CFT EXERCISES 6, 10/3/2022

These are the outstanding problems from the previous sheets, plus some new ones.

1. 2d sigma model with target a Lie group

Let G be a matrix Lie group (e.g. you can take $G = GL_n(\mathbb{R})$) with $\mathfrak{g}$ the corresponding Lie algebra and $(,)$ an ad-invariant nondegenerate pairing on $\mathfrak{g}$. Let $\mu = g^{-1}dg \in \Omega^1(G, \mathfrak{g})$ be the Maurer-Cartan 1-form on G (the unique $\mathfrak{g}$-valued left-invariant 1-form on G, restricting to the tautological map $\mu|_1 \colon T_1 G \to \mathfrak{g}$ at $1 \in G$). Consider the 2d sigma model with target G – the classical field theory on a Riemann surface Σ with the space of fields

$$\mathrm{Fields}_\Sigma = \mathrm{Map}(\Sigma, G)$$

and action functional

$$S(g) = \int_\Sigma \underbrace{\frac{1}{2}(g^*\mu \overset{\wedge}{,} *g^*\mu)}_{L}$$

for $g \colon \Sigma \to G$.

(i) Show that it is a conformal (Weyl-invariant) theory.

(ii) By expressing the Lagrangian density L in local complex coordinates, show that one has

$$L = i(g^{-1}\partial g \overset{\wedge}{,} g^{-1}\bar{\partial} g) \tag{1}$$

where $\partial = dz\frac{\partial}{\partial z}$, $\bar{\partial} = d\bar{z}\frac{\partial}{\partial \bar{z}}$ are the holomorphic/antiholomorphic Dolbeault operators, so that in (1) one has a pairing of a form $g^{-1}\partial g \in \Omega^{1,0}(\Sigma, \mathfrak{g})$ and a form $g^{-1}\bar{\partial} g \in \Omega^{0,1}(\Sigma, \mathfrak{g})$ to a form in $\Omega^2(\Sigma)$.

(iii) Find the Euler-Lagrange equations.

2. Legendre transform

Given a convex smooth function f on a vector space V, its Legendre transform is the function $g \in C^\infty(V^*)$ defined as follows:

$$g(p) = \langle p, v \rangle - f(v)$$

for any $p \in V^*$, where $v = v(p) \in V$ is found from

$$p = df(v).$$

(a) Prove that $f \in C^\infty(V)$ and $g \in C^\infty(V^*)$ are linked by the Legendre transform if and only if one has

$$\mathrm{graph}(df) = \mathrm{graph}(dg) \subset V \times V^*$$

– i.e., f and g are the generating functions for the same exact Lagrangian submanifold in $V \times V^*$, seen respectively as either T^*V or as T^*V^*.

(b) Prove that the Legendre transform is involutive: if g is the Legendre transform of f, then f is the Legendre transform of g.

(c) Compute the Legendre transform of the general quadratic polynomial

$$f(v) = av^2 + bv + c$$

with $a, b, c \in \mathbb{R}$, $a \neq 0$.

3. Example of a Noether current for a mixed source-target symmetry

Consider the free massless scalar field on Euclidean $\mathbb{R}^n$, defined by the action

$$S[\phi] = \int \frac{1}{2} d\phi \wedge * d\phi = \int \frac{1}{2} \partial_\mu \phi \, \partial_\mu \phi \; d^n x$$

And consider the mixed source/target transformation – dilation on $\mathbb{R}^n$ accompanied by a rescaling of the field value

$$x \mapsto x' = \beta x \quad , \quad \phi(x) \mapsto \phi'(x') = \beta^{-\frac{n}{2}+1} \phi(x)$$

with $\beta > 0$ the scaling parameter. Or, equivalently,

$$\phi(x) \mapsto \phi'(x) = \beta^{-\frac{n}{2}+1} \phi(\beta^{-1} x)$$

Show that this is a symmetry (maps solutions of the EL equation to solutions). Show that the corresponding infinitesimal symmetry changes the Lagrangian density by a term of form $d\Lambda$ – and find Λ. Finally, find the Noether current corresponding to the symmetry.

4. Noether current for source symmetries in abelian Chern-Simons theory

Consider abelian Chern-Simons on an oriented 3-manifold M, with fields $A \in \Omega^1(M)$ and action

$$S = \int_M \frac{1}{2} A \wedge dA$$

(a) Show that any vector field r on M is a source symmetry.
(b) Find the conserved current $J_r \in \Omega^{2,0}_{\mathrm{loc}}(M \times \mathrm{Fields}_M)$ associated with r by Noether theorem.
(c) Compare J_r with Show that $J'_r = \iota_{\langle T_{\mathrm{Hilb}}, r\rangle} d\mathrm{vol}_g$ – the conserved current arising from contracting the source symmetry with the Hilbert stress-energy tensor – vanishes.
(d) Are the currents J_r and J'_r equivalent? I.e., do they satisfy

$$J_r - J'_r \underset{EL}{\sim} d(\cdots)$$

5. Standard presentation of $PSL_2(\mathbb{Z})$

Show that the group of Möbius transformations with integer coefficients $PSL_2(\mathbb{Z})$ admits the following presentation:

$$PSL_2(\mathbb{Z}) = \left\langle S, T \;\middle|\; S^2 = 1, (ST)^3 = 1 \right\rangle \tag{2}$$

where the generators are:

$$T : z \mapsto z + 1, \qquad S : z \mapsto -\frac{1}{z}$$

6. Partition function of the harmonic oscillator

Find explicitly the partition function of the (quantum) harmonic oscillator[1] with frequency ω on a circle of length t,

$$Z(t) :=\mathrm{tr}_{\mathcal{H}} e^{-\frac{i}{\hbar}\widehat{H}t}$$

What does this expression look like after Wick rotation, i.e., setting $t = -iT$, with $T > 0$ the "Euclidean time."

[1]Hint: we know the eigenvalue spectrum of the Hamiltonian explicitly.