + \beta L_{-1}^2)|h\rangle = (\alpha 3(L_{-1} + L_{-2}L_1) + \beta(2L_0L_{-1} + L_{-1}L_1L_{-1}))|h\rangle =$$
$$= (\alpha 3L_{-1} + \beta(2L_{-1} + 2L_{-1}L_0 + 2L_{-1}L_0 + L_{-1}^2L_1))|h\rangle = (3\alpha + (4h+2)\beta)|h\rangle,$$

(978)
$$L_2(\alpha L_{-2} + \beta L_{-1}^2)|h\rangle = (\alpha(4L_0 + \frac{c}{2} + L_{-2}L_2) + \beta(3L_1L_{-1} + L_{-1}L_2L_{-1}))|h\rangle =$$
$$= (\alpha(4h + \frac{c}{2}) + \beta(6L_0 + 3L_{-1}L_1 + 3L_{-1}L_1 + L_{-1}^2L_2))|h\rangle = ((4h + \frac{c}{2})\alpha + 6h\beta)|h\rangle.$$

So, the equations on a null vector $L_1|\chi\rangle = 0$, $L_2|\chi\rangle$ are equivalent to a homogeneous system of two linear equations on two coefficients $\alpha, \beta$,

$$3\alpha + (4h+2)\beta = 0, \quad (4h + \frac{c}{2})\alpha + 6h\beta = 0, \tag{979}$$

which has a nonzero solution if and only if the determinant of the coefficient matrix vanishes,

$$\begin{vmatrix} 3 & 4h+2 \\ \frac{c}{2} + 4h & 6h \end{vmatrix} = 0. \tag{980}$$

This is a nontrivial quadratic relation on $c$ and $h$, and as we just showed, $V_{c,h}$ contains a null vector at level $N = 2$ if and only if this relation is satisfied.

For instance, this relation is satisfied for $c = \frac{1}{2}, h = \frac{1}{16}$, which is what allowed us to find a hypergeometric equation on the four-point correlator of fields $\sigma$ in the free fermion CFT in Section 6.3.9.

7.2. **Kac determinant formula.** Consider the "Gram matrix" – the matrix of inner products of level-$N$ descendants of the highest vector $|h\rangle$ in $V_{c,h}$:

$$M^{(N)} = (\langle i|j\rangle)_{i,j} \tag{981}$$

where $i, j$ run over the basis of vectors (966) in $V_{c,h}$. In particular, $M^{(N)}$ is a matrix of size $P(N) \times P(N)$.

**Theorem 7.4** (Kac [20], Feigin-Fuchs [12])**.** *The determinant of the Gram matrix (981) is*

$$\det M^{(N)} = \alpha_N \prod_{\substack{p,q\geq 1 \text{ s.t.} \\ pq \leq N}} (h - h_{p,q}(c))^{P(N-pq)}. \tag{982}$$

*Here*

$$\alpha_N = \prod_{\substack{p,q\geq 1 \text{ s.t.} \\ pq \leq N}} ((2p)^q q!)^{P(N-pq)-P(N-p(q+1))} \tag{983}$$

*is a numerical factor and*

$$h_{p,q}(c) = \frac{((m+1)p - mq)^2 - 1}{4m(m+1)}, \tag{984}$$

*where $m$ is related to the central charge $c$ by*

$$m = -\frac{1}{2} \pm \sqrt{\frac{25-c}{1-c}} \tag{985}$$

*or equivalently*

$$c = 1 - \frac{6}{m(m+1)}. \tag{986}$$

The importance of Kac determinant formula (982) is that says for which $c, h$ the Gram matrix at level $N$ vanishes, which means that $V_{c,h}$ contains a null vector at level $\leq N$. More precisely, Kac formula implies the following:

**Corollary 7.5.** *If $h = h_{p,q}$ (as defined by (984)) for some integers $p, q \geq 1$ then $V_{c,h}$ contains a null vector at level $N = pq$.*

In fact, null, vectors at other levels may also appear (i.e. this is not an "if and only if" statement), however every null vector in $V_{c,h}$ is either covered by Corollary 7.5 or is a descendant of one.

**Example 7.6.** For any $c$ from (984) we have $h_{1,1} = 0$. This corresponds to the fact that $V_{c,0}$ has a null vector at level $N = 1$ for any $c$, cf. Example 7.2.

**Example 7.7.** Consider the case of central charge $c = 1$. By (985), (986) it corresponds to the limiting case $m \to \infty$. In this limit, (984) becomes

$$h_{p,q} = \frac{(p-q)^2}{4}. \tag{987}$$

This implies that for $c = 1$, $h = \frac{n^2}{4}$, with $n = 0, 1, 2, 3, \ldots$, the Verma module $V_{1,h}$ contains an infinite sequence of null vectors at levels $N = p\underbrace{(n+p)}_{q}$, with $p = 1, 2, 3, \ldots$, since $h = \frac{n^2}{4}$ equals $h_{p,q}$ for an infinite sequence of pairs $(p, q)$ of the form $(p, n + p)$.

The following is (a part of) a theorem of Feigin-Fuchs [12].

**Theorem 7.8** (Feigin-Fuchs).
- *$V_{c,h}$ is irreducible if and only if it contains no null vectors and is reducible if and only if $h = h_{p,q}$ for some integers $p, q \geq 1$.*
- *Proper submodules of $V_{c,h}$ are generated by null vectors.*
- *The irreducible highest weight module $M_{c,h}$ for Virasoro algebra at central charge $c$ and with highest weight $h$ has the form*

$$M_{c,h} = V_{c,h}/\mathsf{N} \tag{988}$$

*where $\mathsf{N} \subset V_{c,h}$ is the maximal proper submodule. It can also be realized as the kernel of the sesquilinear form $\langle,\rangle$ on $V_{c,h}$ or equivalently the orthogonal complement of $V_{c,h}$:*

$$\mathsf{N} = \ker\langle,\rangle = V_{c,h}^{\perp}. \tag{989}$$

In Section 7.3 we recall the second part of Feigin-Fuchs theorem giving the full classification of maps (inclusions) between Verma modules at a given $c$, which in particular yields formulae for characters of $M_{c,h}$ and therefore formulae for torus partition functions in a CFT, see Section 7.3.1 and (1019).

I am being redundant here, this was discussed before..

**Example 7.9.** For $c = 1$, $V_{1,h}$ is reducible iff $h = \frac{n^2}{4}$ for some $n = 0, 1, 2, \ldots$. In particular, for $h \neq \frac{n^2}{4}$, one has $M_{1,h} = V_{1,h}$. Reducible Verma modules for $c = 1$ arrange into two sequences connected by inclusions of modules:

$$\begin{gathered} V_{1,0} \leftarrow V_{1,1} \leftarrow V_{1,4} \leftarrow V_{1,9} \leftarrow \cdots \\ V_{1,\frac{1}{4}} \leftarrow V_{1,\frac{9}{4}} \leftarrow V_{1,\frac{25}{4}} \leftarrow V_{1,\frac{49}{4}} \leftarrow \cdots \end{gathered} \tag{990}$$

Irreducible modules for these values of $h$ are obtained by taking the corresponding Verma module and quotienting out the module mapping into it. E.g., $M_{1,0} = V_{1,0}/V_{1,1}$. Null vectors in $V_{1,h}$ are images of highest vectors of Verma modules to the right in the respective sequence, i.e., mapping into $V_{1,h}$, possibly via a sequence of inclusions.

**Example 7.10.** Consider the case $c = \frac{1}{2}$, which corresponds to $m = 3$. One has

$$h_{p,q} = \frac{(4p - 3q)^2 - 1}{48}. \tag{991}$$

The values of $h_{p,q}$ for small $p, q$ are the following.

| $h_{p,q}$ | $p = 1$ | $p = 2$ |
|---|---|---|
| $q = 1$ | $0$ | $\frac{1}{2}$ |
| $q = 2$ | $\frac{1}{16}$ | $\frac{1}{16}$ |
| $q = 3$ | $\frac{1}{2}$ | $0$ |

We recognize these numbers $h_{1,1} = 0$, $h_{2,1} = \frac{1}{2}$, $h_{1,2} = \frac{1}{16}$ as precisely the conformal weights $h$ of primary field $\mathbb{1}$, $\psi$, $\sigma$ in the free fermion CFT. Thus, the corresponding conformal families $M_{\frac{1}{2},h}$ are the ones coming from Verma modules $V_{\frac{1}{2},h}$ containing a null vector, which allows one to write differential equations on correlators of the corresponding primary fields, as we did in Section 6.3.9.

7.3. **Maps between Verma modules.** In this section we follow Feigin-Fuchs [12].

Fix the central charge $c, h \in \mathbb{R}$. Equation $h_{p,q} = h$ with $h_{p,q}$ defined by (984) determines two parallel lines on the $(p, q)$-plane related to one another by reflection $(p, q) \leftrightarrow (-p, -q)$. Pick one of those lines and denote it $l_{c,h}$. The slope of $l_{p,q}$ is $\frac{m+1}{m}$, in particular:

- If $c \leq 1$, the line is real, with positive slope. For $c = 1$ the slope is 1.
- If $c \geq 25$, the line is real, with negative slope. For $c = 25$, the slope is $-1$.
- If $1 < c < 25$, the slope (and the line) is complex.

One is interested in integer points on $l_{p,q}$. The relevant cases (with nomenclature taken from [12]) is:

I $l_{c,h}$ has no integer points.

II $l_{c,h}$ has a single integer point $(a', a'') \in \mathbb{Z}^2$. One distinguishes the following subcases:

$\mathrm{II}_+$ $a'a'' > 0$,

$\mathrm{II}_0$ $a'a'' = 0$,

$\mathrm{II}_-$ $a'a'' < 0$.

III $l_{c,h}$ contains infinitely many integer points. In this case $m \in \mathbb{Q}$ and one has either $c \leq 1$ (subcase $\mathrm{III}_-$) or $c \geq 25$ (subcase $\mathrm{III}_+$). We further distinguish between the subcases according to whether $l_{p,q}$ intersects the coordinate axes at integer points.

$\mathrm{III}^{0,0}_\pm$ $l_{c,h}$ intersects both coordinate axes $q = 0$ and $p = 0$ at integer points. Denote $P$ the middle point of the interval connecting these two intersection points. Enumerate the integer points of the upper half of $l_{c,h}$ (above $P$) as

$$\ldots, (a'_{-1}, a''_{-1}), (a'_0, a''_0), (a'_1, a''_1), \ldots \tag{992}$$

in such order that one has

$$\cdots < a'_{-1}a''_{-1} < a'_0 a''_0 = 0 < a'_1 a''_1 < \cdots \tag{993}$$

In particular, in the case $\mathrm{III}^{0,0}_-$, the sequence (992) is finite on the left and infinite on the right, and vice versa in the case $\mathrm{III}^{0,0}_+$.

III$^0_\pm$ $l_{c,h}$ intersects one of the coordinate axes at an integer point. Then we enumerate all integer points of $l_{c,h}$ (not just half) as in (992), so that (993) holds.

III$_\pm$ $l_{c,h}$ intersects both coordinate axes at non-integer points. Then we enumerate all integer points of $l_{c,h}$ as in (992), so that

$$\cdots < a'_{-1}a''_{-1} < 0 < a'_0a''_0 < a'_1a''_1 < \cdots \tag{994}$$

We also draw a second line $l'_{c,h}$ through the point $(-a'_0, a''_0)$ parallel to $l_{c,h}$ and enumerate its integer points as

$$\ldots, (b'_{-1}, b''_{-1}), (b'_0, b''_0) = (-a'_0, a''_0), (b'_1, b''_1), \ldots \tag{995}$$

so that one has

$$\cdots < b'_{-1}b''_{-1} < b'_0b''_0 = -a'_0a''_0 < 0 < b'_1b''_1 < \cdots \tag{996}$$

**Theorem 7.11** (Feigin-Fuchs)**.** *Fix $c, h \in \mathbb{R}$ and a line $l_{c,h}$ as above. Then:*

- *In cases* I, II$_0$*: $V_{c,h}$ is irreducible and not a proper submodule of any Verma module.*
- *In the case* II$_+$*, $V_{c,h}$ has a single Verma submodule isomorphic to $V_{c,a'a''}$ (which is irreducible) and generated a by a null vector at level $a'a''$; $V_{c,h}$ is not a proper submodule of any Verma module.*
- *In the case* II$_-$*, $V_{c,h}$ is irreducible but can be embedded into $V_{c,h+a'a''}$ and is generated there by a null vector at level $-a'a''$. $V_{c,h}$ cannot be embedded into any other Verma module.*
- *In the cases* III$^{0,0}_\pm$*,* III$^0_\pm$*, there is a sequence of embeddings*

$$\cdots \to V_{c,h+a'_1a''_1} \to V_{c,h} \to V_{c,h+a'_{-1}a''_{-1}} \to \cdots \tag{997}$$

*Modules in this sequence are not related by morphisms with any Verma modules not from this sequence.*

- *In the cases* III$_\pm$ *there is a commutative diagram of embeddings of Verma modules*

$$\begin{tikzcd}
\vdots & \vdots \\
V_{c,h+a'_{-1}a''_{-1}} & V_{c,h+a'_{-2}a''_{-2}} \\
V_{c,h} \arrow[u] \arrow[ur] & V_{c,h+b'_{-1}b''_{-1}+a'_0a''_0} \arrow[u] \arrow[ul] \\
V_{c,h+a'_1a''_1} \arrow[u] \arrow[ur] & V_{c,h+a'_0a''_0} \arrow[u] \arrow[ul] \\
V_{c,h+b'_2b''_2+a'_0a''_0} \arrow[u] \arrow[ur] & V_{c,h+b'_1b''_1+a'_0a''_0} \arrow[u] \arrow[ul] \\
\vdots & \vdots
\end{tikzcd} \tag{998}$$

*Modules in this diagram are not connected by homomorphisms with any other Verma modules. In each piece of the form*

(999)
$$\begin{tikzcd} & A & \\ & B \arrow[u] & C \arrow[ul] \\ & D \arrow[u] \arrow[ur] & E \arrow[u] \arrow[ul] \end{tikzcd}$$

*the images of $B$ and $C$ in $A$ do not contain each other and their intersection is generated by images of $D$ and $E$ in $A$.*

**Example 7.12.** The case $c = 1$, $h = 0$ corresponds is $\mathrm{III}_-^{0,0}$ in Feigin-Fuchs classification and $c = 1$, $h = \frac{n^2}{4}$ with $n = 1, 2, \dots$ is $\mathrm{III}_-^0$. In these cases the sequence (997) is one of the two sequences (990).

**Example 7.13.** For $c = \frac{1}{2}$, $h = 0$ we have the line $l_{\frac{1}{2},0} = \{(p, q) \mid 4p - 3q = 1\}$ corresponding to the case $\mathrm{III}_-$. The integer points on $l_{\frac{1}{2},0}$ are $(1 + 3k, 1 + 4k)$ with $k \in \mathbb{Z}$; arranged in the order (994) they are:

(1000)

| $n$ | 0 | 1 | 2 | 3 | 4 | $\cdots$ |
|---|---|---|---|---|---|---|
| $(a'_n, a''_n)$ | $(1, 1)$ | $(-2, -3)$ | $(4, 5)$ | $(-5, -7)$ | $(7, 9)$ | $\cdots$ |

The parallel line $l'_{\frac{1}{2},0} = \{(p, q) \mid 4p-3q = -7\}$, it has integer points $(-1+3k, 1+4k)$, $k \in \mathbb{Z}$; arranged in the order (996) they are:

(1001)

| $n$ | 0 | 1 | 2 | 3 | 4 | $\cdots$ |
|---|---|---|---|---|---|---|
| $(b'_n, b''_n)$ | $(-1, 1)$ | $(2, 5)$ | $(-4, -3)$ | $(5, 9)$ | $(-7, -7)$ | $\cdots$ |

The diagram of embeddings (998) becomes

(1002)
$$\begin{tikzcd} V_{\frac{1}{2},0} & \\ V_{\frac{1}{2},6} \arrow[u] & V_{\frac{1}{2},1} \arrow[ul] \\ V_{\frac{1}{2},11} \arrow[u] \arrow[ur] & V_{\frac{1}{2},9} \arrow[u] \arrow[ul] \\ \vdots & \vdots \end{tikzcd}$$

One has similar diagrams for $h = \frac{1}{2}$ and for $h = \frac{1}{16}$ (all with $c = \frac{1}{2}$).

7.3.1. *Characters of highest weight modules of Virasoro algebra.* Given a module $W$ of Virasoro algebra with central charge $c$ is defined as

(1003)
$$\chi_W(\mathsf{q}) = \mathrm{tr}_W \mathsf{q}^{L_0 - \frac{c}{24}},$$

with $\mathsf{q}$ a complex parameter with $|\mathsf{q}| < 1$. For a Verma module $V_{c,h}$, one has

(1004)
$$\chi_{V_{c,h}}(\mathsf{q}) = \sum_{N \geq 0} P(N) \mathsf{q}^{h+N-\frac{c}{24}} = \frac{\mathsf{q}^{h+\frac{1-c}{24}}}{\eta(\tau)},$$

where $P(N)$ is the number of partitions and $\eta(\tau)$ is the Dedekind eta-function; $\mathsf{q}$ is related to $\tau \in \Pi_+$ by

$$\mathsf{q} = e^{2\pi i\tau}. \tag{1005}$$

Characters of irreducible highest weight modules $M_{c,h}$ can be obtained using Theorem 7.11.

**Example 7.14.** The character of the irreducible module $M_{\frac{1}{2},0}$ can be obtained from the diagram (1002):

$$\chi_{M_{\frac{1}{2},0}}(\mathsf{q}) = \chi_{V_{\frac{1}{2},0}}(\mathsf{q}) - \chi_{V_{\frac{1}{2},1}}(\mathsf{q}) - \chi_{V_{\frac{1}{2},6}}(\mathsf{q}) + \chi_{V_{\frac{1}{2},9}}(\mathsf{q}) + \chi_{V_{\frac{1}{2},11}}(\mathsf{q}) - \cdots = \tag{1006}$$

$$= \frac{\mathsf{q}^{\frac{1}{48}}}{\eta(\tau)}(1-\mathsf{q}-\mathsf{q}^6+\mathsf{q}^9+\mathsf{q}^{11}-\cdots) = \frac{\mathsf{q}^{\frac{1}{48}}}{\eta(\tau)}\sum_{k\in\mathbb{Z}}\left(\mathsf{q}^{1+(-1+3k)(1+4k)} - \mathsf{q}^{(1+3k)(1+4k)}\right)$$

Characters of irreducible modules $M_{c,h}$ are the conformal blocks for the torus partition function in a CFT. If the space of fields of a CFT with central charge $c, \bar{c}$ contains primary fields $\Phi_i$ with $i \in I$ (the indexing set for primary fields), with conformal weights $(h_i, \bar{h}_i)$, then the space of states (or space of fields) is

$$\mathcal{H} = \bigoplus_{i\in I} M_{c,h_i} \otimes M_{\bar{c},\bar{h}_i} \tag{1007}$$

and the torus partition function is

$$Z(\tau) = \mathrm{tr}_{\mathcal{H}}\mathsf{q}^{L_0-\frac{c}{24}}\bar{\mathsf{q}}^{\overline{L}_0-\frac{\bar{c}}{24}} = \sum_{i\in I}\chi_{M_{c,h_i}}(\mathsf{q})\chi_{M_{\bar{c},\bar{h}_i}}(\bar{\mathsf{q}}) \tag{1008}$$

### 7.4. Minimal models of CFT.

7.4.1. *Unitary minimal models.* The following theorem is due to Friedan-Qiu-Shenker (1984) and Goddard Kent-Olive (1986).

**Theorem 7.15.** *The irreducible highest weight Virasoro module $M_{c,h}$ is unitary (i.e. the sesquilinear form $\langle,\rangle$ is positive definite) if*
*(a) either $c \geq 1$, $h \geq 0$,*
*(b) or $c = 1 - \frac{6}{m(m+1)}$ with $m = 2,3,4,\ldots$ and $h = h_{p,q}$ with $1 \leq p \leq m-1$, $1 \leq q \leq m$.*

Note that for $c$ as in (b) above, one has a symmetry in the table of admissible $h_{p,q}$'s:

$$h_{p,q} = h_{m-p,m+1-q}. \tag{1009}$$

Lecture 35, 11/18/2022

Fix $m = 2,3,4,\ldots$ The "minimal model" $\mathcal{M}(m, m+1)$ is defined[138] as a CFT with central charge

$$c = \bar{c} = 1 - \frac{6}{m(m+1)} \tag{1010}$$

[138]We say "defined" a bit sloppily here. To have a CFT, the definition of the space of states as a Vir $\oplus$ $\overline{\mathrm{Vir}}$-module needs to be supplemented with extra data: OPEs of primary fields, or equivalently, structure coefficients of 3-point correlators of primary fields, which then allows to determine all correlators.

and space of states (or space of fields)

(1011) $$\mathcal{H} = \bigoplus_{1\leq p\leq m-1,\ 1\leq q\leq m\ /\mathbb{Z}_2} M_{c,h_{p,q}} \otimes \overline{M}_{c,h_{p,q}}.$$

Here the sum is over pairs $(p,q)$ where the pairs $(p,q)$ and $(m-p, m+1-q)$ are understood as equivalent; notation "$/\mathbb{Z}_2$" above means that we should take one representative for each equivalence class. Each term in the sum in (1011) is a representation of left and right Virasoro algebra, $\mathrm{Vir}\oplus\overline{\mathrm{Vir}}$, given as a tensor product of two copies of the same irreducible Virasoro module $M_{c,h_{p,q}}$; bar over the second copy of $M$ indicates that we see it as a module over the right (antiholomorphic) copy of Virasoro algebra.

**Example 7.16.** For $m=3$, the minimal model $\mathcal{M}(3,4)$ is the CFT with central charge $\frac{1}{2}$ and three species of primary fields:

- $\Phi_{1,1}$ of conformal weight $h=\bar{h}=h_{1,1}=0$,
- $\Phi_{2,1}$ of conformal weight $h=\bar{h}=h_{2,1}=\frac{1}{2}$,
- $\Phi_{1,2}$ of conformal weight $h=\bar{h}=h_{1,2}=\frac{1}{16}$.

Comparing this to (858), we see that the space of states for $\mathcal{M}(3,4)$ is the even part of the space of states of the free Majorana fermion, and we can identify the fields as

(1012) $$\Phi_{1,1}=\mathbb{1},\quad \Phi_{2,1}=\epsilon,\quad \Phi_{1,2}=\varpi$$

– the identity, "energy" and "spin" fields.

In particular, the CFT minimal model $\mathcal{M}(3,4)$ corresponds to the Ising model at critical temperature (at the point of second-order phase transition), and in particular correlators of $\varpi$ reproduce the correlators of spins in critical Ising model.

The selection rules (so-called "fusion rules") for OPEs are given by the following table

(1013)

| $\times$ | $[\mathbb{1}]$ | $[\epsilon]$ | $[\varpi]$ |
|---|---|---|---|
| $[\mathbb{1}]$ | $[\mathbb{1}]$ | $[\epsilon]$ | $[\varpi]$ |
| $[\epsilon]$ | $[\epsilon]$ | $[\mathbb{1}]$ | $[\varpi]$ |
| $[\varpi]$ | $[\varpi]$ | $[\varpi]$ | $[\mathbb{1}]+[\epsilon]$ |

Here for $\Phi$ a primary field $[\Phi]$ stands for its conformal family (the span of all descendants, or equivalently, the corresponding term in the sum (1011)). For instance, the fusion rule $[\varpi]\times[\varpi]=[\mathbb{1}]+[\epsilon]$ means that in the r.h.s. of the r.h.s of the OPE of any two descendants of $\varpi$ one can find only descendants of $\mathbb{1}$ and of $\epsilon$. We'll comment later on where these selection rules for OPEs come from. link

**Example 7.17.** Case $m=2$ is the "trivial CFT" with $c=0$ and a single conformal family with $h=\bar{h}=h_{1,1}=0$:

(1014) $$\mathcal{H}=M_{0,0}\otimes\overline{M}_{0,0}.$$

In fact the irreducible Virasoro module $M_{0,0}$ consists of just the highest vector $|\mathrm{vac}\rangle$ (or $\mathbb{1}$) and all its descendants are zero.

7.4.2. *General minimal models.* Let $c=1-\frac{6}{m(m+1)}$ with $m\in\mathbb{Q}$ (rational but not necessarily integer). Assume that

(1015) $$\frac{m+1}{m}=\frac{Q}{P}$$

with $Q, P \geq 1$ coprime integers.

As a consequence of Theorem 7.11, one has that for such $c$, the *maximal*[139] reducible highest weight Verma modules are $V_{c,h_{p,q}}$ with $0 \leq p \leq P$, $0 \leq q \leq Q$.

The minimal model $\mathcal{M}(P,Q)$ is defined as a CFT with the space of states (or space of fields)

$$\mathcal{H} = \bigoplus_{1\leq p\leq P-1,\ 1\leq q\leq\ Q-1\ /\mathbb{Z}_2} M_{c,h_{p,q}} \otimes \overline{M}_{c,h_{p,q}}, \tag{1016}$$

where $/\mathbb{Z}_2$ again means that from each equivalence class $(p,q) \sim (P-p, Q-q)$ we need to pick one representative.

The minimal models $\mathcal{M}(P,Q)$ are not unitary (the sesquilinear product on $\mathcal{H}$ is not positive-definite), unless one has $(P,Q) = (m, m+1)$ for $m = 2, 3, \ldots$.

**Example 7.18.** The minimal model $\mathcal{M}(2,5)$ corresponds to $c = -22/5$ (and $m = \frac{2}{3}$) and has two primary fields:

- $\Phi_{1,1} = \mathbb{1}$ of conformal weight $h = \bar{h} = h_{1,1} = 0$,
- $\Phi_{1,2}$ of conformal weight $h = \bar{h} = h_{1,2} = -\frac{1}{5}$.[140]

In particular, it is clear that the model cannot be unitary, since $c < 0$ and there is a field with negative conformal weight (each of these observations separately contradicts unitarity).

**Example 7.19.** The minimal model $\mathcal{M}(4,5)$ is unitary. It has $c = 7/10$ and the array[141] of conformal weights $h_{p,q}$ for admissible $p, q$ is

(1017)

| | $p=1$ | $p=2$ | $p=3$ |
|---|---|---|---|
| $q=1$ | 0 | 7/16 | 3/2 |
| $q=2$ | 1/10 | 3/80 | 3/5 |
| $q=3$ | 3/5 | 3/80 | 1/10 |
| $q=4$ | 3/2 | 7/16 | 0 |

In particular, the model has $6 = 3 \times 4/2$ conformal families/species of primary fields.

Each minimal model $\mathcal{M}(P,Q)$ has a collection of primary field $\Phi_{p,q}$ of conformal weight $(h_{p,q}, h_{p,q})$, with $p, q$ as in the r.h.s. of (1016); $\Phi_{1,1} = \mathbb{1}$ has conformal weight $(0,0)$ and is identified with the identity field.

Some of the fusion rules are:

$$\begin{aligned} [\Phi_{1,1}] \times [\Phi_{p,q}] &= [\Phi_{p,q}], \\ [\Phi_{1,2}] \times [\Phi_{p,q}] &= [\Phi_{p,q-1}] + [\Phi_{p,q+1}], \\ [\Phi_{2,1}] \times [\Phi_{p,q}] &= [\Phi_{p-1,q}] + [\Phi_{p+1,q}]. \end{aligned} \tag{1018}$$

Minimal models of CFT describe different 2d systems of statistical mechanics at the point of second-order phase transition (put another way, they describe universality classes of 2d critical phenomena). For instance, one has the following correspondences were identified between 2d systems at the point of phase-transition and minimal models of CFT:

---

[139] "Maximal" means that they cannot be embedded as proper submodules into any other Verma module

[140] Note that the negative conformal weight means that the correlator of two such fields increases as the fields get farther apart: $\langle \Phi_{1,2}(w)\Phi_{1,2}(z)\rangle = |w-z|^{\frac{4}{5}}$ (cf. Lemma 5.26).

[141] Such arrays for minimal models are called "Kac tables"

| CFT minimal model | phase transition |
|---|---|
| $\mathcal{M}(3,4)$ | Ising model at critical temperature |
| $\mathcal{M}(2,5)$ | Yang-Lee edge singularity |
| $\mathcal{M}(4,5)$ | tricritical Ising model |
| $\mathcal{M}(5,6)$ | 3-state Potts model |
| $\mathcal{M}(6,7)$ | tricritical 3-state Potts model |

*Remark* 7.20. All primary fields in a minimal model $\mathcal{M}(P,Q)$ are highest vectors of reducible Virasoro modules (always corresponding to the case $\mathrm{III}_-$ in Feigin-Fuchs classification, Theorem 7.11) and thus have vanishing descendants. Therefore any 4-point correlation function of primary fields in $\mathcal{M}(P,Q)$ can be reduced to a function $F(\lambda)$ of the cross-ratio $\lambda$ satisfying certain ODE (e.g. a hypergeometric equation in the case of fields $\Phi_{1,2},\Phi_{2,1}$), as in the case of the correlator $\langle\sigma\sigma\sigma\sigma\rangle$ in Section 6.3.9.

