17/01/2018
5

19/01/2018

Step III Use row replacement to create zeros in all positions below the pivot

$$\xrightarrow{r_2 \mapsto r_2 - r_1} \begin{bmatrix} 2 & 4 & 0 & 6 & 0 \\ 0 & -3 & 9 & 3 & 6 \\ 0 & 2 & -6 & -1 & -2 \end{bmatrix}$$

Step IV Cover (or ignore) the row containing the pivot position and all rows above it. Apply Steps I-III to the remaining submatrix. Repeat until there are no nonzero rows to modify.

$$\begin{bmatrix} \cancel{2} & \cancel{4} & \cancel{0} & \cancel{6} & \cancel{0} \\ 0 & -3 & 9 & 3 & 6 \\ 0 & 2 & -6 & -1 & -2 \end{bmatrix} \xrightarrow{r_3 \mapsto r_3 + \frac{2}{3} r_2} \begin{bmatrix} \cancel{2} & \cancel{4} & \cancel{0} & \cancel{6} & \cancel{0} \\ 0 & -3 & 9 & 3 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow[r_2 \mapsto -\frac{1}{3} r_2]{\text{(optional)}} \begin{bmatrix} \cancel{2} & \cancel{4} & \cancel{0} & \cancel{6} & \cancel{0} \\ \cancel{0} & \cancel{1} & \cancel{-3} & \cancel{-1} & \cancel{-2} \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

new pivot

REF!

new pivot

already in REF ⇒ IV stops.

If we want RREF:

Step V • beginning with rightmost pivot and working upward and to the left, create zeros above each pivot. If pivot is not 1, make it 1 by rescaling rows.

$$\begin{bmatrix} 2 & 4 & 0 & 6 & 0 \\ 0 & 1 & -3 & -1 & -2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow[r_2 \mapsto r_2 + r_3]{r_1 \mapsto r_1 - 6 r_3} \begin{bmatrix} 2 & 4 & 0 & 0 & -12 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{r_1 \mapsto r_1 - 4 r_2} \begin{bmatrix} 2 & 0 & 12 & 0 & -12 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \longrightarrow$$

created zeros above pivots

$$\xrightarrow[r_1 \mapsto r_1 \cdot \frac{1}{2}]{\text{rescale}} \begin{bmatrix} 1 & 0 & 6 & 0 & -6 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

← RREF of orig. matrix

Solutions of lin. sys.

Suppose augm. mat. of a sys. has been reduced into RREF

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ \end{array}$$
$$\begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

i.e. system

$$x_1 + 3x_3 = -1$$
$$x_2 + 2x_3 = 5$$
$$0 = 0$$

variables x_1, x_2 corresp. to pivot columns are basic variables, var. x_3 corresp. to non-pivot col. is a "free variable"

can solve for basic var. in terms of free variables:

$$\begin{cases} x_1 = -1 - 3x_3 \\ x_2 = 5 - 2x_3 \\ x_3 \text{ is free (takes any value)} \end{cases}$$

- description of all solutions of the lin. sys.

e.g.: can take $x_3 = 1$
⇒ $(-1-3, 5-2, 1)$ is a solution.

- A system is consistent iff REF of Aug. Mat. does not have a row of form $[0 \cdots 0 \; b]$, $b \neq 0$ ⇔ $0 = b$ contradictory eq.
  i.e. iff the last column is not pivotal.
- solution of a consistent sys. is unique iff there are no free variables, i.e. no non-pivot columns (except last one)

01/19/2018
2

1.3. Vectors equations

- vector = ordered list of numbers.
- column vector = matrix with only one column

vectors in $\mathbb{R}^2$

Ex: $\begin{bmatrix} 1.3 \\ -2 \end{bmatrix}$, $\begin{bmatrix} 5 \\ 0 \end{bmatrix}$ set of vectors with two (real) entries $=: \mathbb{R}^2$

- vectors u, v in $\mathbb{R}^2$ are equal if their ~~respective~~ corresponding entries are equal. E.g. $\begin{bmatrix} 2 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 3 \\ 2 \end{bmatrix}$
- for $u, v \in \mathbb{R}^2$, we can form the sum $u+v$ by adding corresp. entries of u, v

e.g. $\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 2+(-1) \\ 3+5 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$

- for $u \in \mathbb{R}^2$, c a number, the scalar multiple cu is obtained by multiplying each entry of u by c

$3\begin{bmatrix} 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 3\cdot 2 \\ 3\cdot 7 \end{bmatrix} = \begin{bmatrix} 6 \\ 21 \end{bmatrix}$

Ex for $u = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $v = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, find $5u$, $(-2)v$ and $5u+(-2)v$

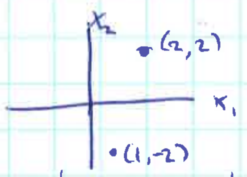
Sol: $5u = \begin{bmatrix} 10 \\ -5 \end{bmatrix}$, $(-2)v = \begin{bmatrix} -6 \\ -4 \end{bmatrix}$, $5u+(-2)v = \begin{bmatrix} 10-6 \\ -5-4 \end{bmatrix} = \begin{bmatrix} 4 \\ -9 \end{bmatrix}$

