CFT EXERCISES 1, 8/29/2022

1. Quantum mechanics of a free particle on a line and on a circle

Quantum mechanics of a free particle on a line[1] $X = \mathbb{R}$ is defined by the space of states $\mathcal{H} = L^2(X)$ with Hermitian inner product $\langle \psi_1, \psi_2 \rangle = \int_X dx\, \bar{\psi}_1(x)\psi_2(x)$, equipped with quantum Hamiltonian[2]

$$\widehat{H} = -\frac{1}{2}\frac{d^2}{dx^2} \tag{1}$$

(a) Show that the evolution operator in time t (the partition function for the *interval* of length t), defined as $U(t) \colon = e^{-i\widehat{H}t}$, is the integral operator with integral kernel

$$K_{\mathbb{R}}(t; x_1, x_0) = (2\pi i t)^{-\frac{1}{2}} e^{i\frac{(x_1 - x_0)^2}{2t}} \tag{2}$$

Hint: for $\psi_0 \in \mathcal{H}$, $\psi(t) \colon = U(t)(\psi_0)$ has to satisfy the equation $(\partial_t + i\hat{H})\psi = 0$ or, in our case, $(\partial_t - \frac{i}{2}\partial_x^2)\psi = 0$. Now solve this equation with initial condition $\psi|_{t=0} = \delta(x - x_0)$ using Fourier transform in x and evaluate the resulting solution at (t, x_1).

(b) Show by a direct computation that (2) satisfies the convolution property

$$\int_{\mathbb{R}} dx_1\, K(t''; x_2, x_1) K(t'; x_1, x_0) = K(t' + t''; x_2, x_0)$$

– This is the sewing axiom $U(t' + t'') = U(t'') \circ U(t')$ expressed in terms of the integral kernel of the evolution operator.

(c) Consider now the quantum mechanics of a particle moving on a circle $X = \mathbb{R}/L\cdot\mathbb{Z}$ with L the length of the circle. The space of states is now is the Hilbert space of L^2 functions on the circle X, with quantum Hamiltonian (1), where now the coordinate x takes values in the circle $\mathbb{R}/L \cdot \mathbb{Z}$. Show that now the integral kernel of the evolution operator $U(t) = e^{-i\widehat{H}t}$ is

$$K_{S^1}(t; x_1, x_0) = \sum_{n=-\infty}^{\infty} K_{\mathbb{R}}(t; x_1 + nL, x_0) \tag{3}$$

(Here one can understand n as the "winding number.")

(d) Show that the evolution operator (in positive real time t) is unitary, both for a particle on a line and a particle on a circle.

(e) Define the partition function on a source/cobordism circle of length t as $Z(t) = \mathrm{tr}_{\mathcal{H}} U(t)$. Show that for real positive t, $U(t)$ is not trace-class. Next, consider evolution in imaginary time $t = -iT$ where T is a positive real number – the

[1] At the classical level, this system has the space of fields $\mathrm{Map}([0,t], X)$ – the mapping space from the source cobordism $[0,t]$ to the target space X – and action functional $S = \int_0^t d\tau \frac{\dot{x}(\tau)^2}{2}$.

[2] We adopt the units where the mass of the particle $m = 1$ and also $\hbar = 1$.

"Euclidean time." Show that, for the quantum particle moving on a circle, $U(-iT) =: U_{\mathrm{E}}(T)$ is trace-class and one has

$$Z_{\mathrm{E}}(T) \colon = \mathrm{tr}_{\mathcal{H}} U_{\mathrm{E}}(T) = L(2\pi T)^{-\frac{1}{2}} \sum_{n=-\infty}^{\infty} e^{-\frac{L^2}{2T}n^2} \tag{4}$$

(f) Show that the quantum Hamiltonian of the particle on a circle is diagonalized in the basis of eigenvectors $\psi_k(x) = e^{\frac{2\pi i k x}{L}}$, with $k \in \mathbb{Z}$, and the corresponding eigenvalues are $E_k = \frac{1}{2}(\frac{2\pi k}{L})^2$. One can think of k as the momentum of the particle (up to normalization).

(g) Using (f) show that

$$\mathrm{tr}_{\mathcal{H}} U_{\mathrm{E}}(T) = \sum_{k=-\infty}^{\infty} e^{-E_k T} = \sum_{k=-\infty}^{\infty} e^{-\frac{4\pi^2 T}{2L^2}k^2} \tag{5}$$

(h) Prove directly that the right hand sides of (4) and (5) give the same function of T, using Poisson summation formula.[3]

(i) Show that for a particle on a line the evolution operator is not trace-class, even for the evolution in Euclidean time, $t = -iT$ with $T > 0$.

2. "Euler characteristic TQFT"

Fix some $D \geq 1$ and two real numbers α, β such that $\alpha + \beta = 1$. Show that the following assignment satisfies Segal's axioms of quantum field theory (especially, multiplicativity, sewing, normalization, naturality):

- A closed oriented $(D-1)$-manifold Γ is assigned the ground field $\mathcal{H}_\Gamma = \mathbb{C}$.
- A D-cobordism $\gamma_{\mathrm{in}} \xrightarrow{\Sigma} \gamma_{\mathrm{out}}$ is assigned a partition function given by multiplication by the number $e^{\chi(\Sigma) - \alpha\chi(\gamma_{\mathrm{in}}) - \beta\chi(\gamma_{\mathrm{out}})}$ where $\chi(\cdots)$ is the Euler characteristic.
- The action ρ of diffeomorphisms on spaces of states is trivial.

Also: what is the pairing between $\mathcal{H}_\gamma$ and $\mathcal{H}_{-\gamma}$ induced by this theory?

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[3] Recall that Poisson summation formula says that if $f(x)$ is a function on $\mathbb{R}$ decaying sufficiently fast at $x \to \pm\infty$, and if $\widetilde{f}(p) = \int_{\mathbb{R}} f(x) e^{2\pi i p x} dx$ is its Fourier transform, then one has $\sum\limits_{k\in\mathbb{Z}} f(k) = \sum\limits_{n\in\mathbb{Z}} \widetilde{f}(n)$. (One can see this as the equality of distributions $\sum_{k\in\mathbb{Z}} \delta(x-k) = \sum_{n\in\mathbb{Z}} e^{2\pi i n x}$ integrated against a test function f.) In the case at hand, we need to specialize to the case $f(x) = e^{-\frac{Ax^2}{2}}$ where $A = \frac{4\pi^2 T}{L^2}$.