**Definition 7.21.** One calls a CFT with finitely many primary fields (or equivalently finitely many conformal families – irreducible summands in the space of states/ space of fields) a *rational* CFT, or RCFT.

Thus, minimal models are the prime examples of rational CFT. On the other hand, free boson (with values in $\mathbb{R}$ or $S^1$) is not rational: it contains infinitely many primary fields.

To define a CFT, one needs to present two pieces of data:

- The space of states $\mathcal{H}$ or equivalently the space of fields $V$ as a $\mathrm{Vir}\oplus\overline{\mathrm{Vir}}$-module (with come central charge $c,\bar{c}$) – in particular, splitting it into irreducible summands, one has conformal families generated by highest weight vectors/primary fields.
- The coefficients in 3-point correlation functions of primary fields (705).

This data allows one to recover all correlation functions of all fields but there are two constraints that the data above must satisfy:

(i) "Crossing symmetry" – a certain quadratic constraint on the coefficients of 3-point functions of primary fields, see Section 7.5.1.
(ii) Modular invariance of genus one partition function.

*Remark* 7.22. If one defines the space of states to be just the single conformal family generated by the identity field $\mathbb{1}$, then the corresponding "CFT" will have correlators and OPEs on the plane but will fail the modular invariance property (unless $c=0$ which is the case of the trivial CFT $\mathcal{M}(2,3)$).

More explicitly, one computes the torus partition function in $\mathcal{M}(P,Q)$ using (1008) and evaluating the characters as in Example 7.14, resulting in the formula

$$Z(\tau)= \tag{1019}$$
$$=\frac{|\mathsf{q}|^{\frac{1-c}{12}}}{|\eta(\tau)|^2}\sum_{1\leq p\leq P-1,\ 1\leq q\leq\ Q-1\ /\mathbb{Z}_2}\left|\mathsf{q}^{h_{p,q}}\sum_{k\in\mathbb{Z}}\left(\mathsf{q}^{pq+(-p+Pk)(q+Qk)}-\mathsf{q}^{(p+Pk)(q+Qk)}\right)\right|^2.$$

This expression is modular invariant (which can be proved by Poisson summation). However, restricting to only the $(p,q)=(1,1)$ term in the sum one obtains a non-modular invariant expression.

7.5. **Correlators and OPEs of primary fields in a general RCFT.** Consider a general CFT. Fix $\{\Phi_p\}_{p\in I}$ an orthonormal basis of primary fields, with $I$ an indexing set.

**Lemma 7.23.** *For $\Phi_1, \Phi_2$ primary fields, the OPE has the form*

$$\Phi_1(w)\Phi_2(z) \sim \sum_{p\in I}\sum_{\vec{k},\vec{\bar{k}}} C_{12p}^{\vec{k},\vec{\bar{k}}}(w-z)^{-h_1-h_2+h_p+|\vec{k}|}(\bar{w}-\bar{z})^{-\bar{h}_1-\bar{h}_2+\bar{h}_p+|\vec{\bar{k}}|}\Phi_p^{\vec{k},\vec{\bar{k}}}(z), \tag{1020}$$

*where:*

- *The first sum is over species primary fields.*
- *The second sum over pairs of nondecreasing sequences $1 \leq k_1 \leq \cdots \leq k_r$ (which we denote $\vec{k}$) and $1 \leq \bar{k}_1 \leq \cdots \leq \bar{k}_s$ (denoted $\vec{\bar{k}}$), with $r, s \geq 0$; we also denoted $|\vec{k}| = k_1 + \cdots + k_r$ and similarly for $\vec{\bar{k}}$; $\Phi_p^{\vec{k},\vec{\bar{k}}}$ is the descendant*

$$\Phi_p^{\vec{k},\vec{\bar{k}}} = L_{-k_r}\cdots L_{-k_1}\overline{L}_{-\bar{k}_s}\cdots\overline{L}_{-\bar{k}_1}\Phi_p \tag{1021}$$

- *The coefficients on the right are*

$$C_{12p}^{\vec{k},\vec{\bar{k}}} = C_{12p}\beta_{12p}^{\vec{k}}\bar{\beta}_{12p}^{\vec{\bar{k}}} \tag{1022}$$

  *where $C_{12p}$ are certain coefficients depending on the triple of primary fields $\Phi_1, \Phi_2, \Phi_p$ and $\beta_{12p}^{\vec{k}}$ a certain family of universal[142] rational functions of $c, h_1, h_2, h_p$ parametrized by the sequence $\vec{k}$; $\bar{\beta}$ is the same family where $\bar{c}, \bar{h}_1, \bar{h}_2, \bar{h}_p$ are used instead.*

One can always assume the normalization $\beta_{12p}^{\varnothing} = 1$.

*Remark* 7.24. (a) "Structure constants" $C_{12p}$ in the r.h.s. of (1022) are the same as the constants appearing in the r.h.s. of the 3-point function (705) of primary fields

$$\langle\Phi_1(w)\Phi_2(z)\Phi_p(x)\rangle. \tag{1023}$$

Expressed another way, it is the matrix element

$$\langle\Phi_p|\widehat{\Phi}_1(1)|\Phi_2\rangle. \tag{1024}$$

It is symmetric under permutations of species $1, 2, p$ (as obvious from the previous interpretation).

(b) Remark that as a consequence of Lemma 7.23, a descendant field $\Phi_p^{\vec{k},\vec{\bar{k}}}$ can appear in the OPE $\Phi_1(w)\Phi_2(z)$ only if the primary field $\Phi_p$ itself appears in that OPE.

(c) From the ansatz (7.23) it is clear that only finitely many descendants of each primary field $\Phi_p$ contribute to the *singular part* of the OPE.

*Sketch of proof of Lemma 7.23.* The exponents in the ansatz (1020) follow immediately from Lemma 5.24. The only thing to check is (1022).

The idea is to consider 3-point correlation functions

$$\langle\Phi_1(w)\Phi_2(z)\Phi_p^{\vec{l},\vec{\bar{l}}}(x)\rangle. \tag{1025}$$

[142] I.e. not depending on any details of the CFT.

for various nondecreasing sequences $\vec{l},\vec{\bar{l}}$ with $|\vec{l}|=|\vec{k}|$, $|\vec{\bar{l}}|=|\vec{\bar{k}}|$. On one hand one can find these correlators explicitly by reducing them to a differential operator acting on $\langle\Phi_1(w)\Phi_2(z)\Phi_p(x)\rangle$ (cf. Example 5.22), resulting expressions of the form

$$\langle\Phi_p^{\vec{l},\vec{\bar{l}}}|\Phi_1(w)|\Phi_2\rangle = C_{12p}\gamma_{12p}^{\vec{l}}\bar{\gamma}_{12p}^{\vec{\bar{l}}} \tag{1026}$$

with $\gamma_{12p}^{\vec{l}}$ some universal rational functions of $c,h_1,h_2,h_p$ depending on the sequence $\vec{l}$, and similarly for $\bar{\gamma}$; for convenience we set $z=0,x\to\infty$ in the correlator (1025). On the other hand one can replace $\Phi_1(w)\Phi_2(z)$ in (1025) with the r.h.s. of (1020) and evaluate the remaining 2-point functions of descendants in terms of elements of the Gram matrix (981):

$$\langle\Phi_p^{\vec{l},\vec{\bar{l}}}|\Phi_1(w)|\Phi_2\rangle = \sum_{\vec{k},\vec{\bar{k}}} C_{12p}^{\vec{k},\vec{\bar{k}}} G^p_{\vec{k},\vec{l}}\,\overline{G}^p_{\vec{\bar{k}},\vec{\bar{l}}}, \tag{1027}$$

where $G^p_{\vec{k},\vec{l}}$ are the matrix elements of the Gram matrix. Here we again set $z=0,x\to\infty$. Comparing the two sides, we obtain the claimed ansatz (1022) with

$$\beta_{12p}^{\vec{k}} = \sum_{\vec{l}}((G^p)^{-1})_{\vec{k},\vec{l}}\,\gamma_{12p}^{\vec{l}}. \tag{1028}$$

$\square$

**Example 7.25.** The first coefficients $\beta_{12p}^{\vec{k}}$ appearing in (1022) are:

$$\beta_{12p}^{\varnothing}=1,$$

$$\beta_{12p}^{\{1\}}=\frac{h_1-h_2+h_p}{2h_p},$$

$$\begin{pmatrix}\beta_{12p}^{\{2\}}\\ \beta_{12p}^{\{1,1\}}\end{pmatrix}=\begin{pmatrix}4h_p+\frac{c}{2} & 6h_p\\ 6h_p & 2h_p(4h_p+2)\end{pmatrix}^{-1}\begin{pmatrix}2h_1-h_2+h_p\\ (-h_1-h_2+h_p)(3h_1-h_2+h_p+1)+6h_1^2\end{pmatrix}.$$

*Remark* 7.26. Assume that the primary field $\Phi_1$ has a vanishing descendant al level $N$ (corresponding to a null vector in the corresponding Verma module). Then by the argument of Example 5.22 there is a degree $\leq N$ differential operator annihilating the 3-point function of primary fields (1023). Combining with the expression (705) for the 3-point function this implies an algebraic equation of degree $\leq N$. Thus, there is an algebraic equation of degree $\leq N$ on the conformal weight $h_p$ of a primary field which (and whose descendant) can appear in the r.h.s. of the OPE (1020).

This is exactly the case in minimal models $\mathcal{M}(P,Q)$ and this is how one obtains "fusions rules" (1018) and, more generally, obtains the result that fields $\Phi_{p,q}$ of the minimal model form a closed algebra under OPEs: no fields with other conformal weights can appear.

7.5.1. *4-point correlator of primary fields.* The correlator of four primary fields in a general CFT is bound by global conformal symmetry to be of the form (712):

$$\langle\Phi_1(z_1)\Phi_2(z_2)\Phi_3(z_3)\Phi_4(z_4)\rangle = \Big(\prod_{1\leq i<j\leq 4} z_{ij}^{\frac{1}{3}\sum_{k=1}^4 h_k-h_i-h_j}\,\bar{z}_{ij}^{\frac{1}{3}\sum_{k=1}^4 \bar{h}_k-\bar{h}_i-\bar{h}_j}\Big)f(\lambda) \tag{1029}$$

with $f$ a smooth function of the cross-ratio $\lambda=\frac{z_{13}z_{24}}{z_{14}z_{23}}\in\mathbb{CP}^1\backslash\{0,1,\infty\}$. We can use Möbius symmetry to fix points $z_2,z_3,z_4$ at $1,0,\infty$, then $z_1$ becomes $\lambda$. Thus, we have

$$\langle\Phi_4|\widehat{\Phi}_2(1)\widehat{\Phi}_1(\lambda)|\Phi_3\rangle = f(\lambda) \tag{1030}$$

Applying the OPE (1020) to the expression $\widehat{\Phi}_1(\lambda)|\Phi_3\rangle$ above, we obtain

$$\begin{aligned} f(\lambda) &= \sum_{p\in I}\sum_{\vec{k},\vec{\bar{k}}} C_{13p}\beta_{13p}^{\vec{k}}\bar{\beta}_{13p}^{\vec{\bar{k}}}\lambda^{-h_1-h_3+h_p+|\vec{k}|}\bar{\lambda}^{\bar{h}_1-\bar{h}_3+\bar{h}_p+|\vec{\bar{k}}|}\langle\Phi_4|\widehat{\Phi}_2(1)|\Phi_p^{\vec{k},\vec{\bar{k}}}\rangle \\ &= \sum_{p\in I} C_{42p}C_{13p}\mathcal{F}_{13}^{24}(p|\lambda)\overline{\mathcal{F}}_{13}^{24}(p|\bar{\lambda}) \end{aligned} \tag{1031}$$

where

$$\mathcal{F}_{13}^{24}(p|\lambda) \colon = \lambda^{-h_1-h_3+h_p}\sum_{K=0}^{\infty}\lambda^K \sum_{\vec{k},\vec{l}\text{ with }|\vec{k}|=|\vec{l}|=K} \beta_{13p}^{\vec{k}}G_{\vec{k},\vec{l}}^{p}\beta_{24p}^{\vec{l}}, \tag{1032}$$

and similarly for $\overline{\mathcal{F}}$. Here $G^p_{\vec{k},\vec{l}}$ is a matrix element of the Gram matrix (981).

The r.h.s. of (1032) is a holomorphic function of $\lambda$ (possibly with monodromy at $\lambda = 0$), the sum over $K$ is absolutely convergent in the unit disk $|\lambda| < 1$. Thus the function $f(\lambda)$ determining the 4-point correlation function is a sum over $I$ (i.e. a finite sum for a rational CFT) of products of certain universal holomorphic and antiholomorphic functions, with coefficients given in terms of coefficients of 3-point functions. This begins to justify the claim that coefficients of 3-point functions determine all correlators in a CFT.

Function (1032) is called the *conformal block* of the 4-point function, cf. (805).

Computation (1031) can be thought of in terms of Segal's axioms, as cutting a 4-punctured sphere $\mathbb{CP}^1$ by a circle $S^1_r$ of radius $|\lambda| < r < 1$ centered at the origin and evaluating the corresponding composition as a sum over the basis in the space of states for the circle $S^1_r$:

$$\langle\Phi_4|\widehat{\Phi}_2(1)\widehat{\Phi}_1(\lambda)|\Phi_3\rangle = \sum_{p\in I}\sum_{\vec{k},\vec{\bar{k}}}\langle\Phi_4|\widehat{\Phi}_2(1)|\Phi_p^{\vec{k},\vec{\bar{k}}}\rangle\langle\Phi_p^{\vec{k},\vec{\bar{k}}}|\widehat{\Phi}_1(\lambda)|\Phi_3\rangle. \tag{1033}$$

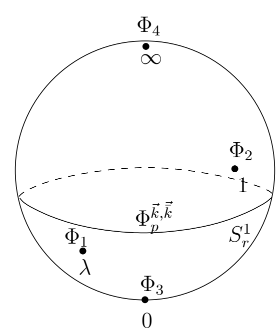



FIGURE 35. Cutting the 4-point correlator on $\mathbb{CP}^1$.

Crossing symmetry. Starting from the 4-point function (1029) and switching the roles of fields $\Phi_2(z_2)$ and $\Phi_3(z_3)$, one obtains another expression for the 4-point function:

$$f(\lambda) = \sum_{p\in I} C_{43p} C_{12p} \mathcal{F}^{34}_{12}(p|1-\lambda)\overline{\mathcal{F}}^{34}_{12}(p|1-\bar{\lambda}). \tag{1034}$$

Expressions (1031) and (1034) must agree in the region where r.h.s. in both cases is defined, i.e., in the region $\{\lambda \in \mathbb{C} \mid |\lambda| < 1,\ |1-\lambda| < 1\}$. This is the so-called "crossing symmetry." In particular it implies nontrivial quadratic relations (a version of associativity constraint) on the coefficients of 3-point functions of primary fields.

In terms of Segal's axioms, crossing symmetry is just the statement that cutting a 4-punctured sphere in two ways yields the same partition function.

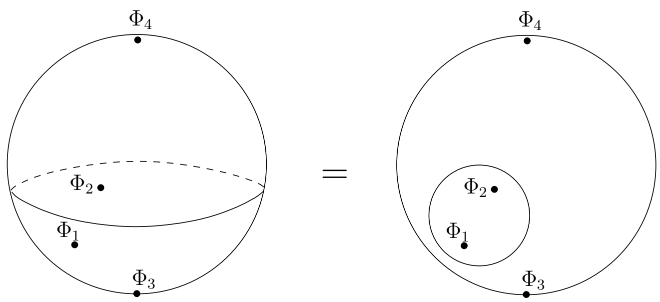



FIGURE 36. Crossing symmetry = cutting the 4-point correlator on $\mathbb{CP}^1$ in two ways.

Replacing punctures by finite circles, the same picture can be regarded as cutting a sphere with four holes into two pairs of pants in two different ways.

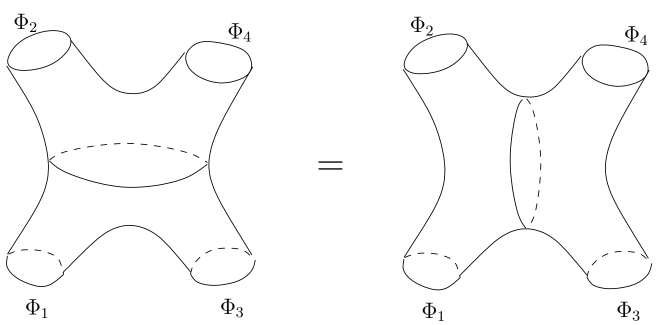



FIGURE 37. Another visualization of crossing symmetry.

Lecture 36, 11/21/2022

## 8. Wess-Zumino-Witten model

8.1. **Affine Lie algebras.** For details on affine Lie algebras we refer to [20], [25], [8].

Fix a compact simple Lie group $G$, denote its Lie algebra $\mathfrak{g}$ and the complexification of the latter $\mathfrak{g}_{\mathbb{C}} = \mathbb{C} \otimes \mathfrak{g}$.

**Definition 8.1.** The *loop group* $LG = \mathrm{Map}(S^1, G)$ is the group of $G$-valued smooth functions on a circle with pointwise multiplication. Its complexified Lie algebra $L\mathfrak{g} = \mathrm{Map}(S^1, \mathfrak{g}_\mathbb{C})$ – the Lie algebra of $\mathfrak{g}_\mathbb{C}$-valued functions on $S^1$ with pointwise Lie bracket is called the *loop Lie algebra.*

One can identify loop Lie algebra with the algebra of $\mathfrak{g}_\mathbb{C}$-valued Laurent polynomials

$$L\mathfrak{g} = \mathfrak{g}_\mathbb{C} \otimes \mathbb{C}[t, t^{-1}] \tag{1035}$$

where $t$ is the complex coordinate on the unit circle $S^1 = \{t \in \mathbb{C} \mid |t| = 1\}$.[143] The Lie bracket in $L\mathfrak{g}$ is

$$[X \otimes f, Y \otimes g] = [X, Y] \otimes fg \tag{1036}$$

for $X, Y \in \mathfrak{g}_\mathbb{C},\ f, g \in \mathbb{C}[t, t^{-1}]$.

**Definition 8.2.** The affine Lie algebra $\widehat{\mathfrak{g}}$ associated with $\mathfrak{g}$ is defined as the unique (up to normalization) central extension $\widehat{\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{C} \cdot \mathbb{K}$ of the loop Lie algebra, equipped with Lie bracket

$$[X \otimes f, Y \otimes g]_{\widehat{\mathfrak{g}}} = [X, Y] \otimes fg + \mathbb{K}\langle X, Y\rangle_\mathfrak{g} \mathrm{res}_{t=0}(df \cdot g). \tag{1037}$$

Here $\mathbb{K}$ is the central element, $\langle , \rangle_\mathfrak{g}$ is the Killing form[144] on $\mathfrak{g}$ and the residue $\mathrm{res}_{t=0}(\cdots)$ returns the coefficient of $t^{-1}dt$ in the 1-form $(\cdots)$.

One can write the Lie bracket (1037) more explicitly:

$$[X \otimes t^n, Y \otimes t^m] = [X, Y] \otimes t^{n+m} + \mathbb{K}\langle X, Y\rangle_\mathfrak{g} n \delta_{n,-m}. \tag{1038}$$

We will be using a shorthand notation $X_n \colon = X \otimes t^n$.

*Remark* 8.3. The statement that (1037) is the *unique* up to normalization central extension of the loop Lie algebra is tantamount to a statement about Lie algebra cohomology:

$$H^2_{\mathrm{Lie}}(L\mathfrak{g}, \mathbb{C}) = \mathbb{C}, \tag{1039}$$

where the nontrivial 2-cocycle is given by the rightmost term in (1037).

The result (1039) uses the fact that $\mathfrak{g}$ is simple. For $\mathfrak{g}$ semisimple with $n$ simple summands $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$, the r.h.s. of (1039) is $\mathbb{C}^n$ – there are $n$ independent 2-cocycles corresponding to Killing forms on $\mathfrak{g}_i$.

*Remark* 8.4. If we set $\mathfrak{g} = \mathbb{R}$ and $\langle X, Y\rangle_\mathfrak{g} = XY$ for $X, Y \in \mathbb{R}$,[145] then (1037) becomes the Lie bracket of the Heisenberg Lie algebra (476), (477), so in this case one has $\widehat{g} = \mathrm{Heis}$.

---

[143] One can choose different completions of the algebra of Laurent polynomials in (1035) corresponding to different regularity assumptions on the allowed maps from $S^1$ to $\mathfrak{g}_\mathbb{C}$, cf. the discussion of models of Witt algebra in Section 2.5.1. We will not dwell on this point.

[144] We assume the normalization of the Killing form $\langle X, Y\rangle_\mathfrak{g} = \mathrm{tr}(XY)$ – the trace of the product in the *fundamental* representation of $\mathfrak{g}$ (e.g. in the 2-dimensional representation for $\mathfrak{g} = \mathfrak{su}(2)$).

[145] This example is somewhat outside the setup of this section: $\mathbb{R}$ is not the Lie algebra of a compact simple group and this choice of $\langle , \rangle_\mathfrak{g}$ is not the Killing form (the Killing form for $\mathfrak{g} = \mathbb{R}$ is zero).

Similarly to the loop Lie algebra $L\mathfrak{g}$, the loop group $LG$ also has a family of central extensions $\widehat{LG}^k$,

$$1 \to \mathbb{C}^* \to \widehat{LG}^k \to LG \to 1, \tag{1040}$$

with the "level" parameter $k = 1, 2, 3, \ldots$; here $\widehat{LG}^k$ is a principal $\mathbb{C}^*$-bundle over $LG$ with first Chern class $c_1 = k \in H^2(LG, \mathbb{Z}) \simeq \mathbb{Z}$.

At the level of Lie algebra, the central extension $\widehat{LG}^k$ corresponds to the affine Lie algebra $\widehat{\mathfrak{g}}$ where $\mathbb{K}$ is identified with $k \cdot \mathrm{Id}$ – an integer multiple of identity (in particular, an $\widehat{LG}^k$-module is automatically a $\widehat{\mathfrak{g}}$-module, with $\mathbb{K}$ acting by $k \cdot \mathrm{Id}$).

8.1.1. *Highest weight modules over* $\widehat{\mathfrak{g}}$. Fix a decomposition

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_+ \oplus \mathfrak{h} \oplus \mathfrak{g}_- \tag{1041}$$

with $\mathfrak{h}$ the Cartan subalgebra, $\mathfrak{g}_+$ the span of positive roots $\{e_\alpha\}_{\alpha>0}$ of $\mathfrak{g}$ and $\mathfrak{g}_-$ the span of negative roots $\{e_\alpha\}_{\alpha<0}$.

Consider the following decomposition of the affine Lie algebra $\widehat{\mathfrak{g}}$:

$$\widehat{\mathfrak{g}} = \underbrace{(\mathfrak{g} \otimes t\,\mathbb{C}[t] \oplus \mathfrak{g}_+)}_{N_+} \oplus \underbrace{(\mathbb{C} \cdot \mathbb{K} \oplus \mathfrak{h})}_{N_0} \oplus \underbrace{(\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}] \oplus \mathfrak{g}_-)}_{N_-}. \tag{1042}$$

A Verma module over $\widehat{\mathfrak{g}}$ is defined (cf. footnote 137) as

$$V^{\widehat{\mathfrak{g}}}_{k,\lambda} = U(\widehat{\mathfrak{g}}) \otimes_{U(N_0 \oplus N_+)} \mathbb{C}_{k,\lambda}. \tag{1043}$$

Here:

- $k \in \mathbb{C}$ is the level[146] and $\lambda = (\lambda^1, \ldots, \lambda^r)$ is a highest weight of $\mathfrak{g}$, with $r = \dim \mathfrak{h}$ the rank of $\mathfrak{g}$. We assume that a basis $\tau^1, \ldots, \tau^r$ in $\mathfrak{h}$ is fixed.
- $\mathbb{C}_{k,\lambda}$ is a 1-dimensional module over $N_0 \oplus N_+$ where $N_+$ acts by zero, $\mathbb{K}$ acts by multiplication by the level $k$ and elements of the Cartan $\tau^i \in \mathfrak{h}$ act by multiplication by $\lambda^i$.
- $U(\cdots)$ is the universal enveloping algebra.

Let us denote by $v$ the highest weight vector in (1043) – the generator of $\mathbb{C}_{k,\lambda}$.

As in Virasoro case, in $V^{\widehat{\mathfrak{g}}}_{k,\lambda}$ one can have null vectors – vectors (distinct from the highest weight vector $v$) annihilated by $N_+$.

The *irreducible* highest weight module (of level $k$, with highest weight $\lambda$) is

$$M^{\widehat{\mathfrak{g}}}_{k,\lambda} = V^{\widehat{\mathfrak{g}}}_{k,\lambda}/\nu \tag{1044}$$

– the quotient of the Verma module by the maximal proper submodule. As in the Virasoro case, $\nu$ can also be described as

- the submodule generated by the null-vectors,
- or equivalently as the kernel of the sesquilinear form on $V^{\widehat{\mathfrak{g}}}_{k,\lambda}$ characterized by the properties $\langle v, v\rangle = 1$, $(X \otimes t^n)^+ = X^+ \otimes t^{-n}$.

*Remark* 8.5. It is convenient to adjoin to $\widehat{\mathfrak{g}}$ an extra generator ("grading operator" or "Euler vector field") $\delta = -t\frac{d}{dt}$ satisfying the commutation relations

$$[\delta, X \otimes t^j] = -jX \otimes t^j, \quad [\delta, \mathbb{K}] = 0. \tag{1045}$$

[146]In the context of Verma modules over $\widehat{\mathfrak{g}}$, the level does not have to be an integer and $\lambda \in \mathbb{C}^r$ can be any vector. However, more detailed structure of the Verma module (e.g. null vectors) is sensitive to integrality of $k$ and to $\lambda$ belonging to the weight lattice of $\mathfrak{g}$.

The algebra $\widehat{\mathfrak{g}} \oplus \mathbb{C} \cdot \delta$ is called the *affine Kac-Moody algebra.*

In a highest weight module $W$, if we set $\delta(v) = 0$, the module becomes $\mathbb{Z}_{\geq 0}$-graded by eigenvalues $n_\delta$ of $\delta$:

$$W = \bigoplus_{n_\delta = 0}^{\infty} W(n_\delta). \tag{1046}$$

We will call $n_\delta$ "depth."[147]

Note that each term $W(n_\delta)$ in the r.h.s. of (1046) carries a representation of $\mathfrak{g}$ (without the hat). In particular, for $W$ the Verma module and $n_\delta = 0$ one has that $V^{\widehat{\mathfrak{g}}}_{k,\lambda}(0)$ is the Verma module $V^{\mathfrak{g}}_\lambda$ of $\mathfrak{g}$ with highest weight $\lambda$ obtained by acting on $v$ by elements of $\mathfrak{g}_-$. Similarly, for the irreducible $\widehat{\mathfrak{g}}$-module one has that $M^{\widehat{\mathfrak{g}}}_{k,\lambda}(0) = M^{\mathfrak{g}}_\lambda$ is the irreducible representation of $\mathfrak{g}$ with highest weight $\lambda$.

Integrable highest weight modules. There is a distinguished set of irreducible highest weight modules over $\widehat{\mathfrak{g}}$ – "integrable highest weight modules" for positive integer level $k = 1, 2, 3, \ldots$ Their equivalent characterizations are:

(i) The module $M^{\widehat{\mathfrak{g}}}_{k,\lambda}$ is integrable if the action of $\widehat{\mathfrak{g}}$ on it integrates to the action of the group $\widehat{LG}^k$.
(ii) (Purely Lie algebraic definition.) The module $M^{\widehat{\mathfrak{g}}}_{k,\lambda}$ is integrable if it satisfies the "local nilpotency condition": for any $u \in M^{\widehat{\mathfrak{g}}}_{k,\lambda}$, any $j \in \mathbb{Z}$ and any root $e_\alpha$ of $\mathfrak{g}$ there exists $N$ such that

$$(e_\alpha \otimes t^j)^N u = 0. \tag{1047}$$

If the irreducible module $M^{\widehat{\mathfrak{g}}}_{k,\lambda}$ is integrable, we will also denote it $H_{k,\lambda}$.

