CFT EXERCISES 8, 11/7/2022

1. Schwarzian derivative

Recall that for a holomorphic map $z \mapsto w(z)$ one defines the Schwarzian derivative as $S(w,z) := \frac{\partial_z^3 w}{\partial_z w} - \frac{3}{2}\left(\frac{\partial_z^2 w}{\partial_z w}\right)^2$. Compute the Schwarzian derivative for $z \mapsto w = \log z$ and for the inverse map, $w \mapsto z = \exp w$.

2. Free boson with values in S^1: partition function on a torus

Recall that for the free boson with values in a circle of radius r the space of states is

$$\mathcal{H} = \mathrm{Span}\{\hat{a}_{-k_r}\cdots\hat{a}_{-k_1}\hat{\bar{a}}_{-l_s}\cdots\hat{\bar{a}}_{-l_1}|e,m\rangle\}_{(e,m)\in\mathbb{Z}^2,\ 1\leq k_1\leq\cdots\leq k_r,\ 1\leq l_1\leq\cdots\leq l_s} \tag{1}$$

It carries a representation of the Lie algebra $\mathrm{Heis}\oplus\overline{\mathrm{Heis}} = \mathrm{Span}\{\hat{a}_n, \hat{\bar{a}}_n\}_{n\in\mathbb{Z}}\oplus\mathbb{C}\cdot\mathbf{1}$, with commutation relations $[\hat{a}_n,\hat{a}_m] = n\delta_{n,-m}\mathbf{1}$, $[\hat{\bar{a}}_n,\hat{\bar{a}}_m] = n\delta_{n,-m}\mathbf{1}$, $[\hat{a}_n,\hat{\bar{a}}_m] = 0$. With respect to this representation, vectors $|e,m\rangle$ are the highest vectors:

$$\hat{a}_{>0}|e,m\rangle = \hat{\bar{a}}_{>0}|e,m\rangle = 0, \tag{2}$$
$$\hat{a}_0|e,m\rangle = \underbrace{\left(\frac{e}{r}+\frac{mr}{2}\right)}_{\alpha_{e,m}}|e,m\rangle,\quad \hat{\bar{a}}_0|e,m\rangle = \underbrace{\left(\frac{e}{r}-\frac{mr}{2}\right)}_{\bar{\alpha}_{e,m}}|e,m\rangle$$

Furthermore, $\mathcal{H}$ carries a representation of two copies of Virasoro algebra, with generators $\hat{L}_n =: \frac{1}{2}\sum_{k\in\mathbb{Z}}\hat{a}_k\hat{a}_{n-k}:$, $\hat{\bar{L}}_n =: \frac{1}{2}\sum_{k\in\mathbb{Z}}\hat{\bar{a}}_k\hat{\bar{a}}_{n-k}:$.

(a) Show that $[\hat{L}_0,\hat{a}_{-p}] = p\,\hat{a}_{-p}$ and similarly $[\hat{\bar{L}}_0,\hat{\bar{a}}_{-p}] = p\,\hat{\bar{a}}_{-p}$.

(b) Show that the state $|e,m\rangle$ is an eigenvector of $\hat{L}_0$ with eigenvalue $\frac{1}{2}\alpha_{e,m}^2$ and an eigenvector of $\hat{\bar{L}}_0$ with eigenvalue $\frac{1}{2}\bar{\alpha}_{e,m}^2$.

(c) Use items (a), (b) to show that the state

$$\hat{a}_{-k_r}\cdots\hat{a}_{-k_1}\hat{\bar{a}}_{-l_s}\cdots\hat{\bar{a}}_{-l_1}|e,m\rangle$$

is an eigenvector of $\hat{L}_0$ with eigenvalue $\frac{1}{2}\alpha_{e,m}^2 + k_1+\cdots+k_r$ and an eigenvector of $\hat{\bar{L}}_0$ with eigenvalue $\frac{1}{2}\bar{\alpha}_{e,m}^2 + l_1+\cdots+l_s$.

(d) Partition function of a conformal field theory on the torus $\mathbb{C}/(\mathbb{Z}\oplus\tau\mathbb{Z})$, with $\tau\in\mathbb{C}, \mathrm{Im}\tau>0$ the modular parameter, is given by the formula (take it as a definition)

$$Z(\tau) = \mathrm{tr}_{\mathcal{H}}\left(q^{\hat{L}_0-\frac{c}{24}}\bar{q}^{\hat{\bar{L}}_0-\frac{\bar{c}}{24}}\right) \tag{3}$$

where $q = e^{2\pi i\tau}$, $\bar{q} = e^{-2\pi i\bar{\tau}}$ is its complex conjugate; $c,\bar{c}$ are the left/right central charges (in our case $c=\bar{c}=1$). Show that in our case of the free boson with values in a circle, (3) becomes

$$Z(\tau,r) = \frac{1}{\eta(\tau)\eta(\bar{\tau})}\sum_{(e,m)\in\mathbb{Z}^2} q^{\frac{1}{2}\alpha_{e,m}^2}\bar{q}^{\frac{1}{2}\bar{\alpha}_{e,m}^2} \tag{4}$$

We made the dependence on the radius of the target circle explicit in the notation. Here

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \tag{5}$$

is the Dedekind eta function.

(e) Show that the partition function (4) satisfies the relation

$$Z(\tau, r) = Z(\tau, \frac{2}{r}) \tag{6}$$

– so-called "T-duality" (in string theory jargon).

3. Poisson summation formula and modular invariance of the torus partition function

Poisson summation formula says that if $f(x)$ is a Schwartz class function on the real line and $\tilde{f}(p) = \int_{-\infty}^{\infty} dx\ e^{2\pi i p x} f(x)$ its Fourier transform, then one has

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{p \in \mathbb{Z}} \tilde{f}(p) \tag{7}$$

(a) [1] Use (7) and *Euler's identity*

$$\prod_{n=1}^{\infty} (1 - q^n) = \sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{3j^2 - j}{2}} \tag{8}$$

to prove that the Dedekind eta function (5) satisfies

$$\eta(-\frac{1}{\tau}) = (-i\tau)^{1/2} \eta(\tau), \quad \eta(\tau + 1) = e^{\frac{2\pi i}{24}} \eta(\tau) \tag{9}$$

(b) Using Poisson summation of (4) in e and m, and using (9), show that the partition function (4) satisfies the modular invariance property:

$$Z(-\frac{1}{\tau}, r) = Z(\tau, r), \quad Z(\tau + 1, r) = Z(\tau, r) \tag{10}$$

I.e., for a fixed r, $Z(\tau, r)$ defines smooth $PSL(2, \mathbb{Z})$-invariant function on the complex upper half-plane Π_+, or equivalently a smooth function on the moduli space of complex structures on a torus $\mathcal{M}_{1,0} = \Pi_+ / PSL(2, \mathbb{Z})$.

(c) Using Poisson summation of (4) in e only, obtain the following asymptotic formula for the case $r \to \infty$:

$$Z(\tau, r) \underset{r \to \infty}{\sim} r \cdot \frac{1}{\sqrt{\mathrm{Im}\tau}\ \eta(\tau) \eta(\bar{\tau})} \tag{11}$$

[1] This item is a bit lengthy and you can jump right to (b) and (c) taking the result (9) for granted on the first pass.