CFT EXERCISES 7, 10/31/2022

1. Vertex operators in scalar field theory

Let $\widehat{V}_\alpha(z) =: e^{i\alpha\widehat{\phi}(z)}$: be the vertex operator in the free scalar field theory with $\alpha \in \mathbb{R}$ a parameter ("charge") and $z \in \mathbb{C}\backslash\{0\}$ a point.

1.1. Prove that one has OPEs

$$\mathcal{R}\widehat{T}(w)\widehat{V}_\alpha(z) \sim \frac{\frac{\alpha^2}{2}\widehat{V}(z)}{(w-z)^2} + \frac{\partial\widehat{V}_\alpha(z)}{w-z} + \text{reg.}, \tag{1}$$

(Hint: use Wick's lemma and explicit presentation of $\widehat{T}$ and $\widehat{V}_\alpha$ in terms of the field operator $\widehat{\phi}$.)

Jointly with the similar OPE $\widehat{\bar{T}}(w)\widehat{V}_\alpha(z)$, (1) implies that V_α is a primary field of conformal weight $(h = \frac{\alpha^2}{2}, \bar{h} = \frac{\alpha^2}{2})$.

1.2. Show that one has

$$\mathcal{R}\widehat{V}_\alpha(w)\widehat{V}_\beta(z) = |w-z|^{2\alpha\beta} : \widehat{V}_\alpha(w)\widehat{V}_\beta(z) : \tag{2}$$

Hint: use explicit expansion of vertex operators in terms of creation/annihilation operators and use the fact that for A, B two operators such that $[A, B]$ commutes with both A and B, one has (from Baker-Campbell-Hausdorff formula) $e^A e^B = e^B e^A e^{[A,B]}$.

From (2) show that for the 2-point correlation function of vertex operators one has

$$\langle V_\alpha(w)V_\beta(w)\rangle = \begin{cases} |w-z|^{-2\alpha^2}, & \text{if } \beta = -\alpha, \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

2. Correlators of primary fields: constraints from global conformal invariance

Let $\Phi_1, \ldots, \Phi_n \in V$ be primary fields in some CFT with conformal weights $(h_i, \bar{h}_i)$. We know from global conformal invariance (Ward identity) that for a Möbius transformation $f \colon \mathbb{CP}^1 \to \mathbb{CP}^1$ one has

$$\langle \Phi_1(z_1)\cdots\Phi_n(z_n)\rangle = \prod_{i=1}^n (\partial f(z_i))^{h_i} (\overline{\partial f(z_i)})^{\bar{h}_i} \langle \Phi_1(f(z_1)), \cdots, \Phi_n(f(z_n))\rangle \tag{4}$$

2.1. n=1. Prove that the 1-point correlation function $\langle \Phi_1(z_1)\rangle$

(i) vanishes unless $h = \bar{h} = 0$,
(ii) is a constant function of z_1 if $h = \bar{h} = 0$

Hint: for (i) consider (4) with f rotations and scalings centered at z_1. For (ii), consider (4) for f a translation.

2.2. n=2. Prove that the 2-point correlation function $\langle\Phi_1(z_1)\Phi_2(z_2)\rangle$

(i) vanishes unless $h_1 = h_2$ and $\bar{h}_1 = \bar{h}_2$,

(ii) is equal to $\frac{C}{(z_1-z_2)^{2h_1}(\bar{z}_1-\bar{z}_2)^{2\bar{h}_1}}$ with C a constant, if $h_1 = h_2$ and $\bar{h}_1 = \bar{h}_2$.

2.3. n=3. Prove that the 3-point correlation function has the form

$$\langle\Phi_1(z_1)\Phi_2(z_2)\Phi_3(z_3)\rangle = C \prod_{1\leq i<j\leq 3} (z_i - z_j)^{-\alpha_{ij}} (\bar{z}_i - \bar{z}_j)^{-\bar{\alpha}_{ij}} \tag{5}$$

with C a constant and α_{ij}, $\bar{\alpha}_{ij}$ some exponents. Find these exponents in terms of conformal weights $h_i, \bar{h}_i$.

3. Szegö kernel

Consider the following holomorphic 2-form on the open configuration of two points on $\mathbb{CP}^1$:[1]

$$\mu = \frac{dz_1 \wedge dz_2}{(z_1 - z_2)^2} \quad \in \Omega^2(C_2(\mathbb{CP}^1)) \tag{6}$$

Prove that it is invariant under Möbius transformations, i.e.,

$$f^*\mu = \mu \tag{7}$$

for f a Möbius transformation (acting diagonally on $C_2(\mathbb{CP}^1)$, $(z_1, z_2) \mapsto (f(z_1), f(z_2))$).

4. Correlators of descendants

Let $\Phi_1, \ldots, \Phi_n$ be primary fields of conformal weights $(h_i, \bar{h}_i)$. Fix $k \geq 1$. Prove that Ward identity implies

$$\langle(L_{-k}\Phi_1)(z_1)\Phi_2(z_2)\cdots\Phi_n(z_n)\rangle = \mathcal{D}\langle\Phi_1(z_1)\cdots\Phi_n(z_n)\rangle \tag{8}$$

with $\mathcal{D}$ some differential operators in variables $z_1, \ldots, z_n$. Find the differential operator $\mathcal{D}$ explicitly.

[1] Usually the *square root* of expression (6), $\frac{d^{\frac{1}{2}}z_1 d^{\frac{1}{2}}z_2}{z_1 - z_2} \in \Gamma(C_2(\mathbb{CP}^1), K^{\otimes\frac{1}{2}} \boxtimes K^{\otimes\frac{1}{2}})$ is called the Szegö kernel.