**Theorem 8.6** (see Kac [20])**.** *There are finitely many integrable highest weight $\widehat{\mathfrak{g}}$-modules for any given positive integer level $k = 1, 2, 3, \ldots$.*

**Example 8.7.** Consider the case $G = SU(2)$. In the complexified Lie algebra $\mathfrak{g}_\mathbb{C} = \mathbb{C} \otimes \mathfrak{su}(2) = \mathfrak{sl}(2, \mathbb{C})$ one can consider the standard basis

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1048}$$

satisfying the commutation relations

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H. \tag{1049}$$

We consider $H$ as the basis vector for the Cartan subalgebra $\mathfrak{h}$, $E$ as the positive root and $F$ the negative root, i.e., the decomposition (1041) is

$$\mathfrak{sl}(2, \mathbb{C}) = \underbrace{\mathbb{C} \cdot E}_{\mathfrak{g}_+} \oplus \underbrace{\mathbb{C} \cdot H}_{\mathfrak{h}} \oplus \underbrace{\mathbb{C} \cdot F}_{\mathfrak{g}_-} \tag{1050}$$

[147] It is not a standard term; we use it because the word "level" already has another meaning in the context of affine Lie algebras.

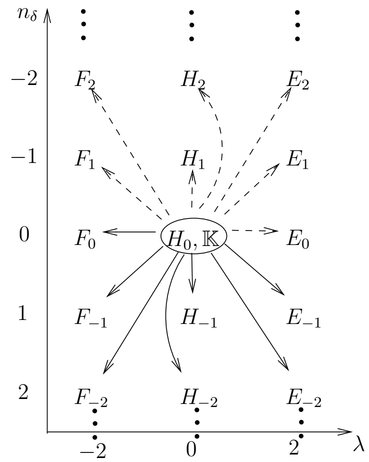



FIGURE 38. Root diagram of $\widehat{\mathfrak{su}(2)}$. Positive roots (basis of $N_+$, cf. (1042)) are indicated by dashed arrows and negative roots (basis of $N_-$) – by solid arrows. The encircled part corresponds to the Cartan subalgebra $N_0$. The diagram extends infinitely vertically.

Fix the level $k = 1, 2, 3, \ldots$. Then the irreducible highest weight module $M^{\widehat{\mathfrak{su}(2)}}_{k,\lambda}$ is integrable if and only if the highest weight $\lambda$ is an integer in the range $0 \le \lambda \le k$. We denote this integrable module $\mathcal{H}_{k,\lambda}$; it can be realized as the quotient of the Verma module $\mathbb{V}^{\widehat{\mathfrak{su}(2)}}_{k,\lambda}$ by the submodule $\nu$ generated by two null vectors[148]

$$\chi = (E_{-1})^{k-\lambda+1} v, \quad \psi = (F_0)^{\lambda+1} v. \tag{1051}$$

At depth $n_\delta = 0$, $\mathcal{H}_{k,\lambda}$ is the standard irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$ of dimension $\lambda + 1$ (or the "representation of spin $\frac{\lambda}{2}$").

As an illustration, consider the case $k = 1$, $\lambda = 0$. Here are the dimensions of first weight spaces (joint eigenspaces of $H$ and $\delta$), a.k.a. multiplicities of weights,

[148] In fact, $\mathbb{V}^{\widehat{\mathfrak{su}(2)}}_{k,\lambda}$ contains other null vectors, but they are contained in $\nu$.

in the Verma module $V_{1,0}^{\widehat{\mathfrak{su}(2)}}$:[149]

(1052)

| $n_\delta \setminus H-$e.v. | $-10$ | $-8$ | $-6$ | $-4$ | $-2$ | $0$ | $2$ | $4$ | $6$ | $8$ | $10$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 | $\boxed{1}$ | | | | | |
| 1 | 3 | 3 | 3 | 3 | 3 | 2 | 1 | | | | |
| 2 | 9 | 9 | 9 | 9 | 8 | 6 | 3 | 1 | | | |
| 3 | 22 | 22 | 22 | 21 | 19 | 14 | 8 | 3 | 1 | | |
| 4 | 51 | 51 | 50 | 48 | 42 | 32 | 19 | 9 | 3 | 1 | |

Here an empty cell means that the corresponding weight space is zero; we are indicating $H$-eigenvalue horizontally and $\delta$-eigenvalue vertically. The boxed entry corresponds to the highest vector $v$. The cell at position $(2i, i)$ corresponds to the weight space $\mathbb{C} \cdot (E_{-1})^i v$; the cell at position $(-2i, 0)$ corresponds to the weight space $\mathbb{C} \cdot (F_0)^i v$.

The similar table of multiplicities for the integrable module $H_{1,0}$ is the following:[150]

(1053)

| $n_\delta \setminus H-$e.v. | $-6$ | $-4$ | $-2$ | $0$ | $2$ | $4$ | $6$ |
|---|---|---|---|---|---|---|---|
| 0 | | | | $\boxed{1}$ | | | |
| 1 | | | 1 | 1 | 1 | | |
| 2 | | | 1 | 2 | 1 | | |
| 3 | | | 2 | 3 | 2 | | |
| 4 | | 1 | 3 | 5 | 3 | 1 | |

This table illustrates e.g. that at the representation of $\mathfrak{sl}(2,\mathbb{C})$ arising at a fixed depth $n_\delta > 0$ is finite-dimensional but generally not irreducible.

For the second integrable module arising at level $k = 1$, $H_{1,1}$, the table of multiplicities is

(1054)

| $n_\delta \setminus H-$e.v. | $-5$ | $-3$ | $-1$ | $1$ | $3$ | $5$ |
|---|---|---|---|---|---|---|
| 0 | | | 1 | $\boxed{1}$ | | |
| 1 | | | 1 | 1 | | |
| 2 | | 1 | 2 | 2 | 1 | |
| 3 | | 1 | 3 | 3 | 1 | |

8.1.2. *Sugawara construction.* Sugawara construction is a realization of Virasoro algebra (with some particular value of of central charge) in terms of quadratic expressions in generators of the affine Lie algebra $\widehat{\mathfrak{g}}$. Put another way, it is an embedding $\mathrm{Vir} \hookrightarrow U(\widehat{\mathfrak{g}})$ of Virasoro into (the degree two part of) the enveloping algebra of $\widehat{\mathfrak{g}}$.

---

[149] The generating function for the numbers in this table is

$$(1-\alpha^2\tau)^{-1}\prod_{n=0}^{\infty}((1-\alpha^2\tau^{2+n})(1-\tau^{1+n})(1-\alpha^{-2}\tau^n))^{-1}.$$

The coefficient of $\alpha^{2k}\tau^l$ in this function is the dimension of the weight space with $H$ eigenvalue $2k$ and $n_\delta = l$. This generating function counts the "nondecreasing" words made out of the ordered alphabet $E_{-1}; F_0, H_{-1}, E_{-2}; F_{-1}, H_{-2}, E_{-3}; F_{-2}, H_{-3}, E_{-4}, \ldots$ (ordered by $n_\delta - \frac{1}{2}(H-\text{eigenvalue})$) – such words give a Poincaré-Birkhoff-Witt basis in $U(N_-)$ and hence in the Verma module.

[150] See Figure 14.4 and Table 15.1 in [8]. For (1054) see Table 15.2 in [8].

Let $\{T^a\}$ be an orthonormal basis in $\mathfrak{g}$ with respect to the Killing form. The quadratic Casimir element

(1055) $$\mathrm{Cas}\colon = \sum_a T^a T^a \in U(\mathfrak{g})$$

acts on the irreducible $\mathfrak{g}$-module with highest weight $\lambda$ by multiplication by a constant $C_\lambda$,

(1056) $$\mathrm{Cas} = C_\lambda \cdot \mathrm{Id} \quad \text{on } M_\lambda^{\mathfrak{g}}.$$

We also denote the normalized trace of the Casimir element in the adjoint representation of $\mathfrak{g}$ by

(1057) $$h^\vee\colon = \frac{\mathrm{tr}_{\mathfrak{g}} \mathrm{ad}(\mathrm{Cas})}{2 \dim \mathfrak{g}}$$

– it is the so-called dual Coxeter number of $\mathfrak{g}$.

**Theorem 8.8** (Sugawara, [36]). *Let $W$ be a highest weight $\widehat{\mathfrak{g}}$-module on which $\mathbb{K}$ acts by multiplication by a number $k \in \mathbb{C}$, $k \neq -h^\vee$. Consider the elements*

(1058) $$L_n = \frac{1/2}{k+h^\vee} \sum_{j\in\mathbb{Z}} \sum_{a=1}^{\dim \mathfrak{g}} : T_j^a T_{n-j}^a : \qquad \in \mathrm{End}(W),$$

*where $T_i^a = T^a \otimes t^i$ and the normal ordering symbol $:\cdots:$ puts $T_{>0}^a$ to the right of $T_{<0}^0$.*[151] *Then:*

*(a) The operators $L_n$ satisfy Virasoro commutation relations with central charge*

(1059) $$c = \frac{k \cdot \dim \mathfrak{g}}{k+h^\vee}.$$

*(b) The commutation relation between operators (1058) and the generators of $\widehat{\mathfrak{g}}$ is*

(1060) $$[L_n, X_j] = -j X_{n+j}$$

*for any $X \in \mathfrak{g}$.*

*(c) If $W = H_{k,\lambda}$ is an integrable $\widehat{\mathfrak{g}}$-module and $v$ is the highest weight vector, then one has*

(1061) $$L_0 v = \frac{\frac{1}{2} C_\lambda}{k+h^\vee} v,$$

*with $C_\lambda$ the value of the quadratic Casimir in the representation $M_\lambda^{\mathfrak{g}}$, as in (1056).*

For the proof see e.g. Theorem 10.1 and Proposition 10.1 in [21].

Comparing (1060), (1061) and (1045) we note that in the decomposition of the integrable module by depth

(1062) $$H_{k,\lambda} = \bigoplus_{n_\delta \geq 0} H_{k,\lambda}(n_\delta),$$

the term $H_{k,\lambda}(n_\delta)$ in the r.h.s. is the eigenspace of $L_0$ with eigenvalue

(1063) $$\Delta + n_\delta,$$

[151] Note that the normal ordering only affects the expression for $L_0$, as $T_j^a$ and $T_{n-j}^a$ commute for $n \neq 0$.

where

$$\Delta = \frac{\frac{1}{2}C_\lambda}{k+h^\vee} \tag{1064}$$

is the constant in (1061). Put another way, one has

$$L_0 = \Delta \cdot \mathrm{Id} + \delta \tag{1065}$$

as an equality of operators on $H_{k,\lambda}$.

They are also $\widehat{\mathfrak{g}}$-primary

Also note that all elements of $H_{k,\lambda}(0)$ are annihilated by $L_{>0}$, i.e. they are all Virasoro-highest weight (or "Virasoro-primary") vectors with $L_0$-eigenvalue $\Delta$. There may also be other Virasoro-primary vectors in $H_{k,\lambda}$ emerging at depths $n_\delta > 0$.

**Example 8.9.** For $\mathfrak{g} = \mathfrak{su}(2)$, one has $h^\vee = 2$ (more generally, for $\mathfrak{g} = \mathfrak{su}(N)$, one has $h^\vee = N$), thus (1058) becomes

$$L_n = \frac{1/2}{k+2} \sum_{j\in\mathbb{Z}} \sum_{a=1}^{3} :T^a_j T^a_{n-j}: . \tag{1066}$$

For the orthonormal basis $\{T^a\}$ in $\mathfrak{su}(2)$, one can choose the appropriately normalized Pauli matrices,

$$T^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad T^2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad T^3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1067}$$

The operators (1066) satisfy Virasoro commutation relations with central charge

$$c = \frac{3k}{k+2}. \tag{1068}$$

For $W = H_{k,\lambda}$ an integrable $\widehat{\mathfrak{su}(2)}$-module, the highest vector satisfies

$$L_0 v = \frac{\frac{1}{4}\lambda(\lambda+2)}{k+2} v, \tag{1069}$$

since for $\mathfrak{g} = \mathfrak{su}(2)$ the value of the quadratic Casimir in an irreducible representation is

$$C_\lambda = \frac{1}{2}\lambda(\lambda+2). \tag{1070}$$

Lecture 37, 11/28/2022

8.2. **Wess-Zumino-Witten model as a classical field theory.** Let $G$ be a compact simple, simply connected matrix group (keeping in mind $G = SU(2)$ in the fundamental representation as the main example).

Consider the following 3-form on $G$:

$$\sigma = \frac{1}{24\pi^2} \mathrm{tr}\left( (X^{-1}dX) \wedge (X^{-1}dX) \wedge (X^{-1}dX) \right) \quad \in \Omega^3(G) \tag{1071}$$

It is known as the Cartan 3-form on $G$; it is left- and right-invariant under $G$-action and represents the image of the generator of $H^3(G,\mathbb{Z}) \simeq \mathbb{Z}$ in de Rham cohomology $H^3(G,\mathbb{R})$. In particular, the form $\sigma$ has integer periods.

**Example 8.10.** For $G = SU(2)$ the group manifold is the 3-sphere and $\sigma$ is a volume form of unit total volume, $\int_G \sigma = 1$. The funny normalization factor in (1071) is tuned so as to have this property.

*Remark* 8.11. The form $\sigma$ is constructed out of the Maurer-Cartan 1-form

$$\mu = X^{-1}dX \quad \in \Omega^1(G,\mathfrak{g}) \tag{1072}$$

– the unique left-invariant $\mathfrak{g}$-valued 1-form form on the group $G$ such that its value at the group unit $\mu|_e\colon \underbrace{T_eG}_{\mathfrak{g}} \to \mathfrak{g}$ is identity. In terms of $\mu$, the Cartan 3-form is

$$\sigma = \frac{1}{48\pi^2}\langle \mu \overset{\wedge}{,} [\mu \overset{\wedge}{,} \mu]\rangle_{\mathfrak{g}}. \tag{1073}$$

<u>The action functional.</u> Let $\Sigma$ be a closed Riemannian surface. Fields of the model are smooth maps

$$g\colon \Sigma \to G \tag{1074}$$

and the action functional is[152]

$$S_\Sigma(g)\colon = -\frac{i}{4\pi}\int_\Sigma \mathrm{tr}\left(g^{-1}\boldsymbol{\partial} g\wedge g^{-1}\bar{\boldsymbol{\partial}} g\right) + \mathrm{WZ}(g) \tag{1075}$$

where the last term is the so-called "Wess-Zumino term." It is defined as

$$\mathrm{WZ}(g)\colon = -\frac{i}{12\pi}\int_B \mathrm{tr}\left(\widetilde{g}^{-1}d\widetilde{g}\right)^{\wedge 3} = -2\pi i\int_B \widetilde{g}^*\sigma \tag{1076}$$

where $B$ is any compact oriented 3-manifold with boundary $\partial B = \Sigma$ (e.g. one can choose $B$ to be a handlebody)[153] and $\widetilde{g}\colon B\to G$ any smooth extension of the map $g\colon \Sigma\to G$ into $B$ ("extension" means that $\widetilde{g}$ must satisfy $\widetilde{g}|_{\partial B} = g$).

**Lemma 8.12.** *For a fixed map $g\colon\Sigma\to G$, the Wess-Zumino term $\mathrm{WZ}(g)$ modulo $2\pi i\mathbb{Z}$ does not depend on the choice of 3-manifold $B$ cobounding $\Sigma$ and on the choice of extension $\widetilde{g}$.*

*Sketch of proof.* Denote by $\mathrm{WZ}^{B,\widetilde{g}}(g)$ the r.h.s. of (1076). Let $B, B'$ be two 3-manifolds cobounding $\Sigma$ and $\widetilde{g},\widetilde{g}'$ some extensions of $g$ from $\Sigma$ into $B$ and into $B'$, respectively. One has

$$\begin{aligned}\mathrm{WZ}^{B,\widetilde{g}}(g) - \mathrm{WZ}^{B',\widetilde{g}'}(g) &= -2\pi i\left(\int_B \widetilde{g}^*\sigma - \int_{B'}(\widetilde{g}')^*\sigma\right) = \\ &= -2\pi i\left(\int_B \widetilde{g}^*\sigma + \int_{\overline{B}'}(\widetilde{g}')^*\sigma\right) = -2\pi i\int_{\check{B}}\check{g}^*\sigma,\end{aligned} \tag{1077}$$

where $\overline{B}'$ is $B'$ with reversed orientation. Here in the last step we defined the closed 3-manifold $\check{B}$ as $B$ glued to $\overline{B}'$ along $\Sigma$, and we defined the "glued" map $\check{g}\colon\check{B}\to G$ as the map whose restrictions to $B$, $\overline{B}'$ are $\widetilde{g}$ and $\widetilde{g}'$, respectively.

[152]Recall that $\boldsymbol{\partial} = dz\frac{\partial}{\partial z}$, $\bar{\boldsymbol{\partial}} = d\bar{z}\frac{\partial}{\partial\bar{z}}$ are the holomorphic and antiholomorphic Dolbeault differentials.

[153]One says "the 3-manifold $B$ *cobounds* the surface $\Sigma$."

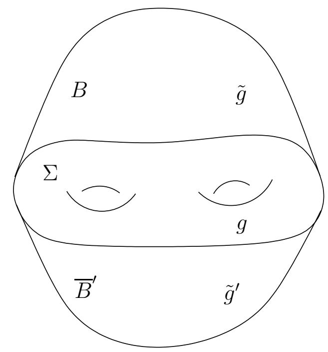



FIGURE 39. Closed 3-manifold $\check{B}$ glued out of $B$ and $\overline{B}'$ along $\Sigma$ and the corresponding glued map to $G$.

Thus, one has

$$\mathrm{WZ}^{B,\widetilde{g}}(g) - \mathrm{WZ}^{B',\widetilde{g}'}(g) = -2\pi i \langle [\check{B}], \check{g}^*[\sigma]\rangle \in 2\pi i\mathbb{Z} \tag{1078}$$

– the pairing (up to normalization) of the fundamental class of the closed 3-manifold $\check{B}$ with the pullback along $\check{g}$ of the integral cohomology class $[\sigma] \in H^3(G,\mathbb{Z})$.[154] □

In particular, this lemma implies that Wess-Zumino-Witten (WZW) action (1075) modulo $2\pi i\mathbb{Z}$ is well-defined (independent of choices of cobounding 3-manifold $B$ and the extension $\widetilde{g}$). Thus, for $k = 1, 2, 3, \ldots$ an integer (the "level" of Wess-Zumino-Witten model), the expression

$$e^{-kS_\Sigma(g)} \tag{1079}$$

is well-defined. This expression is the integrand in the path integral for the Wess-Zumino-Witten model,

$$Z_k(\Sigma) = \text{“}\int_{\mathrm{Map}(\Sigma,G)} \mathcal{D}g\; e^{-kS_\Sigma(g)}.\text{”} \tag{1080}$$

Here the level $k = 1, 2, 3, \ldots$ is a parameter of the theory playing the role of inverse Planck constant, $k = \hbar^{-1}$, see Remark 1.14.

*Remark* 8.13. (a) In the action (1075) the first term is real and the second term is imaginary.
(b) One can write the action (1075) in terms of the Maurer-Cartan 1-form on $G$:

$$S_\Sigma(g) = -\frac{1}{8\pi}\int_\Sigma \langle g^*\mu \overset{\wedge}{,} *_{\mathrm{Hodge}} g^*\mu\rangle_{\mathfrak{g}} \underbrace{- \frac{i}{24\pi}\int_B \widetilde{g}^*\langle \mu \overset{\wedge}{,} [\mu \overset{\wedge}{,} \mu]\rangle_{\mathfrak{g}}}_{\mathrm{WZ}(g)} \tag{1081}$$

The benefit of this rewriting is that it one can use it to define WZW action for non-matrix Lie groups.

[154] By abuse of notations, here $[\sigma]$ stands for the class in $H^3(G,\mathbb{Z})$ whose image in $H^3(G,\mathbb{R})$ is the class of the Cartan 3-form in de Rham cohomology. We also remark that in the special case $G = SU(2)$ the r.h.s. of (1078) admits the interpretation as $-2\pi i$ times the degree of the map $\check{g}\colon \check{B} \to SU(2) \simeq S^3$ between oriented closed 3-manifolds.

(c) Although the Wess-Zumino term is non-local (not an integral over $\Sigma$), its variation is local:

$$\delta \mathrm{WZ} = \frac{i}{4\pi} \int_\Sigma \operatorname{tr} g^{-1}\delta g(\boldsymbol{\partial}(g^{-1}\bar{\boldsymbol{\partial}}g) + \bar{\boldsymbol{\partial}}(g^{-1}\boldsymbol{\partial}g)) \tag{1082}$$

(note that the integral is over $\Sigma$, not over $B$). Putting this together with the variation of the first term of (1075) (let us denote it $E(g)$),

$$\delta E = \frac{i}{4\pi} \int_\Sigma \operatorname{tr} g^{-1}\delta g(\boldsymbol{\partial}(g^{-1}\bar{\boldsymbol{\partial}}g) - \bar{\boldsymbol{\partial}}(g^{-1}\boldsymbol{\partial}g)), \tag{1083}$$

one obtains the variation of the full action (1075) is

$$\delta S_\Sigma = \frac{i}{2\pi} \int_\Sigma \operatorname{tr} (g^{-1}\delta g)\boldsymbol{\partial}(g^{-1}\bar{\boldsymbol{\partial}}g). \tag{1084}$$

An equivalent expression is:

$$\delta S_\Sigma = -\frac{i}{2\pi} \int_\Sigma \operatorname{tr}(\delta g\, g^{-1})\bar{\boldsymbol{\partial}}(\boldsymbol{\partial}g\, g^{-1}). \tag{1085}$$

Euler-Lagrange equation. For the discussion of the Euler-Lagrange equation (especially the holomorphic factorization of solutions (1087)) and symmetries it is convenient to complexify the space of classical fields, i.e., to allow fields $g$ to be maps from $\Sigma$ to the *complexified* group $G_\mathbb{C}$ rather than the compact group $G$.

The Euler-Lagrange equation corresponding to the action (1075) is read off from the fromula for the variation (1084):

$$\boldsymbol{\partial}(g^{-1}\bar{\boldsymbol{\partial}}g) = 0. \tag{1086}$$

Equivalently, the same equation can be written as $\bar{\boldsymbol{\partial}}(\boldsymbol{\partial}g\, g^{-1}) = 0$.

The general solution of the Euler-Lagrange equation (1086) is:

$$g(z) = h_1(z)\overline{h_2(z)}, \tag{1087}$$

where $h_1, h_2 \colon \Sigma \to G_\mathbb{C}$ are two *holomorphic* maps into the complexified group.

*Remark* 8.14. One can consider Wess-Zumino-Witten theory for $G = U(1)$. (This group fails our assumptions: it is neither simple nor simply connected, but nevertheless one can play with it.) Then the field $g \colon \Sigma \to G$ can be parametrized as $g = e^{i\phi}$. The action (1075) is then simply the action of a free boson (with values in $S^1$); the Wess-Zumino term vanishes. Euler-Lagrange equation (1086) becomes the equation of a harmonic function $\Delta\phi = 0$. The factorization (1087) simply becomes the statement that any harmonic function is a sum of a holomorphic and an antiholomorphic function, $\phi(z) = \chi_1(z) + \overline{\chi_2(z)}$.

Symmetry and conserved currents. The action (1075) is invariant under the following transformations of the field:

$$g(z) \mapsto g'(z) = \Omega_1(z)g(z)\overline{\Omega_2(z)} \tag{1088}$$

where $\Omega_1, \Omega_2 \colon \Sigma \to G_\mathbb{C}$ are two arbitrary holomorphic maps.[155]

[155] The transformations (1088) are sometimes called "gauge symmetry" in the literature. We would argue that it is not a very good term here, since the generators of the symmetry are not local: they are holomorphic (rather than, say, smooth) maps from $\Sigma$ to the target, and for holomorphic maps one doesn't have partitions of unity, so one cannot have a bump function as a generator.

The invariance under transformations (1088) corresponds by Noether theorem to having two conserved currents

$$\begin{aligned}\mathbf{J} &= \boldsymbol{\partial} g\cdot g^{-1}\in \Omega^{1,0}(\Sigma,\mathfrak{g}),\\ \overline{\mathbf{J}} &= g^{-1}\bar{\boldsymbol{\partial}} g\in \Omega^{0,1}(\Sigma,\mathfrak{g}),\end{aligned} \tag{1089}$$

satisfying the conservation properties

$$\bar{\boldsymbol{\partial}}\mathbf{J} \underset{EL}{\sim} 0,\quad \boldsymbol{\partial}\overline{\mathbf{J}} \underset{EL}{\sim} 0. \tag{1090}$$

*Remark* 8.15. The action (1075) is the sum of the action of a sigma model with target a group (the natural quadratic "energy of a map") and a seemingly complicated nonlocal cubic term $\mathrm{WZ}(g)$. One might reasonably ask: why add this extra term to the sigma model? The answer is that adding this term actually makes the model much simpler: it creates two separately conserved holomorphic and antiholomorphic Noether currents $\mathbf{J},\overline{\mathbf{J}}$, leads to simpler Euler-Lagrange equation which allows an explicit solution (1087). Ultimately, the addition of the Wess-Zumino term to the model results in the factorization of the model into a holomorphic and an antiholomorphic sector (this statement makes sense both at the classical and at the quantum level).

*Remark* 8.16. One can also write down the currents (1089) without referring to the matrix structure of the group $G$:

$$\mathbf{J}=\frac{1}{2}(\mathrm{id}+i*_{\mathrm{Hodge}})g^*\mu_R,\quad \overline{\mathbf{J}}=\frac{1}{2}(\mathrm{id}-i*_{\mathrm{Hodge}})g^*\mu_L, \tag{1091}$$

where $\mu_L$ is the left-invariant Maurer-Cartan form (1072) and $\mu_R$ is its right-invariant counterpart ($\mu_R = dX\,X^{-1}$ for a matrix group).

Polyakov-Wiegmann formula. For the next discussion it is important to know how the action (1075) interacts with pointwise products of fields (as maps to the group).

**Theorem 8.17** (Polyakov-Wiegmann)**.** *For $\Sigma$ a closed Riemannian surface and $f,g\colon \Sigma\to G$ two maps to the group, one has*

$$S_\Sigma(f\cdot g)=S_\Sigma(f)+S_\Sigma(g)+\underbrace{\frac{i}{2\pi}\int_\Sigma \mathrm{tr}\left(f^{-1}\bar{\boldsymbol{\partial}} f\wedge \boldsymbol{\partial} g\cdot g^{-1}\right)}_{\Gamma_\Sigma(f,g)}. \tag{1092}$$

*Here $\cdot$ in the l.h.s. stands for the pointwise product of maps to $G$.*

Thus, the action is "almost" additive w.r.t. pointwise product of fields, with the defect given by the rightmost term in (1092) which we denoted $\Gamma_\Sigma(f,g)$.

We note that the "defect" $\Gamma_\Sigma$ in (1092) is a 2-cocycle for the group $\mathrm{Map}(\Sigma,G)$ (with trivial coefficients), i.e., for any triple of maps $f,g,h\colon \Sigma\to G$ it satisfies[156]

$$\Gamma_\Sigma(g,h)-\Gamma_\Sigma(fg,h)+\Gamma_\Sigma(f,gh)-\Gamma_\Sigma(f,g)=0. \tag{1093}$$

[156]Indeed, $0=S_\Sigma((fg)h)-S_\Sigma(f(gh))=S_\Sigma(fg)+S_\Sigma(h)+\Gamma_\Sigma(fg,h)-S_\Sigma(f)-S_\Sigma(gh)-\Gamma_\Sigma(f,gh)=\cancel{S_\Sigma(f)}+\cancel{S_\Sigma(g)}+\Gamma_\Sigma(f,g)+\cancel{S_\Sigma(h)}+\Gamma_\Sigma(fg,h)-\cancel{S_\Sigma(f)}-\cancel{S_\Sigma(g)}-\cancel{S_\Sigma(h)}-\Gamma_\Sigma(g,h)-\Gamma_\Sigma(f,gh)=-$l.h.s. of (1093).

8.2.1. *Case of surfaces with boundary.* Here we briefly sketch a geometric construction from [25].

It is not straightforward to generalize the action (1075) to surfaces with boundary, due to the presence of a nonlocal term in the action. It turns out one can still do it, with two caveats:

- one should consider the exponential of the action $e^{-kS_\Sigma}$ instead of the action itself (we assume that the level $k = 1, 2, 3, \ldots$ is fixed),
- instead of obtaining $e^{-kS_\Sigma}$ as a function on the space of fields on a surface with boundary, it will be a section of a certain line bundle over $\mathrm{Fields}_\Sigma$.

