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Perturbative topological quantum field theory on manifolds with boundary

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Abstract

We develop a perturbative quantization scheme producing, out of a classical topological field theory action functional, a quantum field theory on manifolds with boundary, compatible with gluing/cutting. The construction goes through embedding the classical theory in a cohomological symplectic (Batalin-Vilkovisky) formalism, in its form adapted to manifolds with boundary – the BV-BFV formalism of [2]. In a broad class of AKSZ sigma models, explicit spaces of states and partition functions are obtained in terms of configuration space integrals, see [3]. Our general construction also applies to non-topological (geometry dependent) gauge theories.

Programme in progress includes extension to gluing/cutting with corners (and comparison of the output of quantization with Baez-Dolan-Lurie extended TQFT), inclusion of observables supported on embedded submanifolds, studying the applications to algebraic topology. One of the goals of the project is to obtain a mathematically rigorous proof of the long standing conjecture that the large level asymptotics of Reshetikhin-Turaev invariants of closed 3-manifolds coincides with perturbative partition functions of Chern-Simons theory (as obtained, in the case of rational homology spheres, by Axelrod-Singer).

This is a joint work with Alberto S. Cattaneo and Nicolai Reshetikhin.

Background

Functorial field theory

Atiyah (for topological theories) and Segal (for conformal 2D theories) introduced the axiomatic picture of quantum field theory emphasizing the behavior of partition functions w.r.t. gluing. In Atiyah's language, an n-dimensional topological quantum field theory (TQFT) is a functor $\mathrm{Cob}_n \to \mathrm{Vect}$ from the monoidal category of n-cobordisms to the monoidal category of vector spaces. It associates to a closed $(n-1)$-manifold Σ a "space of states" $\mathcal{H}_\Sigma$ and to a cobordism M a linear map between the spaces of states for in- and out-boundaries, $Z_M \in \mathrm{Hom}(\mathcal{H}_{\partial_{\mathrm{in}} M}, \mathcal{H}_{\partial_{\mathrm{out}} M})$, in such a way that gluing of cobordisms is mapped to composition of linear maps and disjoint unions are sent to tensor products. The formalism admits generalizations by allowing a geometric structure on cobordisms and/or changing the target category. Another extension of Atiyah's axiomatics due to Baez-Dolan-Lurie allows gluing/cutting with corners by replacing the source category by an appropriate higher-categorical extension of Cob_n.

Batalin-Vilkovisky formalism

In BV formalism for gauge theories one extends the space of fields to a graded odd-symplectic supermanifold $\mathcal{F}$ and the action functional on fields to the BV action $S \in C^\infty(\mathcal{F})$ such that the master equation $\{S, S\} = 0$ holds (with $\{,\}$ the odd Poisson bracket). This replacement is a crucial tool in constructing the stationary phase path integral for a gauge theory (as a sum of Feynman diagrams).

Classical BV-BFV formalism

In the modification of BV formalism for source manifolds with boundary developed in [2], a **BV-BFV** theory is the following.

- It associates to a closed $(n-1)$-manifold Σ a *phase space* – a *BFV manifold* $(\Phi_\Sigma, \omega_\Sigma, Q_\Sigma, S_\Sigma)$, where
  - Φ_Σ is a dg manifold with cohomological vector field Q_Σ,
  - $\omega_\Sigma = \delta\alpha_\Sigma$ is a degree 0 exact symplectic structure on Φ_Σ compatible with Q_Σ,
  - The *BFV charge* $S_\Sigma \in C^\infty(\Phi_\Sigma)_1$ is the degree 1 Hamiltonian for the cohomological vector field.
- To an n-manifold M with boundary, it associates a *space of fields* – a *BV-BFV manifold* $(\mathcal{F}_M, \omega_M, Q_M, S_M, \pi)$ where
  - $\mathcal{F}_M$ is a dg manifold with cohomological vector field Q_M,
  - ω_M is a degree -1 odd symplectic form,
  - The *action* S_M is a degree 0 function on $\mathcal{F}_M$,
  - $\pi : \mathcal{F}_M \to \Phi_{\partial M}$ is a surjective submersion,
  - Bulk and boundary data are related by structure relations $\pi_* Q_M = Q_{\partial M}$, $\iota_{Q_M}\omega_M = \delta S_M + \pi^*\alpha_{\partial M}$.