Notation can write ~~a vec~~ $\underset{\substack{2\times 1 \\ \text{matrix}}}{\begin{bmatrix} 2 \\ 7 \end{bmatrix}} = (2,7) \neq \underset{\substack{1\times 2 \\ \text{matrix}}}{[2\ 7]}$

comma! shorthand for a col. vector

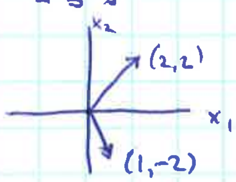
Geometric description of vectors in $\mathbb{R}^2$

point (a,b) on coord plane $\sim \begin{bmatrix} a \\ b \end{bmatrix}$ So $\mathbb{R}^2$ = set of all points on the plane

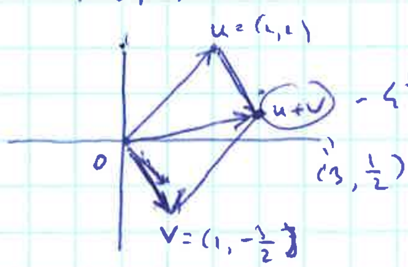


vectors as points

or vectors as arrows



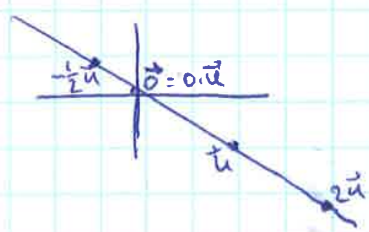
Sum (parallelogram rule) for addition



- 4th vertex of parallelogram with vertices at 0, u and v.

multiples

Ex: $\vec{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Display $\vec{u}$, $2\vec{u}$, $-\frac{1}{2}\vec{u}$



01/19/2018
3

Vectors in $\mathbb{R}^n$

$\mathbb{R}^n$ = set of vectors of form $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ ($n \times 1$ - matrices)

- can add two vectors in $\mathbb{R}^n$ $\vec{u} + \vec{v}$ (vectors must be of same size!!)
- can form a scalar multiple $c \cdot \vec{u}$

* one has commutativity, associativity, distributivity, like for numbers

- zero vector $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\vec{0} + \vec{u} = \vec{u}$
- negative of a vector: $-\vec{u} := (-1)\vec{u} = \begin{bmatrix} -u_1 \\ -u_2 \\ \vdots \\ -u_n \end{bmatrix}$ then $(-\vec{u}) + \vec{u} = \vec{0}$
- also, notation: $\vec{u} + (-1)\vec{v} =: \vec{u} - \vec{v}$.

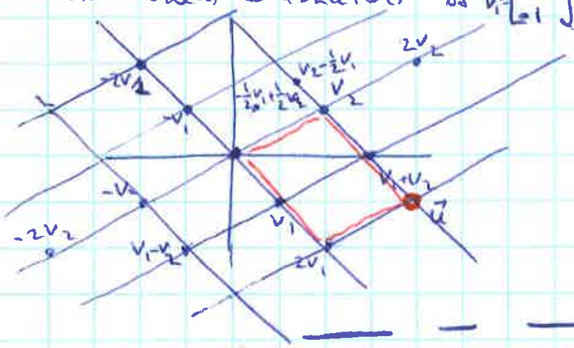
Linear combinations

For vectors $v_1, \dots, v_p \in \mathbb{R}^n$, scalars $c_1, \dots, c_p$, vector

$\boxed{\vec{y} = c_1 \vec{v}_1 + \dots + c_p \vec{v}_p}$ is called a linear combination of $\vec{v}_1, \dots, \vec{v}_p$ with weights $c_1, \dots, c_p$

Ex: some linear comb of v_1, v_2: $3v_1 - \frac{5}{7} v_2$, $\frac{1}{2} v_1 = \frac{1}{2} v_1 + 0 \cdot v_2$, $\vec{0} = 0 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2$

Ex: some linear combinations of $v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$:



Q: Express $\vec{u}$ as a lin. comb. of $\vec{v}_1, \vec{v}_2$.

Sol: by parallelogram rule, $\vec{u} = 2\vec{v}_1 + \vec{v}_2$.

01/24/2018
0

Ex: $\vec{a}_1 = \begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix}$ $\vec{a}_2 = \begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} -1 \\ -1 \\ -4 \end{bmatrix}$

Q: can $\vec{b}$ be generated as a lin. comb. of $\vec{a}_1, \vec{a}_2$, i.e., do weights x_1, x_2 exist, s.t. $x_1 \vec{a}_1 + x_2 \vec{a}_2 = \vec{b}$? $(*)$

05, 03

Sol: $(*)$ means $x_1 \underbrace{\begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix}}_{\vec{a}_1} + x_2 \underbrace{\begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix}}_{\vec{a}_2} = \begin{bmatrix} -1 \\ -1 \\ -4 \end{bmatrix}$

$$lhs = \begin{bmatrix} x_1 \\ -3x_1 \\ -x_1 \end{bmatrix} + \begin{bmatrix} 3x_2 \\ -5x_2 \\ 2x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 3x_2 \\ -3x_1 - 5x_2 \\ -x_1 + 2x_2 \end{bmatrix}$$

So, $(*)$ is true iff $\begin{cases} x_1 + 3x_2 = -1 \\ -3x_1 - 5x_2 = -1 \\ -x_1 + 2x_2 = -4 \end{cases}$ - lin. sys.! $(**)$

Aug. Mat.: $\begin{bmatrix} 1 & 3 & -1 \\ -3 & -5 & -1 \\ -1 & 2 & -4 \end{bmatrix} \underset{\text{row equiv}}{\sim} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 4 & -4 \\ 0 & 5 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & -1 \\ 0 & 5 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$ Sol of $(**)$: $x_1 = 2$, $x_2 = -1$