Let $\Sigma$ be a compact Riemannian surface with $n$ boundary circles. Construct a closed surface $\Sigma'$ by attaching $n$ disks $D_1, \ldots, D_n$ to the boundary of $\Sigma$ (i.e. attach a disk to each boundary circle): $\Sigma' = \Sigma \cup \bigsqcup_{i=1}^n D_i$.

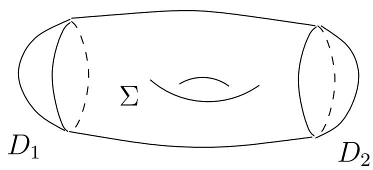



FIGURE 40. Closed surface obtained from $\Sigma$ by attaching disks along boundary circles.

The basic idea is to define the WZW action on a surface with boundary via

$$e^{-kS_\Sigma(g)} \colon = e^{-kS_{\Sigma'}(g')} \tag{1094}$$

where $g$ is a map $\Sigma \to G$ and $g'$ is some extension of $g$ as a map $\Sigma' \to G$ (i.e. an extension of the map $g$ into each disk $D_i$ is to be chosen).

The ambiguity in the choice of the extension $g'$ leads to the idea that the expression $e^{-kS_\Sigma(g)}$ should be understood as taking values in the fiber of the complex line bundle

$$\begin{array}{c} \mathcal{L}^k \boxtimes \cdots \boxtimes \mathcal{L}^k \\ \Big\downarrow \\ LG \times \cdots \times LG \end{array} \tag{1095}$$

over the point $g|_{\partial\Sigma} \in \mathrm{Map}(\partial\Sigma, G) \simeq LG^{\times n}$, i.e., over the boundary value of the map $g$ seen as a collection of loops in $G$.

The complex line bundle over the loop group

$$\mathcal{L}^k \to LG, \tag{1096}$$

several copies of which appear in (1095), is constructed as follows (see [25] for details). Consider the trivial line bundle

$$\mathrm{Map}(D, G) \times \mathbb{C} \to \mathrm{Map}(D, G) \tag{1097}$$

with $D$ the unit disk, and consider the following equivalence relation: two pairs

$$(f_D \colon D \to G, u \in \mathbb{C}) \quad \sim \quad (g_D \colon D \to G, v \in \mathbb{C}) \tag{1098}$$

are considered equivalent if

- $f_D$ and $g_D$ agree on the boundary circle: $f_D|_{\partial D} = g_D|_{\partial D}$,

- one has
$$v = u \cdot e^{-kS_{\mathbb{CP}^1}(h) - k\Gamma_D(f_D, h_D)},$$
where $h\colon \mathbb{CP}^1 \to G$ is defined on $D$ as $f_D^{-1} g_D =: h_D$ and extended by 1 to $\mathbb{CP}^1 \backslash D$; $\Gamma_D$ is given by the same formula as in (1092) (but the integral is over $D$).

Quotienting the line bundle (1097) by this equivalence relation produces a line bundle over $LG$ (loops in $G$ seen as boundary values of functions $f_D$) which we call $\mathcal{L}^k$.

By construction and as a consequence of Polyakov-Wiegmann formula, one indeed has that $e^{-kS_\Sigma(g)}$ defined by (1094) seen as an element in the fiber of (1095) over the boundary value of $g$ is independent of the extension $g'$.[157]

Put another way, the exponentiated action for $\Sigma$ a surface with boundary is not a function on $\mathrm{Fields}_\Sigma = \mathrm{Map}(\Sigma, G)$ but rather is a section of a line bundle,

$$e^{-kS_\Sigma} \in \Gamma(\mathrm{Fields}_\Sigma, \pi^*(\mathcal{L}^k)^{\boxtimes n}) \tag{1099}$$

where

$$\pi\colon \mathrm{Fields}_\Sigma \to \underbrace{\mathrm{Map}(\partial\Sigma, G)}_{\mathrm{Fields}_{\partial\Sigma}} \simeq LG^{\times n} \tag{1100}$$

is the restriction of the map to the boundary. We will denote $\mathcal{L}_{\partial\Sigma} := (\mathcal{L}^k)^{\boxtimes n}$ seen as a line bundle over $\mathrm{Fields}_{\partial\Sigma}$.

*Remark* 8.18. (a) Denoting $\mathcal{L}^1 =: \mathcal{L}$ one has

$$\mathcal{L}^k = \mathcal{L}^{\otimes k}. \tag{1101}$$

Thus, the superscript in $\mathcal{L}^k$ can be interpreted as the tensor power of a special line bundle corresponding to $k = 1$. The first Chern class of the bundle $\mathcal{L}$ in de Rham cohomology $H^2(LG)$ is represented by the 2-form

Maybe write it more cleanly as pull-push in cohomology, not in de Rham representatives?

$$\omega = p_*(\mathrm{ev}^*\sigma) \quad \in \Omega^2(LG), \tag{1102}$$

where $\sigma$ is the Cartan 3-form on $G$ (1071) and $p$ and ev are the projection and evaluation maps in the diagram

$$\begin{array}{ccc} LG \times S^1 & \xrightarrow{\mathrm{ev}} & G \\ {\scriptstyle p}\big\downarrow & & \\ LG & & \end{array} \tag{1103}$$

The map ev evaluates the loop in $G$ at a given point of $S^1$; $p_*$ stands for the fiber integral over $S^1$.

[157] In a bit more detail, one chooses an extension $g'$ of $g\colon \Sigma \to G$ into the disks $D_i$ and thinks of $e^{-kS_\Sigma(g)}$ as a tuple $\left(\{g'|_{D_i}\} \in \mathrm{Map}(D, G)^{\times n}, e^{-kS_{\Sigma'}(g')} \in \mathbb{C}\right)$ up to an equivalence as the one on (1097), extended in an obvious way to $n$ disks.

For instance, if $\Sigma$ has a single boundary circle (i.e. $n = 1$), if $g'$ and $g''$ are two extensions of the map $g\colon \Sigma \to G$ into the single attached disk $D$, one has $g'' = g'h$ with the map $h\colon \Sigma' \to G$ ("discrepancy" of the two extensions) equal to 1 on $\Sigma$ and nontrivial in $D$, the pairs $(g'h|_D, e^{-kS_\Sigma(g'h)})$ and $(g'|_D, e^{-kS_\Sigma(g')})$ are equivalent precisely because by Polyakov-Wiegmann formula one has $e^{-kS_\Sigma(g'h)} = e^{-kS_\Sigma(g')} \cdot e^{-kS_\Sigma(h) - k\Gamma_\Sigma(g', h)}$. We note that in the r.h.s. here $\Gamma_\Sigma$ can be replaced with $\Gamma_D$ and $S_\Sigma(h)$ can be replaced with $S_{\mathbb{CP}^1}(\widetilde{h})$ where $\widetilde{h}$ is the extension of $h|_D$ into $\mathbb{CP}^1 \backslash D$ by 1 (the intuition here is that since $h$ is trivial outside $D$, the surface $\Sigma$ can be replaced by anything, including a complementary disk).

(b) One has a product on the total space of the line bundle $\mathcal{L}^k$ given by

$$(g_1, u_1 e^{-kS_D(g_1)}) * (g_2, u_2 e^{-kS_D(g_2)}) = (g_1 \cdot g_2\ ,\ u_1 u_2 e^{-kS_D(g_1 g_2) - k\Gamma_D(g_1, g_2)}) \tag{1104}$$

with $u_{1,2} \in \mathbb{C}$ and $g_{1,2}\colon D \to G$. Here we understand that on both sides we pass to equivalence classes under (1098). Removing the zero-section from $\mathcal{L}^k$ one obtains a group which is none other than the central extension of the loop group we mentioned in Section 8.1 (see (1040)):

$$\mathcal{L}^k \backslash \{\text{zero-section}\} = \widehat{LG}^k. \tag{1105}$$

Symmetry of the model on a surface with boundary. Fix $\Sigma$ a surface with boundary. One has a left and a right action of the group $\mathrm{Map}(\Sigma, G)$ on $\mathrm{Fields}_\Sigma = \mathrm{Map}(\Sigma, G)$ coming from multiplication in the target $G$ from the left or from the right. One also has left and right actions of the group $\mathrm{Map}(M, G)$ on the space of sections of the line bundle $\mathcal{L}_{\partial\Sigma} \to \mathrm{Fields}_{\partial\Sigma}$. write formulas for the action

The symmetry (1088) for $\Sigma$ with boundary becomes the following statement.

**Lemma 8.19.** *The exponentiated action*

$$e^{-kS_\Sigma} \in \Gamma(\mathrm{Fields}_\Sigma, \pi^* \mathcal{L}_{\partial\Sigma}) \tag{1106}$$

*is*

- *left-invariant under* holomorphic *maps* $\Omega\colon \Sigma \to G_\mathbb{C}$ *and*
- *right-invariant under* antiholomorphic *maps* $\Omega^*\colon \Sigma \to G_\mathbb{C}$,

*where maps act both on the fields and on the bundle* $\mathcal{L}_{\partial\Sigma}$ *in (1106).*

For the proof see [25, Proposition 1.11].

Path integral heuristics. The path integral on a surface with boundary

$$Z(\Sigma) = \int_{g|_{\partial\Sigma} = g_\partial} \mathcal{D}g\, e^{-kS_\Sigma(g)} \quad \in \Gamma(\mathrm{Fields}_{\partial\Sigma}, \mathcal{L}_{\partial\Sigma})^{\mathrm{Hol}(\Sigma, G_\mathbb{C}) \times \mathrm{Antihol}(\Sigma, G_\mathbb{C})} \tag{1107}$$

is to be thought of as averaging the exponentiated action over fields with fixed boundary value $g_\partial \in \mathrm{Fields}_\partial$, and the value of the path integral is not a number but an element in the line $\mathcal{L}_{\partial\Sigma}|_{g_\partial}$. Thus, considering the path integral with all possible boundary conditions one has a section of $\mathcal{L}_{\partial\Sigma}$. By the invariance property of the exponentiated action (Lemma 8.19), this section should be invariant under holomorphic maps $\Sigma \to G_\mathbb{C}$ acting from the left and antiholomorphic maps $\Sigma \to G_\mathbb{C}$ acting from the right. This invariance property of the path integral is a variant of Ward identity. Lecture 38, 11/30/2022

8.3. **Quantum Wess-Zumino-Witten model.** We fix as before a compact simple simply-connected group $G$ and a level $k = 1, 2, 3, \ldots$. It is possible to quantize the classical WZW theory – either by canonical/geometric quantization in the hamiltonian formalism, or by path integral. Here we will just outline the resulting (quantum) CFT.

Space of states/space of fields. The space of states of the model associated to the circle – or equivalently the space of fields $V$ – is

$$\mathcal{H} = \bigoplus_{\lambda \in I_k} H_{k,\lambda} \otimes H^*_{k,\lambda} \tag{1108}$$

where the sum is over integrable highest weight modules of $\widehat{\mathfrak{g}}$ at level $k$ (we denote the set of corresponding highest weights $I_k$); the summand is a tensor product of the integrable module and the dual one, seen as a module over $\widehat{\mathfrak{g}}\oplus\widehat{\overline{\mathfrak{g}}}$ – two copies of the affine Lie algebra.

The space $\mathcal{H}_k$ can be identified with the space of sections of the line bundle $\mathcal{L}^k$ over the loop group (1096) via the inclusion

(1109)
$$\begin{array}{ccc} \bigoplus_{\lambda\in I_k} H_{k,\lambda}\otimes H^*_{k,\lambda} & \to & \Gamma(LG,\mathcal{L}^k) \\ \phi_0\in \mathrm{End}(H_{k,\lambda}) & \mapsto & \Big(\phi(g_+g_0g_-)\colon = \mathrm{tr}_{M^{\mathfrak{g}}_\lambda}(\phi_0\cdot\rho_\lambda(g_0))\cdot e^{-kS_D(g_+g_0g_-)}\Big) \end{array}$$

Here $g_\pm$ are a holomorphic and an antiholomorphic map $D\to G_\mathbb{C}$, taking value $1\in G$ at a base point on the boundary circle $1\in\partial D$; $g_0\colon D\to G_\mathbb{C}$ is a constant map; $\rho_\lambda(g)$ is the linear operator on $M^{\mathfrak{g}}_\lambda$ representing the action of the group element $g$. In (1109) both sides carry a natural action of $\widehat{\mathfrak{g}}\oplus\widehat{\overline{\mathfrak{g}}}$ and these actions are intertwined by the inclusion.

By Sugawara construction, $\mathcal{H}_k$ carries an action of two copies of Virasoro algebra $\mathrm{Vir}\oplus\overline{\mathrm{Vir}}$, with central charges

$$c=\bar{c}=\frac{k\dim G}{k+2}. \tag{1110}$$

Quantum currents. Let $\{T^a\}$ be a fixed orthonormal basis in $\mathfrak{g}$ and let $f^{abc}$ be the structure constants of $\mathfrak{g}$ in this basis defined by $[T^a,T^b]=\sum_c f^{abc}T^c$. Components of Noether currents (1089) become in the quantum setting certain local quantum fields – elements in the space of fields $V$:

$$J^a,\overline{J}^a\in V,\qquad a=1,\ldots,\dim G, \tag{1111}$$

which are holomorphic/antiholomorphic,[158]

$$\bar\partial J^a=0,\quad \partial\overline{J}^a=0 \tag{1112}$$

(as a reflection of the classical conservation laws (1090)) and satisfy the OPEs

$$J^a(w)J^b(z)\ \sim\ \frac{k\delta^{ab}\mathbb{1}}{(w-z)^2}+\frac{\sum_c f^{abc}J^c(z)}{w-z}+\mathrm{reg.}, \tag{1113}$$

$$\overline{J}^a(w)\overline{J}^b(z)\ \sim\ \frac{k\delta^{ab}\mathbb{1}}{(\bar w-\bar z)^2}+\frac{\sum_c f^{abc}\overline{J}^c(z)}{\bar w-\bar z}+\mathrm{reg.}, \tag{1114}$$

$$J^a(w)\overline{J}^b(z)\ \sim\ \mathrm{reg.} \tag{1115}$$

The field $J^a$ acts on the space of states by a local field operator $\widehat{J}^a(z)$; one can introduce the corresponding mode operators $\widehat{J}^a_n\in\mathrm{End}(\mathcal{H})$ as

$$\widehat{J}^a_n\colon=\frac{1}{2\pi i}\oint dz\, z^n\,\widehat{J}(z) \tag{1116}$$

where the integral is over a contour going about the origin. Equivalently, we have

$$\widehat{J}(z)=\sum_{n\in\mathbb{Z}} z^{-n-1}\widehat{J}^a_n. \tag{1117}$$

Repeating the computation of Section 5.2.2, we obtain from the OPE (1113) the commutation relations between the mode operators

$$[\widehat{J}^a_n,\widehat{J}^b_m]=\sum_c f^{abc}\widehat{J}^c_{n+m}+kn\delta_{n,-m}\widehat{\mathbb{1}} \tag{1118}$$

[158]Under a correlator with any collection of test fields, or as local field operators.

Note that these are exactly the commutation relations of the affine Lie algebra $\widehat{\mathfrak{g}}$. Comparing with the notations in (1038), we have the identification $\widehat{J}^a_n = T^a_n = T^a \otimes t^n$. Likewise one introduces the mode operators $\widehat{\overline{J}}^a_n$ for the antiholomorphic current $\overline{J}^a$ which again satisfy the commutation relations of $\widehat{\mathfrak{g}}$ and commute with the mode operators $\widehat{J}^a_n$ (due to (1115)). Therefore, the action of $\widehat{\mathfrak{g}} \oplus \widehat{\overline{\mathfrak{g}}}$ on the space of states is realized by the mode operators generated by the currents $J$, $\overline{J}$.

Similarly to the action on the space of states, we have a local action of $\widehat{\mathfrak{g}}$ on fields at a point $z$ given by local mode operators $J^a_n \in \mathrm{End}(V_z)$ defined by

(1119) $$J^a_n \Phi(z) \colon = \frac{1}{2\pi i} \oint_{\gamma_z} dw\, (w-z)^n J(w)\Phi(z)$$

for any field $\Phi(z) \in V_z$; $\gamma_z$ is a contour going around $z$. Equivalently, the mode operators yield the coefficients in the OPE of a field at $z$ with the current:

(1120) $$J^a(w)\Phi(z) \sim \sum_{n\in\mathbb{Z}} (w-z)^{-n-1} J^a_n \Phi(z).$$

One has a similar local action of $\widehat{\overline{\mathfrak{g}}}$ on $V_z$ generated by local mode operators of $\overline{J}$.

The $\widehat{\mathfrak{g}}$-primary multiplet. Fix $\lambda$ a weight of an integrable $\widehat{\mathfrak{g}}$-module $H_{k,\lambda}$. Let $e^p$ be a basis in the irreducible $\mathfrak{g}$-module $M^{\mathfrak{g}}_\lambda$ (which is also the depth-zero component $H_{k,\lambda}(0)$ of the corresponding integrable $\widehat{\mathfrak{g}}$-module). We have a collection ("multiplet") of $\widehat{\mathfrak{g}} \oplus \widehat{\overline{\mathfrak{g}}}$-primary fields $\phi^{p\bar{p}}_\lambda$ (primary here means "annihilated by $J^a_{>0}, \overline{J}^a_{>0}$") corresponding to coordinates of a vector in

(1121) $$M^{\mathfrak{g}}_\lambda \otimes (M^{\mathfrak{g}}_\lambda)^* = H_{k,\lambda}(0) \otimes H_{k,\lambda}(0)^* \subset V$$

By (1120) and the primary property, we have

(1122) $$J^a(w)\phi^{p\bar{p}}_\lambda(z) \sim \frac{\sum_q (T^a_\lambda)^p_q \phi^{q\bar{p}}_\lambda(z)}{w-z} + \text{reg.}$$

where $T^a_\lambda$ is the matrix representing $T^a \in \mathfrak{g}$ as an operator on $M^{\mathfrak{g}}_\lambda$.

Stress-energy tensor. The quantum stress-energy tensor of the model is the field

(1123) $$T(z) = \frac{1/2}{k+h^\vee} \sum_a : J^a(z) J^a(z) :$$

it satisfies the standard TT OPE (601) with central charge (1110); the expression for $\overline{T}$ is similar (replacing $J$ with $\overline{J}$).

Normal ordering in (1123) refers to the following definition: for local fields $\Phi_1, \Phi_2$ their normally ordered product $: \Phi_1(z)\Phi_2(z) :$ is defined as the constant term in the OPE $\Phi_1(w)\Phi_2(z)$ or equivalently

(1124)
$$: \Phi_1(z)\Phi_2(z) := \lim_{w\to z} (\Phi_1(w)\Phi_2(z) - [\Phi_1(w)\Phi_2(z)]_{\mathrm{sing}}) = \frac{1}{2\pi i} \oint_{\gamma_z} \Phi_1(w)\Phi_2(z),$$

where $[\cdots]_{\mathrm{sing}}$ is the singular part of the OPE and $\gamma_z$ is the contour around $z$.

*Remark* 8.20. The classical Hilbert stress-energy tensor in Wess-Zumino-Witten theory, obtained as a variation w.r.t. the metric, is given by the formula (1123) without the normal ordering and without the $h^\vee$ shift in the denominator. In this regard, the shift by $h^\vee$ should be understood as a quantum correction: it must be incroporated in the quantum picture, otherwise $T$ would not satisfy the OPE of the standard form (601).

*Remark* 8.21. Note that substituting the mode expansion of the current (1117) into the stress-energy tensor (1123) we obtain the Sugawara formula (1058) expressing Virasoro generators in terms of generators of $\widehat{\mathfrak{g}}$:

$$\widehat{L}_n = \frac{1/2}{k+h^\vee} \sum_a : \widehat{J}^a_m \widehat{J}^a_{n-m} : . \tag{1125}$$

In this sense, the construction of the stress-energy tensor (1123) is a restatement of Sugawara construction.

*Remark* 8.22. The counterpart of the formula (1123) in the abelian case $\mathfrak{g} = \mathbb{R}$ is the formula (612) for the free boson. Note that in that case there is no $h^\vee$ shift.

Fields $J^a$ are Virasoro-primary, of conformal weight $(1,0)$. Similarly,fields $\overline{J}^a$ are Virasoro-primary, of conformal weight $(0,1)$.

8.3.1. *Ward identity for $\widehat{\mathfrak{g}}$-symmetry. Knizhnik-Zamolodchikov equations.* As a consequence of Lemma 5.37, one has the Ward identity generated by the holomorphic field $J^a$ as in (725): for a collection of points $z_1, \ldots, z_n \in \mathbb{C}$, $\alpha$ a $\mathfrak{g}$-valued meromorphic function with poles at $z_1, \ldots, z_n$ allowed , $\Phi_1, \ldots, \Phi_n \in V$ a collection of fields, one has

$$\alpha \circ \langle \Phi_1(z_1) \cdots \Phi_n(z_n) \rangle :\, = \sum_{j=1}^n \langle \Phi_1(z_1) \cdots \rho_J^{(z_j)}(\alpha) \circ \Phi_j(z_j) \cdots \Phi_n(z_n) \rangle = 0, \tag{1126}$$

where

$$\rho_J^{(z)}(\alpha) \circ \Phi(z) :\, = \frac{1}{2\pi i} \oint_{\gamma_z} dw\, \alpha^a(w) J^a(w) \Phi(z). \tag{1127}$$

One has a similar Ward identity corresponding to the action on a correlator by an antimeromorphic function using the second current $\overline{J}$.

*Remark* 8.23. One can think of the Ward identity (1126) as corresponding to the expected invariance property of the WZW path integral (1107) where:

- Boundary circles are shrunk to punctures $z_i$.
- We consider the infinitesimal action of the Lie algebra of $\mathfrak{g}$-valued functions, holomorphic in the complement of the punctures, instead of the group of holomorphic maps to the group $G_\mathbb{C}$,

Specializing (1126) to the case $\alpha(w) = \frac{1}{w-z}$ and the collection of fields being the identity field at $z$ and $\widehat{\mathfrak{g}}$-primary fields at $z_1, \ldots, z_n$, we have the identity

$$\langle J^a(z) \phi^{p_1 \bar{p}_1}_{\lambda_1}(z_1) \cdots \phi^{p_n \bar{p}_n}_{\lambda_n}(z_n) \rangle = \sum_{j=1}^n \sum_{q_j} \frac{(T^a_{\lambda_j})^{p_j}_{q_j}}{z - z_j} \langle \phi^{p_1 \bar{p}_1}_{\lambda_1}(z_1) \cdots \phi^{q_j \bar{p}_j}_{\lambda_j}(z_j) \cdots \phi^{p_n \bar{p}_n}_{\lambda_n}(z_n) \rangle \tag{1128}$$

Here we have fixed some weights $\lambda_1, \ldots, \lambda_n$ of $\mathfrak{g}$ corresponding to integrable $\widehat{\mathfrak{g}}$-modules.

One can also obtain this identity by realizing that due to (1122), the l.h.s. has to be a meromorphic function in $z$ with first-order poles at $z = z_1, \ldots, z_n$, with residues controlled by the r.h.s. of (1122). Such a function (decaying as $z \to \infty$) is unique and given by the r.h.s. of (1128).

One can also write the identity (1128) in slightly more pleasing notations:

$$\langle J^a(z)\phi_{\lambda_1}(z_1)\cdots\phi_{\lambda_n}(z_n)\rangle = \sum_{j=1}^n \frac{T^a_{\lambda_j}}{z-z_j}\langle\phi_{\lambda_1}(z_1)\cdots\phi_{\lambda_n}(z_n)\rangle \tag{1129}$$

where

- We denote

$$\phi_\lambda \colon = \sum_{p,\bar{p}} \phi_\lambda^{p\bar{p}} e_p \otimes \bar{e}_{\bar{p}} \in V \otimes (M^{\mathfrak{g}}_\lambda)^* \otimes M^{\mathfrak{g}}_\lambda. \tag{1130}$$

  where $\{e_p\}$ is the basis in $(M^{\mathfrak{g}}_\lambda)^*$ dual to the basis $\{e^p\}$ in $M^{\mathfrak{g}}_\lambda$. (1130) is a vector-valued field – the "full" $\widehat{\mathfrak{g}}$-primary multiplet with weight $\lambda$.
- Both sides of (1129) are valued in tensors

$$\bigotimes_{i=1}^n (M^{\mathfrak{g}}_{\lambda_i})^* \otimes M^{\mathfrak{g}}_{\lambda_i}. \tag{1131}$$

- We understand that the operator $T^a_{\lambda_j}$ is acting in the $j$-th factor in the product (1131).

Knizhnik-Zamolodchikov equations.
As a special case $n=-1$ of the Sugawara construction (1125) one has

$$L_{-1} = \frac{1/2}{k+h^\vee}\sum_{m\in\mathbb{Z}} : J^a_m J^a_{-1-m} : \tag{1132}$$

where we think of both sides as operators acting on the space of fields $V_z$. In particular, for the $\widehat{\mathfrak{g}}$-primary multiplet $\phi_\lambda$, we have

$$L_{-1}\phi_\lambda(z) = \frac{1}{k+h^\vee}\sum_a J^a_{-1}J^a_0\phi_\lambda(z) = \frac{1}{k+h^\vee}J^a_{-1}T^a_\lambda\phi_\lambda(z). \tag{1133}$$

Using this, we have the following:

$$0 = \langle\phi_{\lambda_1}(z_1)\cdots\underbrace{\Big(L_{-1} - \frac{1}{k+h^\vee}\sum_a J^a_{-1}T^a_{\lambda_j}\Big)\phi_{\lambda_j}(z_j)}_{0}\cdots\phi_{\lambda_n}(z_n)\rangle = \tag{1134}$$

$$= \frac{\partial}{\partial z_j}\langle\phi_{\lambda_1}(z_1)\cdots\phi_{\lambda_n}(z_n)\rangle - \sum_a \frac{1}{k+h^\vee}T^a_{\lambda_j}\frac{1}{2\pi i}\oint_{\gamma_{z_j}} dw\langle J(w)\phi_{\lambda_1}(z_1)\cdots\phi_{\lambda_n}(z_n)\rangle.$$

Here we used that $L_{-1}\Phi(z) = \partial\Phi(z)$. Next we deform the integration contour $\gamma_{z_j}$ going around $z_j$ to a collection of contours going around the punctures $z_i$ in negative direction,

$$\gamma_{z_j} \sim \sqcup_{i\neq j}(-\gamma_{z_i}) \tag{1135}$$

Then, using the Ward identity (1129), we obtain the following.

**Theorem 8.24** (Knizhnik-Zamolodchikov [24])**.** *Given the weights $\lambda_1,\ldots,\lambda_n$ of $\mathfrak{g}$ corresponding to integrable $\widehat{\mathfrak{g}}$-modules, the correlator of primary multiplets satisfies the following the system of ODEs* check the sign

$$\underbrace{\left(\frac{\partial}{\partial z_j} + \frac{1}{k+h^\vee}\sum_{i\neq j}\sum_a \frac{T^a_{\lambda_i}T^a_{\lambda_j}}{z_i - z_j}\right)}_{\nabla^{\mathrm{KZ}}_j}\langle\phi_{\lambda_1}(z_1)\cdots\phi_{\lambda_n}(z_n)\rangle = 0, \tag{1136}$$

*for any* $j = 1, \ldots, n$.

One can interpret the result as follows: one has a flat connection

$$\nabla_{\mathrm{KZ}} \colon = \sum_j dz_j \nabla_j^{KZ} + d\bar{z}_j \overline{\nabla}_j^{KZ} \tag{1137}$$

$\mathbb{C}$ or $\mathbb{CP}^1$?

on the the trivial vector bundle over the open configuration space $C_n(\mathbb{C})$ with fiber (1131); here $\nabla_j^{\mathrm{KZ}}$ are the differential operators appearing in the equation (1136). The correlator of $\widehat{\mathfrak{g}}$-primary multiplets $\langle \phi_{\lambda_1}(z_1) \cdots \phi_{\lambda_n}(z_n) \rangle$ is a section of this bundle *horizontal* w.r.t. $\nabla_{\mathrm{KZ}}$.

The flat connection (1137) is known as the Knizhnik-Zamolodchikov (KZ) connection.

For future reference we will introduce a notation for the holomorphic part of the KZ connection

$$\nabla_{\mathrm{KZ}}^{\mathrm{hol}} = \sum_j dz_j \nabla_j^{\mathrm{KZ}} + d\bar{z}_j \frac{\partial}{\partial \bar{z}_j} \tag{1138}$$

as a connection on the vector bundle on $C_n(\mathbb{C})$ with the fiber $\bigotimes_{i=1}^n (M_{\lambda_i}^{\mathfrak{g}})^*$ (i.e. taking only the first factor in each term in (1131)).