  Structure relations imply in particular a from of the BV master equation $\frac{1}{2}\iota_{Q_M}\iota_{Q_M}\omega_M = \pi^* S_{\partial M}$.
- Disjoint unions of n- or $(n-1)$-manifolds are sent to direct products. Reversing the orientation changes the sign of symplectic structures and actions.
- Gluing of n-manifolds along a closed $(n-1)$-manifold $M_1 \cup_\Sigma M_2$ is sent to a (homotopy) fiber product of spaces of fields over the phase space for the gluing interface, $\mathcal{F}_{M_1} \times_{\Phi_\Sigma} \mathcal{F}_{M_2}$.

This construction extends to higher-codimension strata [2]. All AKSZ theories and almost all gauge theories can be embedded in this formalism. It possesses a reduction, passing to the *moduli spaces of dg manifolds*, where Σ gets assigned a symplectic moduli space $\mathcal{M}_\Sigma$ and M gets assigned a degree 1 Poisson moduli space $\mathcal{M}_M$, such that $\pi_* : \mathcal{M}_M \to \mathcal{M}_{\partial_{\mathrm{in}} M} \times \mathcal{M}_{\partial_{\mathrm{out}} M}$ is a Lagrangian hyper-relation between the in- and out-moduli spaces, and the symplectic foliation of the bulk moduli space is given by fibers of π_*.

Quantum picture

A quantum BV-BFV theory [3] is the following association.

- To a closed $(n-1)$-manifold Σ it assigns a cochain complex $\mathcal{H}_\Sigma$ (the *space of states*) with coboundary operator Ω_Σ (the *quantum BFV charge*).
- To an n-manifold M with boundary it assigns
  - a finite-dimensional odd-symplectic manifold $(\mathcal{F}_{\mathrm{r}}, \omega_{\mathrm{r}})$ – the *space of residual fields*,
  - the *partition function* $Z_M \in \mathcal{H}_\Sigma \otimes \mathrm{Dens}^{\frac{1}{2}}(\mathcal{F}_{\mathrm{r}})$ satisfying the *modified quantum master equation*
  $$(i/\hbar\, \Omega_{\partial M} - i\hbar\Delta_{\mathrm{r}})\, Z_M = 0$$
  where Δ_{r} is the BV Laplacian on half-densities on residual fields. Moreover, Z_M is defined modulo $\left(\frac{i}{\hbar}\Omega_{\partial M} - i\hbar\Delta_{\mathrm{r}}\right)$-exact terms, due to ambiguity of gauge-fixing choice involved in quantization.
- Disjoint unions are mapped to tensor products for $\mathcal{H}$ and Z and to direct products for $\mathcal{F}_{\mathrm{r}}$.
- For a glued manifold $M = M_1 \cup_\Sigma M_2$, the partition function is given by the *gluing formula* $Z_M = P_*(Z_{M_1} *_\Sigma Z_{M_2})$ where $*_\Sigma$ is the pairing of states in $\mathcal{H}_\Sigma$ and P_* is the *BV pushforward* of half-densities along the odd-symplectic fibration $\mathcal{F}_{\mathrm{r}}^{M_1} \times \mathcal{F}_{\mathrm{r}}^{M_2} \to \mathcal{F}_{\mathrm{r}}^{M}$.

Admissible spaces of residual fields form a poset. Passing to a smaller model for residual fields corresponds to evaluating the BV pushforward of Z_M w.r.t. the odd-symplectic fibration $\mathcal{F}_{\mathrm{r}} \to \mathcal{F}_{\mathrm{r}'}$. This realizes a version of Wilson's renormalization flow in this picture.

Quantum example 1: abelian BF theory

Fix an n-cobordism M.

