CFT EXERCISES 4, 9/19/2022

(Problems 1–3 are from the previous sheet.)

1. Witt algebra

Show that the generators $l_n = -z^{n+1}\frac{\partial}{\partial z}$ of the Lie algebra W of holomorphic vector fields on $\mathbb{C}^*$ (or equivalently meromorphic vector fields on $\mathbb{CP}^1$ with poles at 0 and ∞ allowed) satisfy the commutation relation

$$[l_n, l_m] = (n-m)l_{n+m}$$

Show that written in the complex coordinate $w = 1/z$ on $\mathbb{CP}^1\{0\}$, l_n takes the form $w^{-n+1}\frac{\partial}{\partial w}$.

Find the pushforward of the vector field l_n under the inversion map $\mathbb{C}^* \to \mathbb{C}^*$, $z \mapsto \frac{1}{\bar{z}}$.

Consider the map $\psi\colon W \to \Gamma(S^1, T\mathbb{C}|_{S^1})$ given by taking the real part of a meromorphic vector field and restricting it to S^1. Show that the preimage by ψ of vector fields on S^1 which are tangent to S^1 is given by elements of W of the form $\sum_n c_n l_n$ with $c_{-n} = -\bar{c}_n$.

2. Conformal symmetry of $\mathbb{R}^{1,1}$

Consider Minkowski plane $\mathbb{R}^{1,1}$ equipped with the metric $g = (dx)^2 - (dy)^2$.

Introduce the "light-cone coordinates" $x^\pm$ on $\mathbb{R}^{1,1}$ given by

$$x^+ = x + y, \quad x^- = x - y.$$

Express the metric g in the light-cone coordinates.

Show that a vector field v on $\mathbb{R}^{1,1}$ is conformal iff it has the form

$$v = v^+(x^+)\partial_+ + v^-(x^-)\partial_-$$

where $v^\pm$ are two smooth functions of a single real variable; $\partial_\pm :=\frac{\partial}{\partial x^\pm} = \frac{1}{2}(\partial_x \pm \partial_y)$. Compute the infinitesimal conformal factor of v.

Show that a diffeomorphism $\phi\colon \mathbb{R}^{1,1} \to \mathbb{R}^{1,1}$ is conformal if and only if it is of the form

$$(x^+, x^-) \mapsto (\phi^+(x^+), \phi^-(x^-)) \quad \text{or} \quad (x^+, x^-) \mapsto (\phi^+(x^-), \phi^-(x^+))$$

when expressed in light-cone coordinates (on both source and target copies of $\mathbb{R}^{1,1}$). Compute the conformal factor in both cases.

3. Liouville theorem, step-by-step

(i) Write the equation $L_\epsilon g = \omega g$ of a conformal vector field $v = v^i\partial_i$ on $\mathbb{R}^{p,q}$ (equipped with the standard metric $g = \eta_{ij}dx^i dx^j$, with $\eta_{ij} = \mathrm{diag}(\underbrace{1,\ldots,1}_p, \underbrace{-1,\ldots,-1}_q)$) in components:[1]

$$\partial_i v_j + \partial_j v_i = \omega \eta_{ij} \tag{1}$$

[1] For simplicity, do this exercise first for the positive signature case, $p = n$, $q = 0$. In particular, then one can forget about the distinction between upper and lower indices.

(ii) Prove:

$$\partial_i v^i = \frac{n}{2}\omega \tag{2}$$

$$\Delta v_i = (1-\frac{n}{2})\partial_i\omega \tag{3}$$

where $n = p+q$ the total dimension and $\Delta = \partial_i\partial^i = \eta^{ij}\partial_i\partial_j$ the Laplacian.

(iii) From (3) obtain:

$$\frac{1}{2}\eta_{ij}\Delta\omega = (1-\frac{n}{2})\partial_i\partial_j\omega \tag{4}$$

$$(n-1)\Delta\omega = 0 \tag{5}$$

(iv) From (4), (5) show that, for $n \notin \{1,2\}$,

$$\partial_i\partial_j\omega = 0 \tag{6}$$

I.e., ω is at most linear in coordinates x^i.

(v) Taking derivatives of (1), show that

$$\partial_i\partial_j v_k = \frac{1}{2}(\partial_i\omega\,\eta_{jk} + \partial_j\omega\,\eta_{ik} - \partial_k\omega\,\eta_{ij}) \tag{7}$$

(vi) From (6), (7) deduce that, for $n \notin \{1,2\}$, we have

$$\partial_i\partial_j\partial_k v_l = 0 \tag{8}$$

I.e., v is at most quadratic in coordinates x^i.

(vii) Assume the most general quadratic ansatz for v and linear ansatz for ω,

$$v_i(x) = a_i + b_{ij}x^j + c_{ijk}x^jx^k \tag{9}$$

$$\omega(x) = 2\mu + 4\nu_i x^i \tag{10}$$

with $a_i, b_{ij}, c_{ijk}, \mu, \nu_i$ some coefficients, and see what constraints does one have on these coefficients from (1).

4. Möbius transformations and the cross-ratio

(a) Prove that the action of $PSL_2(\mathbb{C})$ on $\mathbb{CP}^1$ is 3-transitive: for any two triples of of pairwise distinct points (z_1, z_2, z_3), (z'_1, z'_2, z'_3) in $\mathbb{CP}^1$, one can find an element $\alpha \in PSL_2(\mathbb{C})$ such that

$$z'_i = \alpha \circ z_i, \quad i = 1,2,3 \tag{11}$$

(where α acts by a Möbius transformation).

(b) Prove that $\alpha \in PSL_2(\mathbb{C})$ in (11) is unique (for given triples (z_1, z_2, z_3), (z'_1, z'_2, z'_3)).

(c) Prove that the cross-ratio $[z_1, z_2 : z_3, z_4]$ is a $PSL_2(\mathbb{C})$-invariant function on $C_4(\mathbb{CP}^1)$.

5. Standard presentation of $PSL_2(\mathbb{Z})$

Show that the group of Möbius transformations with integer coefficients $PSL_2(\mathbb{Z})$ admits the following presentation:

$$PSL_2(\mathbb{Z}) = \left\langle S, T \;\middle|\; S^2 = 1, (ST)^3 = 1 \right\rangle \tag{12}$$

where the generators are:

$$T\colon z \mapsto z+1, \qquad S\colon z \mapsto -\frac{1}{z}$$