8.3.2. *Space of conformal blocks. Chiral WZW model.* For $z_1, \ldots, z_n$ distinct points in $\mathbb{CP}^1$, let us denote by $\mathfrak{g}(z_1, \ldots, z_n)$ the Lie algebra of $\mathfrak{g}$-valued meromorphic functions on $\mathbb{CP}^1$ with poles allowed only at $z_1, \ldots, z_n$.

Fix weights $\lambda_1, \ldots, \lambda_n$ of $\mathfrak{g}$ corresponding to integrable modules of $\widehat{\mathfrak{g}}$ at level $k$. Then the Lie algebra $\mathfrak{g}(z_1, \ldots, z_n)$ acts on the tensor product of integrable modules

$$H_{k,\lambda_1} \otimes \cdots \otimes H_{k,\lambda_n} \tag{1139}$$

by

$$\alpha \circ (\psi_1 \otimes \cdots \otimes \psi_n) \colon = \sum_{j=1}^n \psi_1 \otimes \cdots \otimes \rho(\mathrm{Laurent}_{z_j}(\alpha)) \circ \psi_j \otimes \cdots \otimes \psi_n \tag{1140}$$

where $\mathrm{Laurent}_{z_j}(\alpha) = \sum_{m=-N}^\infty \sum_a \alpha_m^a (T^a \otimes t_j^m)$ is the Laurent expansion of $\alpha$ at $z_j$, in powers of $t_j = z - z_j$; this Laurent expansion acts on $H_{k,\lambda_j}$ (this action is denoted by $\rho$ above) via the tautological embedding

$$\mathfrak{g} \otimes \mathbb{C}[t_j^{-1}, t_j]] \hookrightarrow \widehat{\mathfrak{g}}. \tag{1141}$$

**Definition 8.25.** For $\lambda_1, \ldots, \lambda_n$ a collection of weights of $\mathfrak{g}$ corresponding to integrable modules of $\widehat{\mathfrak{g}}$ and a collection of distinct points $z_1, \ldots, z_n \in \mathbb{CP}^1$, the *space of Wess-Zumino-Witten conformal blocks* is defined as the complex vector space

$$\mathcal{B}(z_1, \ldots, z_n; \lambda_1, \ldots, \lambda_n) \colon = \mathrm{Hom}_{\mathfrak{g}(z_1,\ldots,z_n)}(H_{k,\lambda_1} \otimes \cdots \otimes H_{k,\lambda_n}, \mathbb{C}) \tag{1142}$$

– the space of $\mathfrak{g}(z_1, \ldots, z_n)$-equivariant maps between two $\mathfrak{g}(z_1, \ldots, z_n)$-modules, $H_{k,\lambda_1} \otimes \cdots \otimes H_{k,\lambda_n}$ with module structure (1140) and $\mathbb{C}$ as the trivial module.

One can think of elements of (1142) as correlators

$$\langle \psi_1(z_1) \cdots \psi_n(z_n) \rangle^{\mathrm{chiral}} \tag{1143}$$

in the *chiral* WZW model, where the correlators are (possibly multivalued) holomorphic functions on the open configuration space $C_n(\mathbb{CP}^1)$ and only a single copy of $\widehat{\mathfrak{g}}$ (and a single copy of Virasoro) acts on the space of states/space of fields. Thus, in the chiral theory one has

$$V^{\text{chiral}} \simeq \mathcal{H}^{\text{chiral}} = \bigoplus_{\lambda} H_{k,\lambda}. \tag{1144}$$

One can say that the chiral WZW is obtained from usual WZW by setting the antiholomorphic current to zero, $\overline{J} = 0$ (and consequently $\overline{T} = 0$).

The fact that in (1142) the maps are required to be $\mathfrak{g}(z_1, \ldots, z_n)$-equivariant is exactly the statement of Ward identity (1126) for chiral correlators.

Somewhat surprisingly, the space of conformal blocks is finite-dimensional (with dimension depending on the level and the weights). In fact, the inclusion

$$\iota\colon M^{\mathfrak{g}}_{\lambda_1} \otimes \cdots \otimes M^{\mathfrak{g}}_{\lambda_n} \hookrightarrow H_{k,\lambda_1} \otimes \cdots \otimes H_{k,\lambda_n} \tag{1145}$$

of depth-zero subspaces in each integrable module induces an *injective* map

$$i = \iota^*\colon \mathcal{B}(z_1, \ldots, z_n; \lambda_1, \ldots, \lambda_n) \hookrightarrow \mathrm{Hom}_{\mathfrak{g}}(M^{\mathfrak{g}}_{\lambda_1} \otimes \cdots \otimes M^{\mathfrak{g}}_{\lambda_n}, \mathbb{C}). \tag{1146}$$

This map corresponds to considering only correlators of $\widehat{\mathfrak{g}}$-primary chiral fields. The fact that the map $i$ is injective reflects the fact that using the Ward identity one can reduce a correlator of $\widehat{\mathfrak{g}}$-descendants to the correlator of $\widehat{\mathfrak{g}}$-primary fields (similarly to Virasoro case, cf. Example 5.22). From (1146) is is obvious that the space of conformal blocks must be finite-dimensional.

**Example 8.26.** Consider the case $G = SU(2)$ and fix the level $k = 1, 2, 3, \ldots$. The admissible weights corresponding to integrable modules are $\lambda = 0, 1, \ldots, k$.

- For $n = 3$, the space of conformal blocks can be either 0- or 1-dimensional:
  - One has $\mathcal{B}(z_1, z_2, z_3; \lambda_1, \lambda_2, \lambda_3) = \mathbb{C}$ if the "fusion rules" (or "quantum Klebsch-Gordan condition") hold:

$$\lambda_1 + \lambda_2 + \lambda_3 \in 2\mathbb{Z}, \quad |\lambda_1 - \lambda_2| \leq \lambda_3 \leq \lambda_1 + \lambda_2, \quad \lambda_1 + \lambda_2 + \lambda_3 \leq 2k. \tag{1147}$$

  - Otherwise one has $\mathcal{B}(z_1, z_2, z_3; \lambda_1, \lambda_2, \lambda_3) = 0$.
- For a general $n$ one can associate a basis in the space of conformal blocks $\mathcal{B}(z_1, \ldots, z_n; \lambda_1, \ldots, \lambda_n)$ to any trivalent tree with $n$ leaves decorated with $\lambda_1, \ldots, \lambda_n$. Basis vectors in $\mathcal{B}$ corresponds to ways to decorate the internal edges $e$ of the tree by labels $\lambda_e \in \{0, 1, \ldots, k\}$ so that fusion rules (1147) hold at each vertex. The idea behind constructing such a basis is similar to that of Section 7.5.1 and comes from a pair-of-pants decomposition of the surface; edges of the graph correspond to circles we cut along and their decorations correspond to intermediate states we sum over. write better

  In the case when $\mathbb{CP}^1$ is replaced by a Riemannian surface $\Sigma$ of genus $h$, instead of a trivalent tree one should consider decorations of a connected trivalent graph with $h$ loops.
- One has a fascinating explicit formula due to Verlinde [37] for the dimension of the space of $n$-point conformal blocks for $G = SU(2)$, on a surface $\Sigma$ of genus $h$:

$$\dim \mathcal{B}(z_1, \ldots, z_n; \lambda_1, \ldots, \lambda_n) = \sum_{0 \leq \lambda \leq k} (S_{0\lambda})^{2-2h-n} S_{\lambda_1 \lambda} \cdots S_{\lambda_n \lambda}, \tag{1148}$$

  where

$$S_{\lambda\mu} = \sqrt{\frac{2}{k+2}}\, \sin \pi \frac{(\lambda+1)(\mu+1)}{k+2}. \tag{1149}$$

The result comes from a "diagonalization" of the dimension of the space of 3-point conformal blocks:

$$\dim \mathcal{B}(z_1, z_2, z_3; \lambda_1, \lambda_2, \lambda_3) = \\ = \sum_{0\leq\lambda\leq k} \frac{S_{\lambda_1\lambda} S_{\lambda_2\lambda} S_{\lambda_3\lambda}}{S_{0\lambda}} = \begin{cases} 1 & \text{if the fusion rules (1147) hold,} \\ 0 & \text{otherwise} \end{cases} \tag{1150}$$

The matrix $S$ (1149) appearing here can be interpreted as representing the action of the modular $S$-transformation $\tau \to -\frac{1}{\tau}$ on the space of conformal blocks with genus one and no punctures.[159]

The bundle of conformal blocks. Spaces of conformal blocks (1142) with fixed weights $\lambda_1, \ldots, \lambda_n$ and variable points $z_1, \ldots, z_n$ arrange into a complex vector bundle over the open configuration space of $n$ points,

$$\begin{array}{ccc} \mathcal{E}_{\lambda_1\cdots\lambda_n} & \longleftarrow & \mathcal{B}(z_1, \ldots, z_n; \lambda_1, \ldots, \lambda_n) \\ \downarrow & & \\ C_n(\mathbb{CP}^1) & & \end{array} \tag{1151}$$

This vector bundle comes equipped with a flat connection

$$\nabla_{\mathcal{E}} = \sum_{j=1}^{n} dz_j \left( \frac{\partial}{\partial z_j} - L_{-1}^{(j)} \right) + d\bar{z}_j \frac{\partial}{\partial \bar{z}_j}, \tag{1152}$$

where $L_{-1}^{(j)}$ is (the dual of) the Sugawara operator acting on $H_{k,\lambda_j}$. Correlators of chiral WZW model yield a horizontal multivalued section of $\mathcal{E}$. Restricted to depth zero in each integrable module (i.e. restricted to chiral correlators of $\widehat{\mathfrak{g}}$-primary fields), the holomorphic part of the connection $\nabla_{\mathcal{E}}$ becomes the holomorphic part of the Knizhnik-Zamolodchikov connection $\nabla_{\mathrm{KZ}}^{\mathrm{hol}}$ (1138).

Lecture 39, 12/02/2022

8.3.3. *The "holographic" correspondence between 3d Chern-Simons and 2d Wess-Zumino-Witten theories.* Here we quickly mention the remarkable relation between a 3d topological field theory (Chern-Simons theory) on a 3-manifold $M$ and a 2d CFT (Wess-Zumino-Witten model) on the boundary surface $\Sigma = \partial M$. There is a lot of literature on the subject, starting with the seminal work of Witten [40]. The correspondence between WZW and Chern-Simons is an example in the class of so-called "holographic correspondences" between $(d+1)$-dimensional gravity and a $d$-dimensional conformal theory on the boundary.

edit

Fix $G$ a compact, simple, simply connected Lie group with Lie algebra $\mathfrak{g}$ and fix $M$ an oriented compact 3-manifold with the boundary surface $\Sigma$ (possibly disconnected); we assume that $\Sigma$ is equipped with complex structure.

Consider Chern-Simons theory on $M$ with space of fields $\mathrm{Fields}_M^{\mathrm{CS}} = \Omega^1(M, \mathfrak{g}) = \mathrm{Conn}(M, G)$ – the space of connections in the trivial principal $G$-bundle over $M$; we identify connections with their 1-forms on the base. The action functional is

$$S_{\mathrm{CS}}^b(A) \colon = \frac{1}{2\pi} \int_M \mathrm{tr} \left( \frac{1}{2} A \wedge dA + \frac{1}{6} A \wedge [A \stackrel{\wedge}{,} A] \right) + \underbrace{\frac{1}{4\pi} \int_\Sigma \mathrm{tr}\, A^{1,0} \wedge A^{0,1}}_{b(A|_\Sigma)}. \tag{1153}$$

[159] This space is $(k+1)$-dimensional, with a natural basis given by characters of modules $H_{k,\lambda}$ with $0 \leq \lambda \leq k$, cf. Section 7.3.1.

The last term here is a boundary term, depending only on the restriction of $A$ to $\Sigma$ (and the decomposition of that restriction into a $(1,0)$-form and a $(0,1)$-form using the complex structure). The superscript $b$ in the action is to emphasize the presence of the boundary term $b$; $b$ is designed to tweak the Noether 1-form induced by the action to a convenient form: for the variation of the action one has

$$\delta S^b_{\mathrm{CS}} = -\frac{1}{2\pi}\int_M \mathrm{tr}\,\delta A \wedge F_A + \underbrace{\frac{1}{2\pi}\int_\Sigma \mathrm{tr} A^{0,1}\delta A^{1,0}}_{\alpha} \tag{1154}$$

Here the last term is the Noether 1-form on the phase space $\Phi^{\mathrm{CS}}_\Sigma = \Omega^1(\Sigma,\mathfrak{g})$ and the fact that it vanishes on the (Lagrangian) fibers of the fibration

$$p\colon \Omega^1(\Sigma,\mathfrak{g}_\mathbb{C}) \to \Omega^{1,0}(\Sigma,\mathfrak{g}) \tag{1155}$$

implies that one flat connections $A$ are actual critical points of $S^b_{\mathrm{CS}}$ on the subspace of fields with prescribed boundary condition $(A|_\Sigma)^{1,0}$. In particular, one can study the path integral for Chern-Simons theory

$$Z^{\mathrm{CS}}(A^{1,0})\colon = \int_{\mathrm{Conn}(M,G)\ni A \text{ s.t. } (A|_\Sigma)^{1,0}=A^{1,0}} \mathcal{D}A\, e^{ikS^b_{\mathrm{CS}}(A)} \tag{1156}$$

with $k = 1,2,3,\ldots$ the "level" of Chern-Simons theory.

Consider gauge transformations of the connection

$$A \mapsto A^g = g^{-1}Ag + g^{-1}dg \tag{1157}$$

with generator $g\colon M \to G$. If the generator is trivial on the boundary $g|_\Sigma = 1$, one has

$$S^b_{\mathrm{CS}}(A^g) = S^b_{\mathrm{CS}}(A) \mod 2\pi\mathbb{Z}, \tag{1158}$$

i.e., Chern-Simons action is invariant modulo $2\pi\mathbb{Z}$ under gauge transformations *relative to the boundary*. The $2\pi\mathbb{Z}$-ambiguity is the reason why one wants the normalization factor $k$ in the exponential in the path integral (1156) to be an integer – so that the integrand in the path integral is gauge-invariant.

Classical CS-WZW correspondence. If the generator of the gauge transformation is nontrivial on the boundary, one has

$$S^b_{\mathrm{CS}}(A^g) - S^b_{\mathrm{CS}}(A) = iS_{\mathrm{WZW}}(g|_\Sigma) + \frac{1}{2\pi}\int_\Sigma \mathrm{tr}\, A^{1,0}g^{-1}\partial g. \tag{1159}$$

The first term on the r.h.s. is the Wess-Zumino-Witten action evaluated on the boundary restriction of the generator of the gauge transformation. Thus, the defect of gauge-invariance of Chern-Simons theory due to the presence of boundary is given by WZW action on the boundary.

The full r.h.s. of (1159) is sometimes called the *gauged* WZW model. It can also be thought of as the action of the *chiral* WZW model: we can regard the field $A^{1,0}$ as a Lagrange multiplier, integrating it out imposes the vanishing of the antiholomorphic WZW current $\overline{J} = 0$.

Formula (1159) is a manifestation of the Chern-Simons/Wess-Zumino-Witten correspondence at the classical level. A consequence of it is the following: if $M$ is a 3-ball, with $\Sigma = \partial M = \mathbb{CP}^1$, any flat connection on $M$ can be written as $A = g^{-1}dg$ (gauge-equivalent to zero connection) for some $g\colon M \to G$. In this case (1159) implies

$$S^b_{\mathrm{CS}}(A) = iS_{\mathrm{WZW}}(g). \tag{1160}$$

Quantum CS-WZW correspondence. The relation (1159) has a very nontrivial quantum counterpart:

$$\mathcal{B}_\Sigma^{\mathrm{WZW}} = \mathcal{H}_\Sigma^{\mathrm{CS}} \tag{1161}$$

– the space of states that quantum Chern-Simons theory (as an Atiyah's TQFT) assigns to a surface $\Sigma$ is isomorphic to the space of WZW conformal blocks on the surface.

One has a version of this statement with punctures on $\Sigma$. For that one should consider a Wilson graph observable $O_\Gamma$ in the Chern-Simons theory on $M$. Let $\Gamma \subset M$ be an embedded oriented graph in $M$, which is allowed to meet the boundary surface transversally; we treat these boundary points of $\Gamma$ as univalent vertices. Bulk vertices are assumed to be trivalent. Assume that the edges of $\Gamma$ are decorated by weights $\lambda$ of integrable representations of $\mathfrak{g}$[160] and the trivalent vertices are decorated by intertwiners – elements of $\left(M_\lambda^{\mathfrak{g}} \otimes M_{\lambda'}^{\mathfrak{g}} \otimes M_{\lambda''}^{\mathfrak{g}}\right)^{\mathfrak{g}}$, where $\lambda, \lambda', \lambda''$ are the weights decorating the incident edges. At the level of classical field theory, the observable

$$O_\Gamma \colon \mathrm{Fields}_M \to (M_{\lambda_1}^{\mathfrak{g}})^* \otimes \cdots \otimes (M_{\lambda_n}^{\mathfrak{g}})^* \tag{1162}$$

is a function on the space of connections $A$, given by the contraction of holonomies of $A$ along the edges of $\Gamma$, taken in corresponding representations, with the intertwiners at the vertices. In (1162) we are assuming that $\Gamma$ has $n$ boundary vertices at points $z_1, \ldots, z_n \in \Sigma$ and the incident edges are decorated by weights $\lambda_1, \ldots, \lambda_n$.

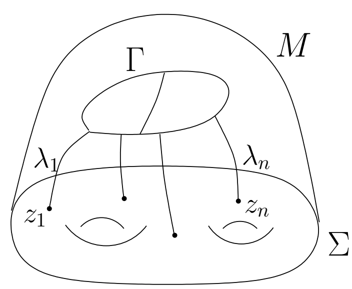



FIGURE 41. Wilson graph observable.

The path integral of Chern-Simons theory with the Wilson graph observable is

$$Z_{M,\Gamma}^{\mathrm{CS}} = \int_{\mathrm{Conn}(G,\Sigma)\ni A \text{ s.t. } (A|_\Sigma)^{1,0}=A^{1,0}} \mathcal{D}A\, e^{ikS_{CS}^b(A)} O_\Gamma(A) \quad \in \\ \in \underbrace{\left(C^\infty(\Omega^{1,0}(\Sigma,\mathfrak{g})) \otimes \mathrm{Hom}(M_{\lambda_1}^{\mathfrak{g}} \otimes \cdots \otimes M_{\lambda_n}^{\mathfrak{g}}, \mathbb{C})\right)^{\mathrm{Map}(\Sigma,G)}}_{\mathcal{H}_\Sigma^{\mathrm{CS},\Gamma}} \tag{1163}$$

where understand that the path integral is a function of the boundary $(1,0)$-form $A^{1,0}$ (the boundary condition of the path integral) and also takes values in a product of representations due to the presence of $O_\Gamma$-observable. The whole expression is expected to be equivariant under gauge transformations, where only the boundary value of the gauge generator matters after averaging over the fields in the bulk, since

[160]We understand that one can switch the orientation of any edge, switching simultaneously the representation $M_\lambda^{\mathfrak{g}}$ to its dual.

the integrand is equivariant (and invariant under gauge transformations relative to the boundary). The expected equivariance property following from (1159) is: double check conventions

$$Z^{\mathrm{CS}}_{M,\Gamma}((A^{1,0})^g) = e^{-kS_{\mathrm{WZW}}(g)+\frac{i}{2\pi}\int_\Sigma A^{1,0}g^{-1}\partial g} \bigotimes_{j=1}^{n} \rho^*_{\lambda_j}(g(z_j)) \circ Z^{\mathrm{CS}}_{M,\Gamma}(A^{1,0}), \tag{1164}$$

where $A^{1,0} \in \Omega^{1,0}(\Sigma, \mathfrak{g})$ is the boundary condition and $g\colon \Sigma \to G$ is the gauge transformation on the boundary; $(A^{1,0})^g = g^{-1}A^{1,0}g + g^{-1}\partial g$ is the chiral gauge transformation on the boundary; $\rho^*_\lambda(g)$ is the operator representing the group element on the module $(M^{\mathfrak{g}}_\lambda)^*$.

The vector space where the path integral takes values is the space of states assigned to the boundary $\Sigma$ by Chern-Simons theory deformed by the observable $\Gamma$. It depends on the positions $z_1, \ldots, z_n \in \Sigma$ of boundary vertices of $\Gamma$ and the corresponding weights $\lambda_1, \ldots, \lambda_n$. The statement of CS-WZW correspondence generalizing (1161) in this setting is:

$$\mathcal{B}^{\mathrm{WZW}}_\Sigma(z_1, \ldots, z_n; \lambda_1, \ldots, \lambda_n) = \mathcal{H}^{\mathrm{CS},\Gamma}_\Sigma \tag{1165}$$

– the Chern-Simons space of states on $\Sigma$ deformed by $\Gamma$ is the space of WZW $n$-point conformal blocks on $\Sigma$. We refer the reader to [15] for details on the correspondence (1165).

*Remark* 8.27. The space of states $\mathcal{H}^{\mathrm{CS}}_\Sigma$ of Chern-Simons theory can also be obtained as as a geometric quantization of the moduli space of flat connections on $\Sigma$ (as a symplectic manifold with singularities, with polarization inferred from complex structure), see [2]. The choice of complex structure serves as a parameter of quantization, and hence one obtains a vector bundle of spaces of states over the moduli space of complex structures

$$\begin{array}{ccc} \mathbb{H} & \longleftarrow & \mathrm{GeomQ}(\mathcal{M}_{\mathrm{flat}}(\Sigma), \mathrm{cx.\,str.\,on}\,\Sigma) \\ \big\downarrow & & \\ \mathcal{M}_\Sigma & & \end{array} \tag{1166}$$

This vector bundle comes with a natural projectively flat connection – the so-called Hitchin connection – allowing one to compare quantizations with different choices of complex structure. This bundle in the case of $\Sigma = \mathbb{CP}^1$ with punctures (and up to reduction by the Möbius group) is the bundle of conformal blocks (1151) with Knizhnik-Zamolodchikov connection.

*Remark* 8.28. The correspondence (1165) allows one to use things known in WZW to make statements about Chern-Simons theory. For instance, from Atiyah's axioms one has that for a closed 3-manifold of the form $M = \Sigma \times S^1$ with $\Sigma$ a closed surface of genus $h$, the Chern-Simons partition function is the dimension of the space of states on $\Sigma$:

$$\begin{aligned} Z^{\mathrm{CS}}_{\Sigma\times S^1} \underset{\mathrm{Atiyah}}{=} \dim \mathcal{H}^{\mathrm{CS}}_\Sigma \underset{\text{holography (1161)}}{=} \dim \mathcal{B}^{\mathrm{WZW}}_\Sigma = \\ \underset{\text{Verlinde (1148)}}{=} \left(\frac{k+2}{2}\right)^{h-1} \sum_{\lambda=0}^{k} \left(\sin \pi \frac{\lambda+1}{k+2}\right)^{2-2h} \end{aligned} \tag{1167}$$

Here we assumed $G = SU(2)$; $k$ is the level.

Likewise, consider the Chern-Simons partition function for the 3-manifold $\Sigma \times S^1$ with observable $\Gamma$ consisting of $n$ circles of the form $\{z_i\} \times S^1$, with $z_1, \ldots, z_n$ an $n$-tuple of distinct points on $\Sigma$, assuming that the circles are decorated with weights $\lambda_1, \ldots, \lambda_n$. By the same logic, this partition function is again given by the Verlinde formula,

$$Z^{\mathrm{CS}}_{\Sigma\times S^1,\Gamma} = \dim \mathcal{B}^{\mathrm{WZW}}_{\Sigma}(z_1, \ldots, z_n; \lambda_1, \ldots, \lambda_n) = \text{r.h.s. of (1148)}. \tag{1168}$$

8.3.4. *Parallel transport of the KZ connection, R-matrix and representation of the braid group.* Consider the Knizhnik-Zamolodchikov connection $\nabla^{\mathrm{hol}}_{\mathrm{KZ}}$ on the depth-zero part of the bundle of $n$-conformal blocks where all weights are the same $\lambda_1 = \cdots = \lambda_n = \lambda$.

$$\mathcal{E}^0_{\lambda\cdots\lambda} \to C_n(\mathbb{C}) \tag{1169}$$

We also restricted the base from $\mathbb{CP}^1$ to $\mathbb{C}$ for the sake of present discussion. Recall that the base $C_n$ here is the space of *ordered* configurations of points. However, since we chose all weights to be the same, we can quotient the bundle by the symmetric group permuting the $n$ points, obtaining a vector bundle

$$\mathcal{E}'^0_{\lambda\cdots\lambda} \to C_n^{\mathrm{unordered}}(\mathbb{C}) \tag{1170}$$

over the unordered configuration space. The connection $\nabla^{\mathrm{hol}}_{\mathrm{KZ}}$ descends to this quotient.

Consider a path

$$\gamma_j(t) = \left(1, \ldots, j-1, j + \frac{1-e^{it}}{2}, j + \frac{1+e^{it}}{2}, j+2, \ldots, n\right), \quad t \in [0, \pi] \tag{1171}$$

in $C_n(\mathbb{C})$ for $j \in \{1, \ldots, n-1\}$ – it interchanges the points $z_j$ and $z_{j+1}$ by a smooth move, i.e., it starts at $P = (1, \ldots, j, j+1, \ldots, n)$ and finishes at $Q = (1, \ldots, j+1, j, \ldots, n)$.

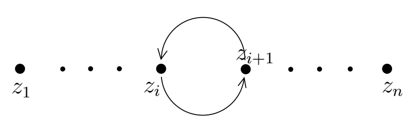



FIGURE 42. Path in the configuration space.

This path descends to a closed loop $\gamma'_j$ in $C_n^{\mathrm{unordered}}(\mathbb{C})$ starting and ending at the point $\{1, \ldots, n\}$. The parallel transport of $\nabla^{\mathrm{hol}}_{\mathrm{KZ}}$ along this loop is an endomorphism of the fiber of $\mathcal{E}'^0_{\lambda\cdots\lambda}$ of the form

$$\underbrace{\mathrm{id} \otimes \cdots \otimes \mathrm{id}}_{j-1} \otimes R \otimes \underbrace{\mathrm{id} \otimes \cdots \otimes \mathrm{id}}_{n-j-1} \quad \in \mathrm{End}((M^{\mathfrak{g}}_\lambda)^{\otimes n}) \tag{1172}$$

with $R$ a certain element

$$R \in \mathrm{End}\left(((M^{\mathfrak{g}}_\lambda)^*)^{\otimes 2}\right) \tag{1173}$$

– it is an example of the so-called "$R$-matrix." (This particular one is the $R$ matrix given by the holonomy of Knizhnik-Zamolodchikov connection.) It satisfies the Yang-Baxter equation

$$(R \otimes \mathrm{id})(\mathrm{id} \otimes R)(R \otimes \mathrm{id}) = (\mathrm{id} \otimes R)(R \otimes \mathrm{id})(\mathrm{id} \otimes R) \tag{1174}$$

by construction – because both sides give the parallel transport along loops in $C_n^{\text{unordered}}(\mathbb{C})$ and the two sides correspond to two *homotopic* loops (recall that $\nabla_{\text{KZ}}^{\text{hol}}$ is a *flat* connection, so the parallel transport does not change under homotopy of the loop).

The fundamental group of $C^{\text{unordered}}(\mathbb{C})$ is also known as the braid group on $n$ strands. Its standard presentation is with $n-1$ generators $c_1, \ldots, c_{n-1}$ subject to relations

$$c_j c_{j+1} c_j = c_{j+1} c_j c_{j+1}, \qquad c_i c_j = c_j c_i \ \text{ if } |i-j| \geq 2. \tag{1175}$$

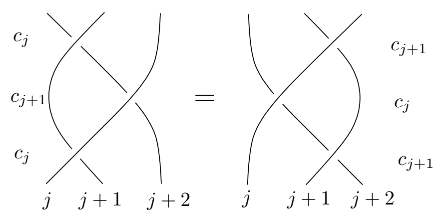



FIGURE 43. Relation in the braid group. One can understand this picture as being in $\mathbb{R} \times \mathbb{C}$. The l.h.s. is the graph of the concatenation of paths $\gamma_j * \gamma_{j+1} * \gamma_j$, and similarly for the r.h.s.