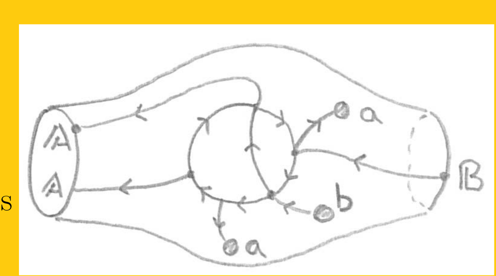
- States on the boundary of M are half-densities on $\mathcal{B} = \Omega^\bullet(\partial_{\mathrm{in}} M)[1] \oplus \Omega^\bullet(\partial_{\mathrm{out}} M)[n-2] \quad \ni (\mathbb{A}, \mathbb{B})$. The coboundary operator on states is the lifting of de Rham operator on ∂M to half-densities.
- Residual fields are constructed in terms of cohomology relative to in/out-boundary, $\mathcal{F}_{\mathrm{r}} = H^\bullet(M, \partial_{\mathrm{in}} M)[1] \oplus H^\bullet(M, \partial_{\mathrm{out}} M)[n-2] \quad \ni (\mathbf{a}, \mathbf{b})$.
- Partition function:
  $$Z_M = \xi \cdot \tau(M, \partial_{\mathrm{in}} M) \cdot \exp \frac{i}{\hbar}\left(\int_{\partial_{\mathrm{in}} M} \mathbf{b}\mathbb{A} + \int_{\partial_{\mathrm{out}} M} \mathbb{B}\mathbf{a} - \int_{\partial_{\mathrm{out}} M \times \partial_{\mathrm{in}} M \ni (x,y)} \mathbb{B}(x)\eta(x,y)\mathbb{A}(y) \right)$$
  Here $\xi \in \mathbb{C}$ is factor depending on relative Betti numbers, containing a mod 16 phase, τ is the relative R-torsion, $\eta \in \Omega^{n-1}(\mathrm{Conf}_2(M))$ is the parametrix for the chain homotopy, contracting differential forms vanishing on ∂_{in} onto relative cohomology (the *propagator*).

This model, and its non-abelian generalization, can be realized on a cobordism endowed with a CW decomposition, with perturbative path integrals replaced by finite-dimensional measure-theoretic integrals [4, 1].

Quantum example 2: Poisson sigma model

Fix M a (2-dimensional) surface with boundary and a Poisson bivector π on $\mathbb{R}^m$. Space of states (but not the coboundary operator Ω) and residual fields are as for m copies of abelian BF theory above.

$$Z_M = \xi^m \cdot \tau(M, \partial_{\mathrm{in}} M)^m \cdot \exp \frac{i}{\hbar} \sum_{\mathrm{graphs}}$$



Here we sum over connected graphs where every internal vertex has two outgoing and arbitrarily many incoming edges; 1-valent vertices on boundaries and leaves (loose half-edges) in the bulk are allowed. The contribution of a graph is given by an integral over the configuration space of points on the surface and is a polynomial in boundary and residual fields. The form integrated is given by a product of propagators, as prescribed by edges of the graph, and of higher derivatives of the Poisson bivector for internal vertices.

The coboundary operator Ω on "nice" states (those not containing products of fields at same point) is given by the standard-ordering quantization ($\mathbb{B} \mapsto -i\hbar\frac{\delta}{\delta\mathbb{A}}$ on ∂_{in}, $\mathbb{A} \mapsto -i\hbar\frac{\delta}{\delta\mathbb{B}}$ on ∂_{out}) of $\int_\partial \mathbb{B}^i d\mathbb{A}_i + \frac{1}{2}\Pi^{ij}(\mathbb{B})\mathbb{A}_i\mathbb{A}_j$, where $\Pi^{ij} = \frac{x^i * x^j - x^j * x^i}{i\hbar}$ is the deformation of π by the associated Kontsevich's star-product.

Theorem: *This data satisfies the axioms of the quantum picture above (the master equation and the gluing rule).*

Research in progress

- Extend the construction to allow gluing/cutting with corners, compare the output we get with Baez-Dolan-Lurie extended TQFT framework. Calculate partition function for the fundamental building block (e.g. a ball with stratified boundary) in interesting examples.
- Incorporate observables into the picture, apply to the study of cohomology of spaces of embeddings.
- Apply our machine to Chern-Simons theory. In particular, prove the coincidence of large-level asymptotics of Reshetikhin-Turaev invariants with perturbative Chern-Simons invariants.
- Calculate the cohomology of Ω in the Poisson sigma model and compare with the geometric quantization of the symplectic groupoid.

References

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