The construction above gives a representation of the braid group on the space

$$((M_\lambda^{\mathfrak{g}})^*)^{\otimes n}, \tag{1176}$$

with the generator $c_j$ represented by the element (1172) – by the $R$-matrix acting in the $j$-th and $(j+1)$-st factors of the representation space (1176). The first relation in (1175) holds due to the Yang-Baxter equation (1174) and the second relation is obvious by construction (1172).

*Remark* 8.29. Let $\gamma$ be a loop in $C_n^{\text{unordered}}(\mathbb{CP}^1)$ or equivalently a braid. Gluing the top and the bottom of the braid, we obtain a link $L$ in the 3-manifold $M = \mathbb{CP}^1 \times S^1$. Let $\Xi \in \text{End}(\mathcal{B}(1, 2, \ldots, n; \lambda, \ldots, \lambda))$ be the parallel transport of the connection (1152) along $\gamma$. Then by the argument analogous to Remark 8.28 one has

$$Z^{\text{CS}}_{\mathbb{CP}^1 \times S^1, L} = \text{tr}_{\mathcal{B}(1,2,\ldots,n;\lambda,\ldots,\lambda)} \Xi. \tag{1177}$$

Here we think of $L$ as a special type of Wilson graph (a disjoint union of circles), with all components of the link decorated by the weight $\lambda$. Given a presentation of $\gamma$ seen as a braid in terms of generators $c_j$, $\gamma = c_{j_1} \cdots c_{j_r}$, the endomorphism $\Xi$ can be written as a product of $R$-matrices,

$$\Xi = R_{j_1} \cdots R_{j_r}, \tag{1178}$$

where the subscript $j$ means that the $R$-matrix acts on the $j$-th and $(j+1)$-st factors. Here a remark is that although the r.h.s. of (1178) is an endomorphism of (1176), it actually stabilizes the image of the inclusion (1146) and hence determines an endomorphism of the space of conformal blocks.

## 9. A-model

The A-model introduced by Witten in [38] is an example of a 2d topological conformal field theory which contains a special class $Q$-closed observables (so-called evaluation observables) whose correlators yield closed forms on the moduli space $\mathcal{M}_{g,n}$. Integrated over $\mathcal{M}_{g,n}$, these correlators yield interesting integer numbers – Gromov-Witten invariants – solutions of a certain class of enumerative geometric problems. Moreover, field-theoretic origin of these numbers (ultimately, Segal's axioms) result in an equation on Gromov-Witten invariants – the Witten-Dijkgraaf-Verlinde-Verlinde or WDVV equation – which allows in some cases to fully compute the Gromov-Witten invariants, see [26].

Closed forms on the moduli space from TCFT correlators. First, recall Remark 6.18: in any TCFT given a collection of $Q$-cocycles $\Phi_1, \ldots, \Phi_n$, the correlator of their total descendants

$$\langle \widetilde{\Phi}_1(z_1) \cdots \widetilde{\Phi}_n(z_n) \rangle \tag{1179}$$

yields a closed form (under de Rham differential) on the moduli space $\mathcal{M}_{0,n}$, which can subsequently be integrated over relevant cycles to yield interesting periods.

More generally, for a surface $\Sigma$ of general genus $g$, given $Q$-closed fields $\Phi_1, \ldots, \Phi_n \in V$, one can consider the correlator

expand? prove closedness?

$$\langle G(x_1) \cdots G(x_p) \overline{G}(y_1) \cdots \overline{G}(y_q) \Phi_1(z_1) \cdots \Phi_n(z_n) \rangle. \tag{1180}$$

It can be understood (via integrating the positions of $G, \overline{G}$ fields against Beltrami differentials $\mu, \bar{\mu}$, cf. Section 2.8.3) as a closed form of type $(p,q)$ on $\mathcal{M}_{g,n}$.

### 9.1. 2d cohomological field theories.

Given a TCFT, restricting to correlators of $Q$-cocycles (extended to total descent towers as in (1179)), one obtains a simpler structure called a *cohomological field theory*.[161]

The following definition is from Kontsevich-Manin [26, section 6.1].

**Definition 9.1.** A 2d cohomological field theory is the following data:

- A $\mathbb{Z}$-graded complex vector space $W$ with an inner product $\langle , \rangle$.[162]
- A collection of linear maps (correlators)

$$I_{g,n} \colon W^{\otimes n} \to H^\bullet(\overline{\mathcal{M}}_{g,n}) \tag{1181}$$

  with $g, n \geq 0$ satisfying

$$2 - 2g - n < 0 \tag{1182}$$

  ("stability" condition). I.e., $I_{g,n}$ maps an $n$-tuple of elements of $W$ to a de Rham cohomology class of the Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of the moduli spaces of complex structures.

The collection of maps $I_{g,n}$ is assumed to satisfy the following factorization axioms.

(i) Let $S = \{i_1, \ldots, i_{n_1}\} \subset \{1, \ldots, n\}$ be a subset with $n_1$ elements and $S^c = \{j_1, \ldots, j_{n_2}\}$ its complement, with $n_2 = n - n_1$ elements. For $g_1 + g_2 = g$, let

$$\partial^{\mathrm{I}}_{g_1;S} \overline{\mathcal{M}}_{g,n} \simeq \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \tag{1183}$$

[161] Here we are making an implicit assumption that the correlators extend to the Deligne-Mumford compactification of the moduli spaces $\mathcal{M}_{g,n}$.

[162] In the cohomological field theory associated with a TCFT, one should think of $W$ as the $Q$-cohomology of the space of fields of the TCFT, $W_{\mathrm{CohFT}} = H_Q(V_{\mathrm{TCFT}})$.

be the Deligne-Mumford compactification stratum of complex codimension 1 (a.k.a. "compactification divisor"), corresponding to nodal curves where one component has genus $g_1$ and contains punctures from the subset $S$, plus the "node" or "neck" puncture and the second component similarly has genus $g_2$ and contains punctures from $S^c$, plus the "neck" puncture.[163] Then the factorization axiom is:

$$I_{g,n}(\Phi_1,\ldots,\Phi_n)\Big|_{\partial^{\mathrm{I}}_{g_1;S}\overline{\mathcal{M}}_{g,n}} = \\ = \sum_{k,l} I_{g_1,n_1+1}(\Phi_{i_1},\ldots,\Phi_{i_{n_1}},e_k)h^{kl}I_{g_2,n_2+1}(\Phi_{j_1},\ldots,\Phi_{j_{n_2}},e_l) \tag{1184}$$

Here $\Phi_1,\ldots,\Phi_n \in W$ any elements. We also introduced a basis $\{e_k\}$ in $W$ and $h^{kl}$ is the inverse matrix of the inner product in this basis $h_{kl} = \langle e_k, e_l\rangle$.

(ii) Consider the second type of Deligne-Mumford compactification stratum, corresponding to introducing a neck on a handle,

$$\partial^{\mathrm{II}}\overline{\mathcal{M}}_{g,n} \simeq \overline{\mathcal{M}}_{g-1,n+2}. \tag{1185}$$

The corresponding factorization axiom is:

$$I_{g,n}(\Phi_1,\ldots,\Phi_n)\Big|_{\partial^{\mathrm{II}}\overline{\mathcal{M}}_{g,n}} = \sum_{k,l} h^{kl} I_{g-1,n+2}(\Phi_1,\ldots,\Phi_n,e_k,e_l). \tag{1186}$$

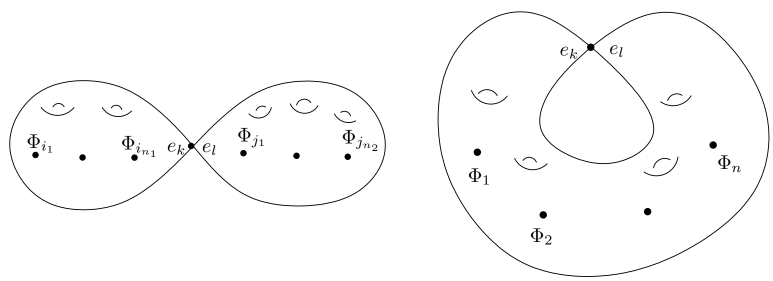



FIGURE 44. Factorization on nodal curves.

*Remark* 9.2. Thinking of a cohomological field theory as a reduction of a "parent" TCFT by passing to $Q$-cohomology, the factorization axioms above are a consequence of Segal's sewing axiom for the parent TCFT.

9.2. **Gromov-Witten cohomological field theory.** Fix a compact Kähler manifold $X$ (the target). We will assume that the Kähler symplectic form $\omega$ on $X$ has integer periods.[164]

We will be constructing a cohomological field theory in the sense of Definition 9.1 with the space of fields $W = H^\bullet_{\mathrm{de\,Rham}}(X)$. This cohomological field theory, called

[163] See Remark 2.65.

[164] In fact, the story of this section goes through under much milder assumptions: one just needs to require $X$ to be a symplectic manifold with compatible almost complex structure, such that the symplectic form has integer periods. The stronger assumption that $X$ is Kähler comes from the field theory side, where one wants to start with a sigma-model, cf. Section 9.3 (in the original approach [38], with $\mathcal{N} = (2,2)$ supersymmetric sigma-model).

Gromov-Witten theory, arises as a reduction by passing to $Q$-cohomology from a certain TCFT – the A-model, which is a sigma-model with target $X$ (coupled to certain extra fields).

Let $\Sigma$ be a closed Riemannian surface. For any smooth map $\phi\colon \Sigma \to X$, we define the *degree* of $\phi$ as

$$d = \int_\Sigma \phi^*\omega \quad \in \mathbb{Z}. \tag{1187}$$

Let us denote by $\mathrm{Hol}_d(\Sigma, X)$ the space of holomorphic maps $\phi\colon \Sigma \to X$ of a fixed degree $d$.

The space $\mathrm{Hol}_d(\Sigma, X)$ is finite-dimensional for any $d \in \mathbb{Z}$;[165] it vanishes for $d < 0$ and consists of constant maps for $d = 0$:

$$\mathrm{Hol}_0(\Sigma, X) = X. \tag{1188}$$

**Example 9.3.** Let the surface be $\Sigma = \mathbb{CP}^1$ with homogeneous coordinates $(z_0 : z_1)$ and let the target be $X = \mathbb{CP}^N = (\mathbb{C}^{N+1}\backslash\{0\})/\mathbb{C}^*$ with homogeneous coordinates $(u_0 : \cdots : u_N)$. We assume that the target $\mathbb{CP}^N$ is equipped with the symplectic structure $\omega_0 = \omega_{\mathrm{FS}}$ the Fubini-Study 2-form normalized to have unit integral over $\mathbb{CP}^1 \subset \mathbb{CP}^N$. We describe degree $d$ holomorphic maps $\mathbb{CP}^1 \to \mathbb{CP}^N$ as degree $d$ polynomial maps

$$\mathbb{C}^2\backslash\{0\} \to \mathbb{C}^{N+1}\backslash\{0\} \tag{1189}$$

where we subsequently quotient both sides by $\mathbb{C}^*$.

Thus, a degree $d$ holomorphic map $\mathbb{CP}^1 \to \mathbb{CP}^N$ is given as

$$u_p = A_p(z_0, z_1), \quad 0 \leq p \leq N, \tag{1190}$$

where $A_0, \ldots, A_p$ are homogeneous polynomials of degree $d$ in $z_0, z_1$. Tuples of polynomials $\{A_p\}$ and $\{A'_p\}$ determine the same map $\mathbb{CP}^1 \to \mathbb{CP}^n$ if and only if $A'_0 = cA_0, \ldots, A'_n = cA_n$ for some $c \in \mathbb{C}^*$. Also, a tuple $\{A_p\}$ determines a map $\mathbb{CP}^1 \to \mathbb{CP}^N$ if and only if the polynomials $\{A_p\}$ do not have a common nontrivial root $(z_0, z_1)$ – if they do, then there is a point of $\mathbb{C}^2\backslash\{0\}$ which is mapped to $\{0\} \in \mathbb{C}^{N+1}$, which does not correspond to any point in $\mathbb{CP}^N$. Such tuples $\{A_p\}$ correspond to so-called *Drinfeld's quasimaps* $\mathbb{CP}^1 \to \mathbb{CP}^N$; they are not however holomorphic maps in the usual sense (in particular they cannot be evaluated at all points of the source), so we will discard them. In summary, the space of holomorphic maps of degree $d$ is

$$\begin{aligned} \mathrm{Hol}_d(\mathbb{CP}^1, \mathbb{CP}^N) &= \\ = \{(A_p(z_0,z_1) = \sum_{j=0}^d a_{pj} z_0^j z_1^{d-j})_{p=0,\ldots,n} &\mid \{A_p\} \text{ do not have common roots}\} \,/\, \mathbb{C}^* \\ &= \mathbb{CP}^{(d+1)(N+1)-1}\backslash\mathbb{D} \end{aligned} \tag{1191}$$

where $a_{pj} \in \mathbb{C}$ are the coefficients of the polynomials – thus in order to specify a holomorphic map $\mathbb{CP}^1 \to \mathbb{CP}^N$ we need to specify the $(N+1) \times (d+1)$ array of coefficients $a_{pj}$, modulo scaling them all by a number $c \in \mathbb{C}^*$, which yields the projective space $\mathbb{CP}^{(d+1)(N+1)-1}$. We denoted the set of "prohibited" configurations corresponding to quasimaps by $\mathbb{D}$ – it is a subvariety in $\mathbb{CP}^{(d+1)(N+1)-1}$ of positive

[165] For this statement, compactness of $X$ is crucial.

codimension and can be described as $\mathbb{D} \simeq \mathbb{CP}^1 \times \mathbb{CP}^{d(N+1)-1}$ (the first factor in the r.h.s. gives the point on the source where the common root occurs).

As a further simplicifation, consider the case $N = 1$. Then degree $d$ holomorphic maps $\mathbb{CP}^1 \to \mathbb{CP}^1$ are described by

$$(1 : z) \mapsto (1 : \frac{A_1(z)}{A_0(z)}) \tag{1192}$$

where $A_0$ and $A_1$ are two degree $d$ polynomials in the variable $z$ without common roots. For instance, for $d = 1$ the holomorphic maps are

$$(1 : z) \mapsto (1 : a\frac{z-b}{z-c}) \tag{1193}$$

with parameters $a, b, c \in \mathbb{C}$ such that $a \neq 0$ and $b \neq c$ (otherwise it is a quasimap).

9.2.1. *Genus zero case.* Let $\Sigma = \mathbb{CP}^1$. We have a diagram of maps

$$\begin{tikzcd} C_n(\Sigma) \times \mathrm{Hol}_d(\Sigma, X) \arrow[r, "\mathrm{ev}"] \arrow[d, "p"'] & \underbrace{X \times \cdots \times X}_{n} \\ C_n(\Sigma) & \end{tikzcd} \tag{1194}$$

Here ev is the evaluation map, evaluating the holomorphic map on an $n$-tuple of points in $\Sigma$,

$$\mathrm{ev}\colon ((z_1, \ldots, z_n), \phi) \mapsto (\phi(z_1), \ldots, \phi(z_n)). \tag{1195}$$

The vertical map $p$ in (1194) is the projection onto the first factor.

*Remark* 9.4. The two objects in the right column in (1194) admit a certain compactification (we will leave it as a black box and denote it by an overline) such that the maps ev, $p$ extend to it.[166]

Fix a collection of closed forms on the target, $\alpha_1, \ldots, \alpha_n \in \Omega^\bullet_{\mathrm{cl}}(X)$. Then one can define

$$I_{0,n,d}(\alpha_1, \ldots, \alpha_n) = \int_{\mathrm{Hol}_d(\Sigma,X)} \mathrm{ev}^*(\pi_1^*(\alpha_1) \wedge \cdots \wedge \pi_n^*(\alpha_n)) \quad \in \Omega^\bullet_{\mathrm{cl}}(C_n(\Sigma)), \tag{1196}$$

where $\pi_i\colon X^n \to X$ is the projection onto the $i$-th factor. The form (1196) has the following properties:

(i) it is closed and its cohomology class is depends only on the cohomology classes of forms $\alpha_i$ – this fact follows from Stokes' theorem for fiber integrals and relies on the existence of compactifications, cf. Remark 9.4.
(ii) The form (1196) extends to a closed form on the Fulton-MacPherson compactified configuration space $\overline{C}_n(\Sigma)$.
(iii) The form is also basic w.r.t. Möbius transformations (which act diagonally in the top right corner in (1194) and in the obvious way on the configuration space). Therefore, the $I_{0,n,d}$ descends to a closed from on the moduli space $\overline{\mathcal{M}}_{0,n}$:

$$I_{0,n,d}(\alpha_1, \ldots, \alpha_n) \in \Omega^\bullet_{\mathrm{cl}}(\overline{\mathcal{M}}_{0,n}), \tag{1197}$$

[166]The compactification of the configuration space $C_n(\Sigma)$ is due to Fulton-MacPherson. The compactification of $C_n(\Sigma) \times \mathrm{Hol}_d(\Sigma, X)$ is a special case of Kontsevich's compactification of the moduli space of stable maps.

and by (i) above, the construction descends to de Rham cohomology:

$$I_{0,n,d}([\alpha_1],\ldots,[\alpha_n]) \in H^\bullet(\overline{\mathcal{M}}_{0,n}). \tag{1198}$$

This is the so-called Gromov-Witten cohomology class.

The genus zero part of the Gromov-Witten cohomological field theory is then defined as

$$I_{0,n}([\alpha_1],\ldots,[\alpha_n]) := \sum_{d\geq 0} q^d I_{0,n,d}([\alpha_1],\ldots,[\alpha_n]), \tag{1199}$$

where $q$ is a formal (infinitesimal) generating parameter.

*Remark* 9.5. The A-model, the "parent" TCFT for the Gromov-Witten hohomological field theory, contains a class of $Q$-closed observables: for each closed form $\alpha$ on the target one has an "evaluation observable" $O_\alpha \in V$, see Section 9.3.3. The cohomology class (1199) is the cohomology class of the $n$-point correlator on $\mathbb{CP}^1$,

$$\langle \widetilde{O}_{\alpha_1} \cdots \widetilde{O}_{\alpha_n} \rangle, \tag{1200}$$

where tilde means the full descendant, cf. (1179).

Lecture 40, 12/5/2022

9.2.2. *General genus.* Let $\Sigma$ be a closed oriented smooth surface of any genus $g$ and fix $d \geq 0$. One has a fiber bundle over the moduli space of complex structures on $\Sigma$ with fiber over $J \in \mathcal{M}_\Sigma$ the space of holomorphic maps (w.r.t. to the complex structure $J$ on $\Sigma$) to $X$ of degree $d$:

$$\begin{tikzcd} \mathcal{M}_\Sigma(X,d) \arrow[d] & \mathrm{Hol}_d(\Sigma,X) \arrow[l] \\ \mathcal{M}_\Sigma & \end{tikzcd} \tag{1201}$$

We have the "forgetful" map

$$r\colon \mathcal{M}_{\Sigma,n} \to \mathcal{M}_\Sigma \tag{1202}$$

from the moduli space with $n$ marked points to the moduli space without marked points, given by forgetting the marked points. The pullback of the bundle (1201) along the forgetful map $\mathcal{M}_{\Sigma,n}(X,d) := r^*\mathcal{M}_\Sigma(X,d) \to \mathcal{M}_{\Sigma,n}$ fits into the diagram similar to (1194):

$$\begin{tikzcd} \mathcal{M}_{\Sigma,n}(X,d) \arrow[r, "\mathrm{ev}"] \arrow[d, "p"'] & X^n \\ \mathcal{M}_{\Sigma,n} & \end{tikzcd} \tag{1203}$$

where ev evaluates the holomorphic map at the $n$ marked points. Again, there exists a compactification of the objects in the right column of the diagram – Kontsevich's moduli space of stable maps at the top and Deligne-Mumford compactification of $\mathcal{M}_{g,n}$ at the bottom – such that the maps ev, $p$ extend to the compactifications:[167]

check the footnote

[167] Very roughly, the idea is that in addition to adjoining Deligne-Mumford compactification strata (nodal curves) coming from the compactification of the base, in the total space one needs to blow up the configurations where a quasimap point in the space of holomorphic maps coincides with a marked point on the surface (i.e. exactly the situations where the evaluation of a map at a marked point becomes problematic).

(1204)
$$\begin{array}{ccc} \overline{\mathcal{M}}_{g,n}(X,d) & \xrightarrow{\mathrm{ev}} & X^n \\ {\scriptstyle p}\big\downarrow & & \\ \overline{\mathcal{M}}_{g,n} & & \end{array}$$

Here we put the genus of $\Sigma$ instead of $\Sigma$ as index.

Given closed forms $\alpha_1, \ldots, \alpha_n \in \Omega^\bullet_{\mathrm{cl}}(X)$, we construct a form

(1205) $$I_{g,n,d}(\alpha_1, \ldots, \alpha_n) = \int_{\mathrm{Hol}_d(\Sigma,X)} \mathrm{ev}^*(\pi_1^*(\alpha_1) \wedge \cdots \wedge \pi_n^*(\alpha_n)) \quad \in H^\bullet(\mathcal{M}_{g,n})$$

As in genus zero case, by Stokes' theorem and due to the existence of compactifications, this form is closed and its cohomology class depends only on the cohomology classes of $\alpha_i$; thus the construction descends to cohomology. Also, the form (1205) extends to the compactification of the moduli space of complex structures $\overline{\mathcal{M}}_{g,n}$.

This leads to the following definition at the level of cohomology.

**Definition 9.6.** Genus $g$ Gromov-Witten classes are defined via the diagram (1204) as

(1206) $$I_{g,n,d}([\alpha_1], \ldots, [\alpha_n]) = p_* \mathrm{ev}^*(\pi_1^*[\alpha_1] \wedge \cdots \pi_n^*[\alpha_n]) \quad \in H^\bullet(\overline{\mathcal{M}}_{g,n}),$$

where $[\alpha_1], \ldots, [\alpha_n] \in H^\bullet(X)$ are any de Rham cohomology classes of the target $X$.

As a generalization of (1199) to any genus, Gromov-Witten cohomological field theory is defined by summing the classes (1206) over the degree $d$, weighed with $q^d$,

(1207) $$I_{g,n}([\alpha_1], \ldots, [\alpha_n]) \colon = \sum_{d \geq 0} q^d I_{g,n,d}([\alpha_1], \ldots, [\alpha_n]).$$

**Theorem 9.7.** *The cohomology classes $I_{g,n}$ satisfy the factorization properties (1184), (1186).*

*Idea of proof.* Fix the numbers $g, n, d \geq 0$, fix a splitting of genus $g = g_1 + g_2$ and a splitting of the set of marked points into complementary subsets $\{1, \ldots, n\} = S \sqcup S^c$. Consider a compactification stratum $\partial_{g_1,S}\overline{\mathcal{M}}_{g,n}$ of the moduli space of complex structures. The restriction of the bundle (1204) to it is[168]

(1208) $$p^{-1}(\partial_{g_1,S}\overline{\mathcal{M}}_{g,n}) \simeq \bigsqcup_{d_1+d_2=d} \overline{\mathcal{M}}_{g_1,S\cup q}(X,d_1) \times_X \overline{\mathcal{M}}_{g_2,S^c \cup q^*}(X,d_2)$$

Here $q, q^*$ are the names of the nodal point as point seen as a marked point on either component of the nodal curve; the fiber product in the r.h.s. is w.r.t. evaluations at $q$ and at $q^*$, respectively. The evaluation map on the r.h.s. lands in $X^S \times \Delta \times X^{S^c}$ where $\Delta \subset X \times X$ is the diagonal.

Fix the cohomology classes $[\alpha_1], \ldots, [\alpha_n] \in H^\bullet(X)$. Then we have

(1209)
$$\mathrm{ev}^*(\prod_{i=1}^n \pi_i^*[\alpha_i])\Big|_{p^{-1}(\partial_{g_1,S}\overline{\mathcal{M}}_{g,n})} = \sum_{k,l} \mathrm{ev}^*_{S\cup q}(\prod_{i\in S} \pi_i^*[\alpha_i]\cdot \pi_q^* e_k) h^{kl} \mathrm{ev}^*_{S^c\cup q^*}(\prod_{i \in S^c} \pi_i^*[\alpha_i]\cdot \pi_{q^*}^* e_l)$$

[168]The intuition is that a holomorphic map $\phi$ from a nodal curve $\Sigma = \Sigma' \cup_q \Sigma''$ to $X$ is given by a pair of holomorphic maps, $\phi'$ on $\Sigma'$ and $\phi''$ on $\Sigma''$ agreeing at the node $q$. The degree of $\phi$ splits as the degree of $\phi'$ plus the degree of $\phi''$.

where $e_k$ is a basis in $H^\bullet(X)$ and $h^{kl}$ is the inverse matrix of Poincaré pairing; ev in the l.h.s. is for holomorphic maps out of the whole nodal curve $\Sigma$ and in the r.h.s. we have maps ev for the two components of $\Sigma$. Here we used the fact that the cohomology class of $X \times X$ Poincaré dual to the homology class of the diagonal $\Delta \subset X \times X$ is $\sum_{k,l} h^{kl} e_k \otimes e_l$. The appearance of this class in the r.h.s. of (1209) effectively forces $q$ and $q^*$ to map to the same point in $X$.

Pushing forward (i.e. performing the fiber integral) the l.h.s. of (1209) to the Deligne-Mumford stratum $\partial_{g_1,S}\overline{\mathcal{M}}_{g,n}$ and pushing forward the r.h.s. to the product $\overline{\mathcal{M}}_{g_1,S\cup q} \times \overline{\mathcal{M}}_{g_2,S^c\cup q^*}$, and summing over the degree $d$ (and the splittings $d = d_1+d_2$) with weight $q^d$, we obtain the desired factorization property (1184):
(1210)

$$I_{g,n}([\alpha_1],\ldots,[\alpha_n])|_{\partial_{g_1,S}\overline{\mathcal{M}}_{g,n}} = \sum_{k,l} I_{g_1,n_1+1}(\{[\alpha_i]\}_{i\in S}, e_k) h^{kl} I_{g_2,n_2+1}(\{[\alpha_i]\}_{i\in S^c}, e_l)$$

The factorization property on the second type of Deligne-Mumford strata (1186) is proved similarly.

□

**Definition 9.8.** For a collection of cohomology classes $[\alpha_1],\ldots,[\alpha_n] \in H^\bullet(X)$. The genus $g$, $n$-point Gromov-Witten invariant of degree $d$ is defined as the pairing of the Gromov-Witten class (1206) with the fundamental class of the moduli space $\overline{\mathcal{M}}_{g,n}$:

$$\mathrm{GW}_{g,n,d}([\alpha_1],\ldots,[\alpha_n])\colon = \int_{\overline{\mathcal{M}}_{g,n}} I_{g,n,d}([\alpha_1],\ldots,[\alpha_n]) \quad \in \mathbb{C} \tag{1211}$$

9.2.3. *Enumerative meaning of Gromov-Witten classes.* Fix $c_1,\ldots,c_n \in C_\bullet(X,\mathbb{Z})$ – a collection of cycles in $X$ and let $[\delta_{c_i}] \in H^\bullet(X)$ be the Poincaré dual classes to the homology classes of $c_i$; $[\delta_{c_i}]$ can be represented in de Rham cohomology by a (cohomologically smeared) Dirac delta-form on $c_i$, hence the notation.

Recall that for $c \subset X$ a $k$-cycle in a smooth $N$-manifold $X$, the delta-form $\delta_c$ is the distributional $(N-k)$-form characterized by the property

$$\int_X \delta_c \wedge \alpha = \int_c \alpha|_c \tag{1212}$$

for any $\alpha \in \Omega^k(X)$. A cohomologically smeared $\delta$-form on $c$ is a smooth form with the same property which is only required to hold for $\alpha$ a *closed* $k$-form.

The Gromov-Witten invariant

$$\mathrm{GW}_{g,n,d}([\delta_{c_1}],\ldots,[\delta_{c_n}]) \quad \in \mathbb{Q} \tag{1213}$$

is the "virtual" count of holomorphic curves in $X$ of genus $g$ and degree $d$ passing through the cycles $c_1,\ldots,c_n$. This number is an integer for zero genus. Generally, for higher genus, it is a rational number: holomorphic maps $\phi$ in this virtual count should be weighed with $\frac{1}{|\mathrm{stab}(\phi)|}$ – the inverse of the number of conformal
check the footnote automorphisms $\Sigma$ commuting with $\phi$.[169]

PICTURE

[169]A related point: compactified moduli spaces $\overline{\mathcal{M}}_{g,n}(X)$ have orbifold singularities which lead to having the "virtual" fundamental class defined over $\mathbb{Q}$ rather than $\mathbb{Z}$.

9.2.4. *Quantum cohomology ring.* Consider de Rham cohomology of $X$ as a $\mathbb{Z}_2$-graded[170] vector space $H^\bullet(X)$ equipped with an inner product $\langle,\rangle$ given by Poincaré pairing $\langle[\alpha_1],[\alpha_2]\rangle = \int_X \alpha_1 \wedge \alpha_2$ and equipped with a bilinear map

$$m\colon H(X) \otimes H(X) \to H(X) \tag{1214}$$

characterized by

$$\langle m([\alpha_1],[\alpha_2]),[\alpha_3]\rangle :\, = \sum_{d\geq 0} q^d \mathrm{GW}_{0,3,d}([\alpha_1],[\alpha_2],[\alpha_3]) \tag{1215}$$

with $q$ the generating parameter as in (1199). Note that Gromov-Witten classes in the r.h.s. here are elements of $H^\bullet(\overline{\mathcal{M}}_{0,3})$, i.e., numbers (since $\overline{\mathcal{M}}_{0,3}$ is a point). If $\alpha_i$ are integer classes then the Gromov-Witten invariant $GW_{0,3,d}$ is an integer.

**Definition 9.9.** The bilinear operation $m\colon H(X)\otimes H(X) \to H(X)$ defined by (1215) is called the "quantum product" on the cohomology $H(X)$. Te quantum product endows the cohomology $H(X)$ with the structure of a $\mathbb{Z}_2$-graded ring called the "quantum cohomology ring."

Note that due to (1188) one has

$$\mathrm{GW}_{0,3,0}([\alpha_1],[\alpha_2],[\alpha_3]) = \int_X \alpha_1 \wedge \alpha_2 \wedge \alpha_3. \tag{1216}$$

Thus, the $q^0$ term in $m$ is the usual cup product while $q^{>0}$ terms comprise a deformation of the cup product by the data of (genus-zero, three-point) Gromov-Witten classes.

Implicitly present in the definition above (in the words "ring" and "product") is the following.

**Lemma 9.10.** *The operation $m$ defined by (1215) is supercommutative and associative.*

Supercommutativity is obvious from the definition of Gromov-Witten classes. Associativity is not obvious and is a consequence of the WDVV equation (1230).

**Example 9.11.** Let $X = \mathbb{CP}^1$. The space of holomorphic maps of degree $d$ is given by (1191):

$$\mathrm{Hol}_d(\mathbb{CP}^1,\mathbb{CP}^1) = \mathbb{CP}^{2d+1}\backslash \mathbb{D} \tag{1217}$$

– it is a manifold of real dimension $2(2d+1) = 4d+2$. Thus the Gromov-Witten invariants

$$\mathrm{GW}_{0,3,d}(\alpha_1,\alpha_2,\alpha_3) = \int_{\mathrm{Hol}_d(\mathbb{CP}^1,\mathbb{CP}^1)} \mathrm{ev}^*(\pi_1^*\alpha_1 \wedge \pi_2^*\alpha_2 \wedge \pi_3^*\alpha_3). \tag{1218}$$

Note that for this number to be nonzero it is necessary that the dimension of the space over which we integrate is equal to the degree of the form we are integrating:

$$4d+2 = |\alpha_1| + |\alpha_2| + |\alpha_3|, \tag{1219}$$

where $|\alpha|$ is the de Rham degree of the form $\alpha$.

The cohomology of $\mathbb{CP}^1$ is spanned by two classes, $[1] \in H^0(\mathbb{CP}^1)$ and $[\omega] \in H^2(\mathbb{CP}^1)$ – the class of the Fubini-Study 2-form normalized to have unit volume.

[170] The reason for why we only consider the mod 2 projection of the natural $\mathbb{Z}$-grading on cohomology is elucidated in Example 9.11 below: $\mathbb{Z}$-grading is not preserved by the deformation of the cup product we are describing.

Choosing $\alpha_{1,2,3}$ in (1218) to be the basis classes in $H^\bullet(\mathbb{CP}^1)$ we observe that there are only two possibilities (up to permutations) to satisfy (1219):

$$\mathrm{GW}_{0,3,0}([1],[1],[\omega]) = 1, \tag{1220}$$

$$\mathrm{GW}_{0,3,1}([\omega],[\omega],[\omega]) = 1. \tag{1221}$$

Note that (1220) corresponds to the usual cup product in cohomology ($[1]\cup[1]=[1]$, or $[1]\cup[\omega]=[\omega]$). On the other hand, (1221) is the number of degree 1 holomorphic maps $\mathbb{CP}^1 \to \mathbb{CP}^1$ (i.e., Möbius transformations) mapping three marked points in the source $\mathbb{CP}^1$ into three fixed points $c_1, c_2, c_3$ in the target $\mathbb{CP}^1$ in general position (we then think of $c_i$ as zero-cycles with $[\omega]$ the Poincaré dual cohomology class for each $c_i$). There is exactly one such map.

To summarize the result, the quantum product in the cohomology of $\mathbb{CP}^1$ is given by the following multiplication table.

$$m([1],[1])=[1], \quad m([1],[\omega])=[\omega], \quad m([\omega],[\omega]) = q\cdot[1]. \tag{1222}$$

Note that due to the last relation the quantum product does not preserve the de Rham degree. In this particular example, $X=\mathbb{CP}^1$, one can prescribe degree $-4$ to $q$ and then $m$ preserves the $\mathbb{Z}$-degree.

9.2.5. *Gromov-Witten potential.* Fix a basis $e_1,\ldots,e_s$ for $H^\bullet(X)$. The function

$$\Phi(t_1,\ldots,t_s)\colon = \sum_{n_1,\ldots,n_s\geq 0}\sum_{d\geq 0} \frac{t_1^{n_1}\cdots t_s^{n_s}}{n_1!\cdots n_s!} q^d\, \mathrm{GW}_{0,\sum n_i,d}(\underbrace{e_1,\ldots,e_1}_{n_1},\ldots,\underbrace{e_s,\ldots,e_s}_{n_s}) \tag{1223}$$

of the generating parameters $t_1,\ldots,t_s$ is called the Gromov-Witten potential. Here we understand that the variable $t_a$ is even (commuting) if $e_a \in H^{\mathrm{even}}(X)$ and $t_a$ is odd if $e_a \in H^{\mathrm{odd}}(X)$. Thus, $\Phi$ is a generating function for Gromov-Witten invariants.

One can think of $t_1,\ldots,t_n$ as coordinates on $H^\bullet(X)$, i.e., coordinates of the vector $\beta=\sum_a t_a e_a \in H^\bullet(X)$. Then one can also write $\Phi$ as

$$\Phi(t_1,\ldots,t_s) = \sum_{n\geq 0}\sum_{d\geq 0}\frac{q^d}{n!}\mathrm{GW}_{0,n,d}(\underbrace{\beta,\ldots,\beta}_{n}) \tag{1224}$$

One can treat parameters $t_a$ as formal (i.e. treat $\Phi$ as a formal power series in $t_a$'s), however the sum over $n$ is actually convergent for $\beta$ in some open set $U$ in $H^\bullet(X)$.

<u>"Big" quantum product.</u> One defines the "big quantum product" as a family parametrized by $\beta=\sum_a t_a e_a \in H^\bullet(X)$ of supercommutative associative products on cohomology

$$m_\beta\colon H(X)\otimes H(X) \to H(X) \tag{1225}$$

defined by

$$\langle m_\beta(\alpha_1,\alpha_2),\alpha_3\rangle\colon = \sum_{n\geq 0}\sum_{d\geq 0}\frac{q^d}{n!}\mathrm{GW}_{0,n+3,d}(\alpha_1,\alpha_2,\alpha_3,\underbrace{\beta,\ldots,\beta}_{n}), \tag{1226}$$

for any $\alpha_{1,2,3}\in H(X)$. Thus, it is the construction of the quantum product (1215) deformed by the class $\beta \in H(X)$.

Note that the big quantum product can be written in terms of the third derivative of the potential $\Phi$:

$$\langle m_\beta(e_a, e_b), e_c\rangle = \frac{\partial^3\Phi}{\partial t_a \partial t_b \partial t_c}, \tag{1227}$$

for any $a, b, c = 1, \ldots, s$; both sides are understood as functions of $\beta \in U \subset H(X)$.

The big quantum product endows an open subset of cohomology $U \subset H(X)$ with the structure of a *Frobenius manifold*.

The following definition is due to Dubrovin [9, 10].

**Definition 9.12.** A Frobenius manifold is a manifold $Y$ equipped with the following data:

- Affine flat structure on $Y$ and a compatible (flat) Riemannian metric $h$.
- For each $\beta \in Y$, the tangent space $T_\beta Y$ is equipped with a commutative associative product

$$m_\beta \colon T_\beta Y \otimes T_\beta Y \to T_\beta Y \tag{1228}$$

  compatible with $h$, in the sense that $h(m_\beta(x, y), z) = h(x, m_\beta(y, z))$.
- A potential $\Phi \in C^\infty(Y)$ such that

$$h(m(u, v), w) = u \circ v \circ w \circ \Phi \tag{1229}$$

  for any triple of flat vector fields $u, v, w$ on $Y$.

This definition has a straightforward $\mathbb{Z}_2$-graded generalization. To see the big quantum product as equipping an open set in $H(X)$ with the structure of a Frobenius manifold, one should consider the ring of scalars to be formal power series in $q$.

9.2.6. *WDVV equation.* Let $h^{ab}$ be the inverse matrix of the Poincaré pairing in the basis $\{e_a\}$ in $H(X)$. The following theorem is due to Witten-Dijkgraaf-Verlinde-Verlinde [41].

**Theorem 9.13.** *Gromov-Witten potential $\Phi$ satisfies the following differential equation:*

index $d$ is unfortunate here, can be confused with the degree

$$\sum_{c,d} \frac{\partial^3\Phi}{\partial t_a \partial t_b \partial t_c} h^{cd} \frac{\partial^3\Phi}{\partial t_d \partial t_e \partial t_f} = \sum_{c,d} \frac{\partial^3\Phi}{\partial t_e \partial t_b \partial t_c} h^{cd} \frac{\partial^3\Phi}{\partial t_d \partial t_a \partial t_f} \tag{1230}$$

*(the r.h.s. is the l.h.s. with indices $a, e$ switched).*

The equation (1230) is known as Witten-Dijkgraaf-Verlinde-Verlinde (or WDVV) equation. It is a consequence of the factorization properties of Gromov-Witten classes on compactification divisors in $\overline{\mathcal{M}}_{g,n}$ (Theorem 9.7) and certain relations between homology classes of these divisors – so-called Keel's relations, see Section 9.2.10 below.

9.2.7. *Example of Gromov-Witten potential: $X = \mathbb{CP}^1$.* Consider the example $X = \mathbb{CP}^1$. In this case the cohomology $H(X)$ has a basis $[1], [\omega]$ (with $\omega$ the Fubini-Study 2-form normalized to have unit volume); let us denote the corresponding generating parameters $t_0, t_1$. We already know the numbers $\mathrm{GW}_{0,3,d}$ from (1220), (1221).

**Lemma 9.14.** *Gromov-Witten invariants for $n \geq 4$ points are*

$$\mathrm{GW}_{0,n,d}(\underbrace{[\omega],\ldots,[\omega]}_{k},\underbrace{[1],\ldots,[1]}_{l}) = \begin{cases} 1 & \text{if } l=0, d=1, \\ 0 & \text{otherwise} \end{cases} \tag{1231}$$

*here $k+l=n$.*

*Proof.* If $l>0$, $p_*\mathrm{ev}^*$ is a class on $\overline{\mathcal{M}}_{0,n}$ coming as a pullback of a class from $\overline{\mathcal{M}}_{0,k}$ via the map forgetting the $l$ points mapping to [1]. Being a pullback, it integrates to zero on $\mathcal{M}_{0,n}$.

For the case $l=0$ (and hence $k=n$), we have a balancing condition (degree of the form) = (dimension of $\mathrm{Hol}_d$)+(dimension of $\mathcal{M}_{0,n}$):

$$2n = 2(2d+1)+2(n-3) \quad \Leftrightarrow \quad d=1. \tag{1232}$$

In the case $k=n$, $d=1$ – the only case when we might get a nontrivial Gromov-Witten invariant, we are counting the number of Möbius transfromations $\mathbb{CP}^1 \to \mathbb{CP}^1$ that take points $(0,1,\infty,z_4,\ldots,z_n)$ to points $(u_1,\ldots,u_n)$ where $u_i$ are fixed distinct points on the target and $z_4,\ldots,z_n$ are arbitrary (integrated over when we integrate over $\mathcal{M}_{0,n}$ in (1213)). There is exactly one such map. □

As a corollary, the Gromov-Witten potential for $X=\mathbb{CP}^1$ is

$$\Phi(t_0,t_1) = \frac{t_0^2 t_1}{2} + \sum_{n\geq 3} q\frac{t_1^n}{n!} = \frac{t_0^2 t_1}{2} + q\left(e^{t_1}-1-t_1-\frac{t_1^2}{2}\right). \tag{1233}$$

The big quantum product is given on basis elements by

$$m_\beta([1],[1]) = [1], \quad m_\beta([1],[\omega]) = [\omega], \quad m_\beta([\omega],[\omega]) = qe^{t_1}\cdot[1], \tag{1234}$$

where the reference point is $\beta = t_0[1]+t_1[\omega] \in H(\mathbb{CP}^1)$.

9.2.8. *Example of Gromov-Witten potential: $X=\mathbb{CP}^2$.* We proceed to the case $X=\mathbb{CP}^2$. We refer to original paper [26] for details. One has three basis cohomology classes: $[1],[\omega],[\omega^2]$ where again $\omega$ is the Fubini-Study 2-form normalized to have unit period on $\mathbb{CP}^1 \subset \mathbb{CP}^2$. Let us denote the corresponding generating parameters $t_0,t_1,t_2$.

**Theorem 9.15** (Kontsevich-Manin [26]). *(i) The Gromov-Witten potential for $X=\mathbb{CP}^2$ has the form*

$$\Phi(t_0,t_1,t_2) = \frac{t_0^2 t_2}{2} + \frac{t_0 t_1^2}{2} - q\frac{t_2^2}{2} + \sum_{d\geq 1}\frac{\mathcal{N}(d)}{(3d-1)!}q^d t_2^{3d-1}e^{dt_1}, \tag{1235}$$

*where $\mathcal{N}(d)$ is the number of rational (i.e. genus zero) holomorphic curves of degree $d$ in $\mathbb{CP}^2$ passing through $3d-1$ points in general position.*

*(ii) The numbers $\mathcal{N}(d)$ satisfy $\mathcal{N}(1)=1$ and the recurrence relation*

$$\mathcal{N}(d) = \sum_{k+l=d}\mathcal{N}(k)\mathcal{N}(l)k^2 l\left(l\begin{pmatrix}3d-4\\3k-2\end{pmatrix} - k\begin{pmatrix}3d-4\\3k-1\end{pmatrix}\right) \tag{1236}$$

*for $d \geq 2$. These two properties define the numbers $\mathcal{N}(d)$ completely. In particular, the first numbers are:*

(1237)

| $d$ | 1 | 2 | 3 | 4 | 5 | $\cdots$ |
|---|---|---|---|---|---|---|
| $\mathcal{N}(d)$ | 1 | 1 | 12 | 620 | 87304 | $\cdots$ |

In particular $\mathcal{N}(1) = 1$ is the number of degree 1 curves (lines) in $\mathbb{CP}^2$ through 2 (generic) points, $\mathcal{N}(2) = 1$ is the number of conics through 5 points, $\mathcal{N}(3) = 12$ is the number of *rational* cubics through 8 points,[171] etc.

The term $-q\frac{t_2^2}{2}$ in (1235) is inconsequential, it cancels a similar term with the opposite sign present in the sum over $d$; it is put there so that $\Phi$ does not have terms of degree $< 3$ in $t$'s (cf. the stability condition (1182): we only consider GW invariants in genus zero for $n \geq 3$).

*Sketch of proof.* (i) Consider the Gromov-Witten invariant

$$\mathrm{GW}_{0,n,d}(\underbrace{[1],\ldots,[1]}_{n_0},\underbrace{[\omega],\ldots,[\omega]}_{n_1}.\underbrace{[\omega^2],\ldots,[\omega^2]}_{n_2}) \tag{1238}$$

for $n \geq 4$; we understand that $n = n_0 + n_1 + n_2$. The number (1238) vanishes for $n_0 > 0$ by the same argument as in (1231) for $l > 0$. The balancing condition between the form degree and the dimension of the space over which it is integrated is

$$\underbrace{2n_1 + 4n_2}_{\text{form degree}} = \underbrace{2(3(d+1)-1)}_{\dim_{\mathbb{R}} \mathrm{Hol}} + \underbrace{2(n-3)}_{\dim_{\mathbb{R}} \mathcal{M}_{0,n}} \quad \Leftrightarrow \quad n_2 = 3d-1 \tag{1239}$$

Denote

$$\mathcal{N}(d) \colon = \mathrm{GW}_{0,3d-1,d}(\underbrace{[\omega^2],\ldots,[\omega^2]}_{3d-1}) \tag{1240}$$

If we insert $n_1$ additional copies of the class $[\omega]$ (Poincaré dual to the class of a hyperplane $H \subset \mathbb{CP}^2$ of complex codimension 1) into the Gromov-Witten invariant (1240), the number (1240) gets multiplied by $d^{n_1}$, since a curve of degree $d$ intersects the hyperplane $H$ $d$ times.

This analysis, together with the straightforward case $n = 3$ results in the ansatz (1235).

(ii) The recurrence relation (1236) is an immediate consequence of the WDVV equation (1230), from substituting the ansatz (1235) into it.

□

Lecture 41, 12/7/2022

9.2.9. *Keel's theorem.* For a subset $S \subset \{1,\ldots,n\}$, let us denote by $D_S \in H_\bullet(\overline{\mathcal{M}}_{0,n})$ the homology class of Deligne-Mumford compactification stratum $\partial_{0,S}$ (1183) of the compactified moduli space $\overline{\mathcal{M}}_{0,n}$. We will denote $S^c$ the complement of $S$ in $\{1,\ldots,n\}$.

**Theorem 9.16** (Keel [23])**.** *Homology of the moduli space $\overline{\mathcal{M}}_{0,n}$ is generated by classes $D_S$ with $S$ subsets of $\{1,\ldots,n\}$ such that $|S|,|S^c| \geq 2$, modulo the following relations:*

- *$D_S = D_{S^c}$.*
- *For $i,j,k,l$ distinct,*

$$\sum_{i,j\in S,\ k,l\in S^c} D_S = \sum_{i,k\in S,\ j,l\in S^c} D_S = \sum_{i,l\in S,\ j,k\in S^c} D_S \tag{1241}$$

- *$D_S \cap D_T = 0$ unless $S \subset T$ or $T \subset S$.*

---

[171]One can find a cubic through 9 points in general position, but it will (in general position) have genus one, not zero.

*In the relation (1241) the summation in the left term is over partitions of* $\{1,\ldots,n\}$ *into two subsets* $S, S^c$ *such that* $S$ *contains* $i,j$ *and* $S^c$ *contains* $S^c$, *and similarly for the middle and the right terms.*

**Example 9.17.** Consider the case $n=4$. Non-compactified moduli space $\mathcal{M}_{0,4} = C_4(\mathbb{CP}^1)/PSL(2,\mathbb{C})$ can be identified with sphere with three punctures, $\mathbb{CP}^1\{0,1,\infty\}$ (fixing three of the marked points to $0,1,\infty$, the modulus is the position of the fourth point), cf. (209). Deligne-Mumford compactification fills the the three punctures with the compactification strata $\partial_{0,\{1,4\}}$, $\partial_{0,\{2,4\}}$, $\partial_{0,\{3,4\}}$ (configurations where $z_4$ approaches $z_1=0$, $z_2=1$ or $z_3=\infty$), see Figure 15. The compactified moduli space $\overline{\mathcal{M}}_{0,4}$ is just a sphere $\mathbb{CP}^1$ and all three Deligne-Mumford strata are in the same homology class – the class of a point in $\mathbb{CP}^1$. Thus, one indeed has

conventions in Fig. 15 are a bit different.

$$D_{\{1,4\}} = D_{\{2,4\}} = D_{\{3,4\}} \tag{1242}$$

which is the Keel's relation (1241) for $n=4$.

9.2.10. *Explanation of WDVV equation from Keel's theorem and factorization of GW classes.* Consider the moduli space $\overline{\mathcal{M}}_{0,n+4}$ with marked points labeled $\{A,B,E,F,1,\ldots,n\}$. Fix $a,b,e,f\in\{1,\ldots,s\}$ a quadruple of basis elements in $H(X)$. Restricting the Gromov-Witten class to Deligne-Mumford compatification strata of $\overline{\mathcal{M}}_{0,n+4}$, we obtain

$$\sum_{S\subset\{1,\ldots,n\}} \int_{D_{S\cup\{A,B\}}} I_{0,n+4}(e_a,e_b,e_e,e_f,\underbrace{\beta,\ldots,\beta}_{n}) \underset{\text{factorization (1210)}}{=} \tag{1243}$$

$$= \sum_{S\subset\{1,\ldots,n\}}\sum_{c,d}\int_{\overline{\mathcal{M}}_{0,S\cup\{A,B,C\}}} I_{0,|S|+3}(e_a,e_b,\underbrace{\beta,\ldots,\beta}_{|S|},e_c)\, h^{cd}\int_{\overline{\mathcal{M}}_{0,S^c\cup\{D,E,F\}}} I_{0,|S^c|+3}(e_e,e_f,\underbrace{\beta,\ldots,\beta}_{|S^c|},e_d)$$

$$= \sum_{n_1+n_2=n}\frac{n!}{n_1!n_2!}\sum_{c,d}\mathrm{GW}_{0,n_1+3}(e_a,e_b,\underbrace{\beta,\ldots,\beta}_{n_1},e_c)h^{cd}\mathrm{GW}_{0,n_2+3}(e_e,e_f,\underbrace{\beta,\ldots,\beta}_{n_2},e_d)$$

In this computation we called $D,E$ the nodal point seen as a marked point on the two components of the curve. Note that by Keel's relation (1241), expression (1243) doesn't change if we switch $A\leftrightarrow E$ and $a\leftrightarrow e$: under this switch, both the cohomology class in the l.h.s. and the homology class $\sum_S D_{S\cup\{A,B\}}$ it is paired with are invariant – the former trivially and the latter by Keel's theorem.

Summing (1243) over $n\geq 0$ with weight $\frac{1}{n!}$, we obtain the l.h.s. of the WDVV equation (1230). Switching $a\leftrightarrow e$ (which doesn't change the expression by the argument above), we obtain the r.h.s. of WDVV.

PICTURE: nodal curve with two groups of points corresponding to (1243).

9.3. **A-model.** For details on the A-model we refer to Witten's original papers [38, 42]. For the viewpoint on the A-model as calculating the Euler class of a vector bundle over the mapping space whose section is the holomorphicity equation, see [4].

Fix a Riemannian surface $\Sigma$ and a target Kähler manifold $X$. We will assume that the Kähler symplectic form $\omega$ on $X$ has integral periods.

We will use local complex coordinates on the target: holomorphic coordinates $x^i$ and antiholomorphic coordinates $x^{\bar{i}}$; we will denote the real coordinates on the target $x^I$ (equivalently, one may think of $x^I$ as holomorphic and antiholomorphic coordinates jointly). The action functional of the $A$-model is

(1244) $$S = \int_\Sigma \frac{i}{2} g_{IJ}\boldsymbol{\partial}\phi^I \bar{\boldsymbol{\partial}}\phi^J + \psi_i^{(1,0)}\overline{\mathbf{D}}\chi^i - \psi_{\bar{i}}^{(0,1)}\mathbf{D}\chi^{\bar{i}} + iR^{i\bar{i}}{}_{j\bar{j}}\psi_i^{(1,0)}\psi_{\bar{i}}^{(0,1)}\chi^j\chi^{\bar{j}}$$

The fields are

- A smooth map $\phi\colon \Sigma \to X$.
- An odd (anticommuting) field

(1245) $$\chi \in \Gamma(\Sigma, \phi^* TX).$$

- Odd $(1,0)$- and $(0,1)$-form fields

(1246) $$\psi^{(1,0)} \in \Omega^{1,0}(\Sigma, \phi^*(T^{1,0})^* X), \quad \psi^{(0,1)} \in \Omega^{0,1}(\Sigma, \phi^*(T^{0,1})^* X).$$

One can assign $\mathbb{Z}$-grading to fields (ghost number):

(1247) $$\mathrm{gh}(\phi) = 0, \;\; \mathrm{gh}(\chi) = 1, \;\; \mathrm{gh}(\psi^{1,0}) = \mathrm{gh}(\psi^{0,1}) = -1.$$

In the action (1244), $g = g(\phi)$ is the Riemannian metric on the target $X$ pulled back to $\Sigma$ by the map $\phi$;

(1248) $$\overline{\mathbf{D}}\chi^i = \bar{\boldsymbol{\partial}}\chi^i + \Gamma^i_{jk}(\phi)\bar{\boldsymbol{\partial}}\phi^j\chi^k, \quad \mathbf{D}\chi^{\bar{i}} = \boldsymbol{\partial}\chi^{\bar{i}} + \Gamma^{\bar{i}}_{\bar{j}\bar{k}}(\phi)\boldsymbol{\partial}\phi^{\bar{j}}\chi^{\bar{k}}$$

are the Dolbeault operators on $\Sigma$ twisted by the pullback of the Levi-Civita connection $\nabla_{LC}$ on $X$; $R = R(\phi)$ is the pullback of the Riemann curvature tensor on $X$ to $\Sigma$.

The first term of the action (1244) is the action of a sigma-model with target $X$ seen as a Riemannian manifold; one can rewrite it as

(1249) $$\int_\Sigma \frac{i}{2} g_{IJ}\boldsymbol{\partial}\phi^I\bar{\boldsymbol{\partial}}\phi^J = \int_\Sigma i g_{\bar{i}j}\boldsymbol{\partial}\phi^{\bar{i}}\bar{\boldsymbol{\partial}}\phi^j + \underbrace{\frac{1}{2}\int_\Sigma \phi^*\omega}_{S_{\mathrm{top}}}$$

Here $\omega = \frac{i}{2} g_{i\bar{j}} dx^i \wedge dx^{\bar{j}}$ is the Kähler symplectic form on $X$. The last term in the r.h.s. of (1249) is "topological": it depends only on the homotopy class of the map $\phi$ (and the cohomology class of $\omega$). In particular, $S_{\mathrm{top}}$ is a locally constant function on the space of fields.

The space of fields is equipped with a degree $-1$ odd derivation $Q$ acting by

(1250) $$\begin{aligned} Q\phi^I &= \chi^I, \;\; Q\chi^I = 0, \\ Q\psi_i^{(1,0)} &= -i g_{i\bar{j}}\boldsymbol{\partial}\phi^{\bar{j}} + \Gamma^k_{ij}\chi^j\psi_k^{(1,0)}, \\ Q\psi_{\bar{i}}^{(0,1)} &= -i g_{\bar{i}j}\bar{\boldsymbol{\partial}}\phi^j + \Gamma^{\bar{k}}_{\bar{i}\bar{j}}\chi^{\bar{j}}\psi_{\bar{k}}^{(0,1)}. \end{aligned}$$

The operator $Q$ squares to zero modulo equations of motion,[172]

(1251) $$Q^2 \underset{EL}{\sim} 0.$$

One can in fact massage the model (construct a "first-order" action) to make $Q$ square to zero on the nose, see Section 9.3.4.

The crucial property of the action (1244) is that it is $Q$-exact, up to the topological term:

(1252) $$S \underset{EL}{\sim} S_{\mathrm{top}} + Q(R)$$

[172] More precisely, here and in (1252), we only need the part of the Euler-Lagrange equations arising as the variation of $S$ w.r.t. fields $\psi^{(1,0)}, \psi^{0,1}$. These equations read $\overline{\mathbf{D}}\chi^i + iR^{i\bar{i}}{}_{j\bar{j}}\psi_{\bar{i}}^{(0,1)}\chi^j\chi^{\bar{j}} = 0$ and $\mathbf{D}\chi^{\bar{i}} - iR^{i\bar{i}}{}_{j\bar{j}}\psi_i^{(1,0)}\chi^j\chi^{\bar{j}} = 0$.

where

$$R = \int_\Sigma -\frac{1}{2}\psi_i^{(1,0)}\bar{\boldsymbol{\partial}}\phi^i + \frac{1}{2}\psi_{\bar{i}}^{(0,1)}\boldsymbol{\partial}\phi^{\bar{i}} \tag{1253}$$

Again, the equality (1252) is true only modulo equation of motion but becomes true everywhere on the space of fields in the version of Section 9.3.4.

*Remark* 9.18. In the language of TCFT, the operator $Q$ is given by integrating around a field the conserved current $\mathbb{J} = \mathbf{J} + \overline{\mathbf{J}}$, cf. (911), where

normalizations for $J, G$?

$$\mathbf{J} = g_{i\bar{j}}\chi^i\boldsymbol{\partial}\phi^{\bar{j}}, \quad \overline{\mathbf{J}} = g_{\bar{i}j}\chi^{\bar{i}}\bar{\boldsymbol{\partial}}\phi^j. \tag{1254}$$

The currents $\mathbf{J}, \overline{\mathbf{J}}$ are conserved separately: $\bar{\boldsymbol{\partial}}\mathbf{J} \underset{EL}{\sim} 0$, $\boldsymbol{\partial}\overline{\mathbf{J}} \underset{EL}{\sim} 0$.

The fields $G$, $\overline{G}$ – the $Q$-primitives of the components of the stress-energy tensor (909) are given by

$$G(dz)^2 = \psi_i^{(1,0)}\boldsymbol{\partial}\phi^i, \quad \overline{G}(d\bar{z})^2 = \psi_{\bar{i}}^{(0,1)}\bar{\boldsymbol{\partial}}\phi^{\bar{i}}. \tag{1255}$$

*Remark* 9.19. The A-model is described by somewhat lengthy formulae due to the involvement of target geometry. For a flat target all formulae simplify drastically. E.g., the action (1244) becomes simply a free (quadratic) action

$$S = S_{\mathrm{top}} + \int_\Sigma i g_{i\bar{j}}\boldsymbol{\partial}\phi^{\bar{i}}\bar{\boldsymbol{\partial}}\phi^j + \psi_i^{(1,0)}\bar{\boldsymbol{\partial}}\chi^i - \psi_{\bar{i}}^{(0,1)}\boldsymbol{\partial}\chi^{\bar{i}}, \tag{1256}$$

with $g_{\bar{i}j}$. In fact there is a very interesting class of cases where the target is compact and admits a flat metric everywhere except for finitely many points – toric manifolds $X$. In this case one can study the A-model as a free theory with special observables corresponding to the preimages of the special points in $X$ where the metric is singular. This approach is due to Frenkel-Losev [13].

9.3.1. *Path integral heuristics: independence on the target geometric data.* The fact (1252) leads to the following expectation about the A-model path integral: the correlator of any collection of $Q$-exact observables $\Phi_1, \ldots, \Phi_n$ should be invariant under deformations of the geometric data on the target, except for the possible change of the topological term. More precisely, one can split the correlator into contributions of different homotopy classes of the map $\phi\colon \Sigma \to X$:

$$\begin{aligned} \langle \Phi_1(z_1)\cdots\Phi_n(z_n)\rangle &= \int_{\mathrm{Fields}} e^{-S}\Phi_1(z_1)\cdots\Phi_n(z_n) = \\ &= \sum_{[\phi]\in[\Sigma,X]} e^{-S_{\mathrm{top}}([\phi])}\int_{\mathrm{Fields}_{[\phi]}} e^{-Q(R)}\Phi_1(z_1)\cdots\Phi_n(z_n) = \\ &= \sum_{[\phi]\in[\Sigma,X]} e^{-S_{\mathrm{top}}([\phi])}\langle\Phi_1(z_1)\cdots\Phi_n(z_n)\rangle_{[\phi]} \end{aligned} \tag{1257}$$

where $[\Sigma, X]$ is the set of homotopy classes of maps. Then the expectation is that for $Q$-closed observables $\Phi_i$ and for a path $(g_t, J_t, \omega_t)$ of Kähler data on $X$ with parameter $t$, the contribution

$$\langle\Phi_1(z_1)\cdots\Phi_n(z_n)\rangle_{[\phi]} \tag{1258}$$

of a homotopy class into the correlator (1257) does not depend on $t$.

The logic is that one differentiates the path integral over a given homotopy class in (1257) in the parameter $t$ of the family which results in a $Q$-exact expression (modulo Euler-Legrange equations) under the path integral; such expressions are expected to have zero averages over the space of fields.

9.3.2. *A-model as an integral representation for the delta-form on holomorphic maps.* If we rescale the target metric $g \to \frac{1}{\epsilon} g$ with $\epsilon$ a constant, the action becomes

(1259)
$$S^\epsilon = \underbrace{\frac{1}{2\epsilon}\int_\Sigma \phi^*\omega}_{S^\epsilon_{\mathrm{top}}} + \underbrace{\int_\Sigma \underbrace{\frac{i}{\epsilon} g_{\bar{i}j}\boldsymbol{\partial}\phi^{\bar{i}}\bar{\boldsymbol{\partial}}\phi^j}_{(I)} + \psi^{(1,0)}_i \overline{\mathbf{D}}\chi^i - \psi^{(0,1)}_{\bar{i}}\mathbf{D}\chi^{\bar{i}} + i\epsilon R^{i\bar{i}}{}_{j\bar{j}}\psi^{(1,0)}_i\psi^{(0,1)}_{\bar{i}}\chi^j\chi^{\bar{j}}}_{S'_\epsilon}$$

In the limit $\epsilon \to 0$ the dominating term (I) in the action essentially enforces the constraint $\bar{\boldsymbol{\partial}}\phi^i = 0$, i.e., enforces the holomorphicity property of the map $\phi\colon \Sigma \to X$.

More precisely, integrating out fields $\psi^{(1,0)}, \psi^{(0,1)}$, we obtain a cohomologically smeared delta-form on the space of smooth maps $\Sigma \to X$ supported on holomorphic maps:

(1260)
$$\int \mathcal{D}\psi^{(1,0)}\mathcal{D}\psi^{(0,1)}\, e^{-S'_\epsilon} = \delta^\epsilon_{\mathrm{Hol}(\Sigma,X)} \quad \in \Omega(\mathrm{Map}(\Sigma,X)).$$

In this identification, one identifies the field $\chi^I$ as

(1261)
$$\chi^I = d_{\mathrm{Map}}\phi^I \quad \in T^*_\phi \mathrm{Map}(\Sigma, X)$$

– a 1-form/covector on the mapping space; $d_{\mathrm{Map}}$ stands for de Rham operator on the mapping space. The parameter $\epsilon$ in (1260) serves a "smearing" parameter, with $\epsilon \to 0$ limit being the "true" (non-smeared) distributional delta-form.

Prototype of Mathai-Quillen representative. Given a function $f\colon M \to \mathbb{R}$ (assume that it is smooth, with nonvanishing differential on its zero-locus), one has the following cohomologically smeared delta-form on the hypersurface $f^{-1}(0) \subset M$:

(1262)
$$\delta^\epsilon_{f^{-1}(0)} = (2\pi\epsilon)^{-\frac{1}{2}} e^{-\frac{f(x)^2}{2\epsilon}} df \in \Omega^1_{\mathrm{cl}}(M)$$

with $\epsilon > 0$ a smearing parameter. In the limit $\epsilon \to 0$ this form distributionally converges to true delta-form $\delta_{f^{-1}(0)}$. The form (1262) can be written as a Berezin integral over an auxiliary odd (anticommuting) variable $\psi$:

(1263)
$$\delta^\epsilon_{f^{-1}(0)} = (2\pi\epsilon)^{-\frac{1}{2}} \int D\psi\; e^{-\frac{f(x)^2}{2\epsilon} + \psi df}.$$

More generally, for $f\colon M \to \mathbb{R}^k$ a smooth function with surjective differential on $f^{-1}(0)$, the zero-locus is a submanifold of codimension $k$ and one has the following smeared delta-form on it:

(1264)
$$\delta^\epsilon_{f^{-1}(0)} = (2\pi\epsilon)^{-\frac{k}{2}} \int \prod_{a=1}^k D\psi_a\; e^{-\frac{||f(x)||^2}{2\epsilon} + \psi_a df^a} \in \Omega^k_{\mathrm{cl}}(M),$$

where we introduced $k$ auxiliary odd variables $\psi^a$.

Mathai-Quillen representative of the Euler class of a vector bundle. Let $E \to M$ be a real oriented vector bundle of rank $k$ over a manifold $M$. Assume that $E$ is equipped with fiberwise metric $g$, a connection $\nabla$ compatible with the metric and a section $s\colon M \to E$. Consider the following differential form:

(1265)
$$S_{MQ} = \frac{1}{2\epsilon} g(s,s) + i\langle \psi, \nabla s\rangle - \frac{\epsilon}{2}\langle \psi, F_\nabla(g^{-1}(\psi))\rangle =$$

$$= \frac{1}{2} g_{ab} s^a s^b + i\psi_a (ds^a + A^a{}_b s^b) - \frac{\epsilon}{4} g^{bc} F^a{}_c \psi_a \psi_b \quad \in \Omega^\bullet(M, \wedge^\bullet E).$$

Here we think of odd variables $\psi_a$ as generators of the exterior algebra of the fiber, $\wedge^\bullet E_x$ (put another way $\psi_a$ are coordinates on the parity-reversed dual fiber $\Pi E_x^*$). In the second line we rewrote $S_{MQ}$ explicitly in a local trivialization of $E$; $A^a{}_b$ are the components of the local connection 1-form, $F_\nabla \in \Omega^2(M, \mathrm{End}(E))$ is the curvature 2-form of the connection and $F^a{}_c \in \Omega^2(M)$ are its components. The smearing parameter $\epsilon$ in (1265) corresponds to scaling the fiber metric $g \mapsto \frac{1}{\epsilon} g$.

Even more epxlicitly, using local coordinates $x^i$ on $M$, (1265) can be written as

(1266) $$S_{MQ} = \frac{1}{2} g_{ab} s^a s^b + i\psi_a (\partial_i s^a + A_i{}^a{}_b s^b)\chi^i - \frac{\epsilon}{4} g^{bc} F_{ij}{}^a{}_c \psi_a \psi_b \chi^i \chi^j,$$

where we denoted $\chi^i \colon = dx^i$.

Consider the fiber Berezin integral

(1267) $$\Xi = \left(\frac{i}{\sqrt{2\pi\epsilon}}\right)^k \int_{\text{fiber of } \Pi E^* \to M} D\psi \, e^{-S_{MQ}} \quad \in \Omega^k(M).$$

Here $D\psi \in \Gamma(M, \wedge^k E^*)$ is the fiber Berezinian (fermionic integration measure) induced from the fiber metric and the orientation of the fiber.

**Theorem 9.20** (Mathai-Quillen [30])**.**

- *Form $\Xi$ is closed.*
- *Changing the data $s, g, \nabla, \epsilon$ changes $\Xi$ by an exact form, $\Xi \mapsto \Xi + d(\cdots)$.*
- *The class of $\Xi$ in de Rham cohomology $H^k(M)$ is the Euler class of the bundle $E \to M$.*[173]
- *If the section $s$ intersects the zero-section of $E$ transversally, then one has*

(1268) $$\lim_{\epsilon \to 0} \Xi = \delta_{s^{-1}(0)}$$

*where the limit is understood in distributional sense.*

In particular, the form (1267) is a cohomologically smeared delta-form on the zero-locus of the section $s$; $\Xi$ is known as the Mathai-Quillen representative of the Euler class of the bundle $E \to M$.

Mathai-Quillen construction has a straightforward modification to complex vector bundles equipped with hermitian fiber metric.

A-model as a Mathai-Quillen representative. Consider the vector bundle $\mathcal{E}$ over the space of smooth maps $M = \mathrm{Map}(\Sigma, X)$ where the fiber over the map $\phi$ is

(1269) $$E_\phi = \Omega^{0,1}(\Sigma, \phi^* T^{1,0} X)$$

The bundle $E$ is equipped with:

- A natural section $s = \bar{\boldsymbol{\partial}} \colon M \to E$. Note that the zero-locus of this section is the submanifold of holomorphic maps inside smooth maps, $\mathrm{Hol}(\Sigma, X) \subset \mathrm{Map}(\Sigma, X)$.

[173] Recall that for rank $k$ oriented real vector bundle $E$ over a closed manifold $M$, the Euler class $e$ is the cohomology class of $M$ Poincaré dual to the homology class of the zero-locus of a generic section $s \colon M \to E$ ("generic" here means "transversal to the zero-section"). More precisely (to take signs into account), "zero-locus" should be understood as the intersection of the graph of $s$ with the graph of the zero-section. An equivalent definition: consider the Thom class of $E$ – the cohomology class of the total space $\tau \in H^k(E)$ with the property that its pushforward to $M$ by the bundle projection is the constant function 1. Then the Euler class is the pullback $e = s^*\tau$ of the Thom class by an (arbitrary) section $s : M \to E$ (here one doesn't need a transversality condition).

- A natural fiber hermitian metric given by $\langle \xi, \rho \rangle = \int_\Sigma g(\xi \overset{\wedge}{,} \bar{\rho})$ for $\xi, \rho \in E_\phi$, with $g$ the metric on the target.
- A connection compatible with fiber metric, induced from Levi-Civita connection on $X$.

Comparing (1267) and the l.h.s. of (1260), we observe that the integral over the field $\psi$ in the A-model can be formally identified with the Mathai-Quillen representative of the Euler class of the vector bundle (1269) over the space of smooth maps, or, put differently, with the cohomologically smeared delta-form on the cycle of holomorphic maps inside smooth maps.

9.3.3. *Evaluation observables.* Consider the evaluation map

$$\mathrm{ev}\colon \Sigma \times \mathrm{Map}(\Sigma, X) \to X \tag{1270}$$

Given a differential form $\alpha$ on $X$

$$\alpha = \alpha_{I_1 \cdots I_p}(x) dx^{I_1} \cdots dx^{I_p} \in \Omega^p(X), \tag{1271}$$

one defines the corresponding *evalutation observable*[174] $\widetilde{O}_\alpha(z)$ at a point $z \in \Sigma$ as

$$\begin{aligned} \widetilde{O}_\alpha(z)\colon = \mathrm{ev}^*\alpha|_z &= \alpha_{I_1\cdots I_p}(\phi)(\chi^{I_1} + d\phi^{I_1}) \cdots (\chi^{I_p} + d\phi^{I_p})\big|_z \\ &\in \Omega^\bullet(\mathrm{Map}(\Sigma, X)) \otimes \wedge^\bullet T^*_z\Sigma \quad \subset C^\infty(\mathrm{Fields}_\Sigma) \otimes \wedge^\bullet T^*_z \Sigma \end{aligned} \tag{1272}$$

Thus, $\widetilde{O}_\alpha$ is a form on $\Sigma$ depending on field configuration, or more specifically on the fields $\phi$, $\chi = d_{\mathrm{Map}}\chi$ and first derivatives of $\phi$ at the point $z$. Evaluation observable can be split according to the de Rham degree on $\Sigma$,

$$\widetilde{O}_\alpha = O^{(0)}_\alpha + O^{(1)}_\alpha + O^{(2)}_\alpha. \tag{1273}$$

The following is checked by a direct computation.

**Lemma 9.21.** *Evaluation observables satisfy the following properties:*

$$(d+Q)\widetilde{O}_\alpha = \widetilde{O}_{d_X\alpha}, \tag{1274}$$

$$QO^{(0)}_\alpha = O^{(0)}_{d_X\alpha}, \tag{1275}$$

$$\widetilde{O}_\alpha = e^\Gamma O^{(0)}_\alpha, \tag{1276}$$

*where $d, d_X$ are the de Rham differentials on the source and the target, respectively; $\Gamma$ is the descent operator (938).*

In particular, if $\alpha$ is a *closed* form on $X$, then $O^{(0)}_\alpha$ is $Q$-closed and $\widetilde{O}_\alpha$ is $(d+Q)$-closed and is the total descendant of $O^{(0)}_\alpha$, cf. (945).

*Remark* 9.22. It is natural to identify the total de Rham differential on $\Sigma \times \mathrm{Map}(\Sigma, X)$ with $d+Q$, rather than $d-Q$. Thus, in this section we are using a different sign convention than in Section 6.6 for the descent equations (928), (946): $(d+Q)\widetilde{O} = 0$, or $dO^{(k-1)} = -QO^{(k)}$.

[174]Here we think of $\tilde{O}_\alpha$ as an observable in the sense of classical field theory, which can then be put into the path integral. Tilde in the notation refers to the fact that it is a nonhomogeneous form on $\Sigma$ which we will in a moment identify as a total descendant (945), for $\alpha$ closed.

Gromov-Witten classes as correlators of evaluation observables. Consider for simplicity the case $\Sigma = \mathbb{CP}^1$. Given a collection of closed forms on the target, $\alpha_1, \ldots, \alpha_n \in \Omega_{\mathrm{cl}}(X)$, the correlator of the corresponding evaluation observables in the path integral formalism is

$$(1277)\quad \langle \widetilde{O}_{\alpha_1} \cdots \widetilde{O}_{\alpha_n} \rangle = \int \mathcal{D}\phi \mathcal{D}\chi \int \mathcal{D}\psi \; e^{-S} \widetilde{O}_{\alpha_1} \cdots \widetilde{O}_{\alpha_n} =$$
$$\underset{(1260)}{=} \sum_{d \geq 0} e^{-\frac{d}{2\epsilon}} \int_{\mathrm{Map}_d(\Sigma,X)} \delta^{\epsilon}_{\mathrm{Hol}(\Sigma,X)} \widetilde{O}_{\alpha_1} \cdots \widetilde{O}_{\alpha_n} =$$
$$= \sum_{d \geq 0} q^d \left( \int_{\mathrm{Hol}_d(\Sigma,X)} \pi_1^* \mathrm{ev}^* \alpha_1 \wedge \cdots \wedge \pi_n^* \mathrm{ev}^* \alpha_n + d\Big( \cdots \Big) \right) \quad \in \Omega_{\mathrm{cl}}(\overline{C}_n(\Sigma)).$$

Here in the second line, the prefactor is the exponential of the topological term in the action, $e^{-S_{\mathrm{top}}}$, evaluated on maps of degree $d$ (defined by (1187)); we also identify this prefactor as $q^d$ with

$$(1278)\qquad q \colon = e^{-\frac{1}{2\epsilon}}.$$

In the second step in (1277) we consider the limit $\epsilon \to 0$ in the path integral over $\mathrm{Map}_d(\Sigma, X)$, which localizes the integral to holomorphic maps; however the change of $\epsilon$ induces a shift of the value of the integral by a closed form on the configuration space (since we are looking at a *fiber* integral over $C_n(\Sigma) \times \mathrm{Map}(\Sigma, X) \to C_n(\Sigma)$ of a closed form changed by an exact form – such a change induces an exact change of the fiber integral). The cohomology class of the correlator (1277) is the genus zero Gromov-Witten class (1199).

9.3.4. *A-model in first-order formalism.* The first-order action for the A-model is

$$(1279)\quad S^{\text{first-order}} = S_{\mathrm{top}}(\phi) +$$
$$+ \int_\Sigma -p_i^{(1,0)} \bar{\boldsymbol{\partial}} \phi^i + p_{\bar{i}}^{(0,1)} \boldsymbol{\partial} \phi^{\bar{i}} + i g^{i\bar{j}} p_i^{(1,0)} p_{\bar{j}}^{(0,1)} + \psi_i^{(1,0)} \overline{\mathbf{D}} \chi^i - \psi_{\bar{i}}^{(0,1)} \mathbf{D} \chi^{\bar{i}} + i R^{i\bar{i}}{}_{j\bar{j}} \psi_i^{(1,0)} \psi_{\bar{i}}^{(0,1)} \chi^j \chi^{\bar{j}}$$

with $S_{\mathrm{top}}(\phi)$ the topological term as in (1249). Here the fields are as in (1244), plus two new "momentum" fields (even, of ghost number 0):

$$(1280)\qquad p^{(0,1)} \in \Omega^{0,1}(\Sigma, \psi^*(T^{1,0})^* X), \quad p^{(1,0)} \in \Omega^{1,0}(\Sigma, \psi^*(T^{0,1})^* X)$$

Integrating out the fields, $p^{(1,0)}, p^{(0,1)}$, one obtains back the action (1244):

$$(1281)\qquad \int \mathcal{D}p^{(1,0)} \, \mathcal{D}p^{(0,1)} e^{-S^{\text{first-order}}} = e^{-S}.$$

The odd derivation $Q$ acts on fields of the first-order theory as

(1282)
$$Q\phi^I = \chi^I, \quad Q\chi^I = 0,$$
$$Q\psi_i^{(1,0)} = p_i^{(1,0)} + \Gamma^k_{ij} \chi^j \psi_k^{(1,0)}, \quad Q\psi_{\bar{i}}^{(0,1)} = p_{\bar{i}}^{(0,1)} + \Gamma^{\bar{k}}_{\bar{i}\bar{j}} \chi^{\bar{j}} \psi_{\bar{k}}^{(0,1)},$$
$$Qp_i^{(1,0)} = \Gamma^k_{ij} \chi^j p_k^{(1,0)} - R^j{}_{ik\bar{k}} \psi_j^{(1,0)} \chi^k \chi^{\bar{k}}, \quad Qp_{\bar{i}}^{(0,1)} = \Gamma^{\bar{k}}_{\bar{i}\bar{j}} \chi^{\bar{j}} p_{\bar{k}}^{(0,1)} - R^{\bar{j}}{}_{\bar{i}k\bar{k}} \psi_{\bar{j}}^{(0,1)} \chi^k \chi^{\bar{k}}$$

The operator $Q$ squares to zero

$$(1283)\qquad Q^2 = 0$$

and one has
(1284)
$$S^{\text{first-order}} = S_{\text{top}}+Q\left(\int_\Sigma -\psi_i^{(1,0)}\bar{\boldsymbol{\partial}}\phi^i + \psi_{\bar{i}}^{(0,1)}\boldsymbol{\partial}\phi^{\bar{i}} + \frac{i}{2}g^{i\bar{j}}\psi_i^{(1,0)}p_{\bar{j}}^{(0,1)} - \frac{i}{2}g^{\bar{i}j}\psi_{\bar{i}}^{(0,1)}p_j^{(1,0)}\right).$$

Both equalities (1283), (1284) hold strictly, not just modulo Euler-Lagrange equations.

The counterpart of currents (1254) in the first-order formalism is

$$\mathbf{J} = \chi^i p_i^{(1,0)}, \quad \overline{\mathbf{J}} = \chi^{\bar{i}} p_{\bar{i}}^{(0,1)}, \tag{1285}$$

whereas formulae (1255) for $G, \overline{G}$ do not change.

## References


[1] I. M. Anderson, "Introduction to the variational bicomplex." (1992).

[2] S. Axelrod, S. Della Pietra, E. Witten, "Geometric quantization of Chern-Simons gauge theory," J. Diff. Geom. 33.3 (1991) 787–902.

[3] M. Atiyah, "Topological quantum field theory." Publications Mathématiques de l'IHÉS 68 (1988) 175–186.

[4] M. Atiyah, L. Jeffrey, "Topological Lagrangians and cohomology." Journal of Geometry and Physics 7.1 (1990) 119–136.

[5] S. Barannikov, M. Kontsevich, "Frobenius manifolds and formality of Lie algebras of polyvector fields," Internat. Math. Res. Notes 14(1998), 201–215.

[6] A. A. Belavin, A. M. Polyakov, A. B. Zamolodchikov, "Infinite conformal symmetry in two-dimensional quantum field theory," Nuclear Physics B 241.2 (1984) 333–380.

[7] A. S. Cattaneo, P. Mnev, N. Reshetikhin, "Perturbative quantum gauge theories on manifolds with boundary." Communications in Mathematical Physics 357, no. 2 (2018) 631–730.

[8] P. Di Francesco, P. Mathieu, D. Sénéchal, "Conformal field theory." Springer Science & Business Media, 2012.

[9] B. Dubrovin, "Integrable systems in topological field theory." Nucl. Phys. B379 (1992) 627–689.

[10] B. Dubrovin, "Geometry of 2D topological field theories." In: Springer LNM, 1620 (1996) 120–348.

[11] B. Farb, D. Margalit. "A primer on mapping class groups," Princeton university press (2011).

[12] B. L. Feigin, D. B. Fuchs, "Verma modules over the Virasoro algebra." Topology. Springer, Berlin, Heidelberg (1984) 230–245.

[13] E. Frenkel, A. Losev, "Mirror symmetry in two steps: A–I–B," Communications in mathematical physics 269.1 (2007) 39–86.

[14] K. Gawedzki, "Lectures on conformal field theory." No. IHES-P-97-02. SCAN-9703129, 1997.

[15] K. Gawedzki, A. Kupiainen, "SU(2) Chern-Simons theory at genus zero," Communications in mathematical physics 135.3 (1991) 531–546.

[16] E. Getzler, "Batalin-Vilkovisky algebras and two-dimensional topological field theories," Communications in mathematical physics 159.2 (1994) 265–285

[17] P. Ginsparg, "Applied conformal field theory," arXiv preprint hep-th/9108028 (1988).

[18] R. Iraso, P. Mnev, "Two-dimensional Yang-Mills theory on surfaces with corners in Batalin–Vilkovisky formalism." Communications in Mathematical Physics 370, no. 2 (2019) 637–702.

[19] Th. Johnson-Freyd, "Feynman-diagrammatic description of the asymptotics of the time evolution operator in quantum mechanics." Letters in Mathematical Physics 94, no. 2 (2010) 123–149.

[20] V. G. Kac, "Highest weight representations of infinite dimensional Lie algebras", Proc. Internat. Congress Mathematicians (Helsinki, 1978).

[21] V. G. Kac, A. K. Raina, "Bombay lectures on highest weight representations of infinite dimensional Lie algebras." World scientific, 1987.

[22] S. Kandel, P. Mnev, K. Wernli, "Two-dimensional perturbative scalar QFT and Atiyah-Segal gluing." arXiv preprint arXiv:1912.11202 (2019).

[23] S. Keel, "Intersection theory of moduli space of stable N-pointed curves of genus zero," Trans. AMS 330.2 (1992) 545–574.
[24] V. G. Knizhnik, A. B. Zamolodchikov, "Current Algebra and Wess–Zumino Model in Two-Dimensions," Nucl. Phys. B, 247.1 (1984) 83–103.
[25] T. Kohno, "Conformal field theory and topology." American Mathematical Soc., 2002.
[26] M. Kontsevich, Yu. Manin, "Gromov-Witten classes, quantum cohomology, and enumerative geometry," Commun. Math. Phys. 164 (1994) 525–562.
[27] A. S. Losev, Lectures on topological quantum field theory, 2008 (lectures given online in Russian).
[28] A. S. Losev, P. Mnev, D. R. Youmans, "Two-dimensional abelian BF theory in Lorenz gauge as a twisted N=(2, 2) superconformal field theory," Journal of Geometry and Physics 131 (2018) 122–137.
[29] A. S. Losev, P. Mnev, D. R. Youmans, "Two-dimensional non-abelian BF theory in Lorenz gauge as a solvable logarithmic TCFT," Communications in Mathematical Physics 376.2 (2020) 993–1052.
[30] V. Mathai, D. Quillen, "Superconnections, Thom classes, and equivariant differential forms," Topology, 25.1 (1986) 85–110.
[31] L. Onsager, "Crystal statistics. I. A two-dimensional model with an order-disorder transition," Physical Review, Series II, 65 (3–4) (1944) 117–149.
[32] R. C. Penner, "Decorated Teichmüller theory." Vol. 1. European Mathematical Society, 2012.
[33] N. Reshetikhin, "Lectures on quantization of gauge systems," In: *New Paths Towards Quantum Gravity.* Springer, Berlin, Heidelberg (2010) 125–190.
[34] M. Schottenloher, "A mathematical introduction to conformal field theory." Vol. 759. Springer (2008).
[35] G. Segal, "The definition of conformal field theory," Differential geometrical methods in theoretical physics. Springer, Dordrecht (1988) 165–171.
[36] H. Sugawara, "A field theory of currents," Phys. Rev. 170, 1659 (1968).
[37] E. Verlinde, "Fusion rules and modular transformations in 2D conformal field theory," Nuclear Physics B, 300.3 (1988) 360–376.
[38] E. Witten, "Topological sigma models," Commun. Math. Phys. 118 (1988) 411–449.
[39] E. Witten, "Topological quantum field theory," Comm. Math. Phys. Volume 117, Number 3 (1988) 353–386
[40] E. Witten, "Quantum field theory and the Jones polynomial." Communications in Mathematical Physics 121.3 (1989) 351–399.
[41] E. Witten, "Two-dimensional gravity and intersection theory on moduli space." Surveys in Diff. Geom. 1 (1991) 243–310.
[42] E. Witten, "Mirror manifolds and topological field theory." arXiv:hep-th/9112056 (1991).
[43] E. Witten, "Superstring perturbation theory revisited," arXiv:1209.5461 (2012).