3.3 **Cramer's rule; volume and linear transformations** 02/16/2018



$A$ $n\times n$ matrix, $\vec{b} \in \mathbb{R}^n$. Set $A_i(\vec{b}) = [\vec{a}_1 \cdots \underset{\substack{\uparrow \\ \text{col } i}}{\vec{b}} \cdots \vec{a}_n]$

**THM** (Cramer's rule)

For $A$ invertible $n\times n$, $\vec{b} \in \mathbb{R}^n$, the unique solution $\vec{x}$ of $A\vec{x} = \vec{b}$ has entries

$$\boxed{x_i = \frac{\det A_i(\vec{b})}{\det A}}, \quad i = 1 \dots n$$

Ex: $\begin{aligned} 4x_1 + 5x_2 &= 2 \\ 2x_1 + 3x_2 &= 6 \end{aligned}$ ($\uparrow \vec{b}$) Solve using Cramer's rule: $A = \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}$, det $= 2$; $A_1(\vec{b}) = \begin{bmatrix} 2 & 5 \\ 6 & 3 \end{bmatrix}$, det $= -24$; $A_2(\vec{b}) = \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix}$, det $= 20$

$$x_1 = \frac{-24}{2} = -12$$
$$x_2 = \frac{20}{2} = 10$$

Ex: For which $s$ (parameter), system $\begin{aligned} 3s\,x_1 - 2x_2 &= 1 \\ -6x_1 + s\,x_2 &= 2 \end{aligned}$ (a) has a unique solution? write the solution using Cramer's rule

Sol: $A = \begin{bmatrix} 3s & -2 \\ -6 & s \end{bmatrix}$ $\quad A_1(\vec{b}) = \begin{bmatrix} 1 & -2 \\ 2 & s \end{bmatrix}$ $\quad A_2(\vec{b}) = \begin{bmatrix} 3s & 1 \\ -6 & 2 \end{bmatrix}$

$\det A = 3s^2 + 12 = 3(s-2)(s+2)$ $\quad$ det $= s+4$ $\quad$ det $= 6s + 6 = 6(s+1)$

(a): $\neq 0$ iff $s \neq \pm 2$

(b): $x_1 = \dfrac{s+4}{3(s-2)(s+2)}$

$x_2 = \dfrac{6(s+1)}{3(s-2)(s+2)} = 2\dfrac{s+1}{(s-2)(s+2)}$.

## Formula for $A^{-1}$

For $A$ invertible $n\times n$ matrix, (cofactors)

$$A^{-1} = \frac{1}{\det A} \underbrace{\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}}_{\text{"adjugate" of } A, \ \operatorname{adj} A}$$

or equivalently $(A^{-1})_{ij} = \dfrac{C_{ji}}{\det A}$

Ex: $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -3 \\ 2 & 1 & -6 \end{bmatrix}$ find $(A^{-1})_{12}$

Sol: $\det A = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & -3 \\ 0 & 1 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1$. $\quad C_{21} = -\det A_{21} = -\begin{vmatrix} 1 & 1 \\ 1 & -6 \end{vmatrix} = 7 \quad \Rightarrow (A^{-1})_{12} = \dfrac{C_{21}}{\det A} = 7$

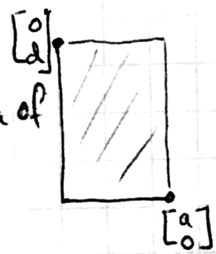
02/16/2018
2

## Determinants as area or volume

**Thm:** (a) If $A = [\vec{a}_1 \vec{a}_2]$ is a $2\times 2$ matrix, the area of the parallelogram determined by $\vec{a}_1, \vec{a}_2$ is $|\det A|$

(b) If $A = [\vec{a}_1 \vec{a}_2 \vec{a}_3]$ is a $3\times 3$ matrix, the volume of the parallelipiped determined by $\vec{a}_1, \vec{a}_2, \vec{a}_3$ is $|\det A|$.

Ex: $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ Area of



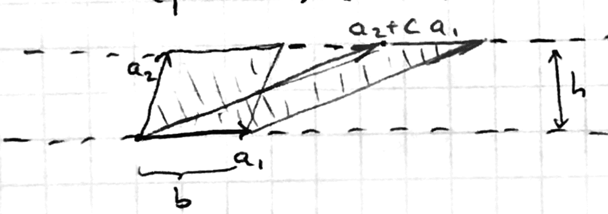

$= |ad| = |\det A|$

**Idea of proof** (of (a)): $A \sim$ diagonal matrix $\begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$

(i) col. replacements, (ii) col. interchanges } don't change neither $|\det A|$, nor Area

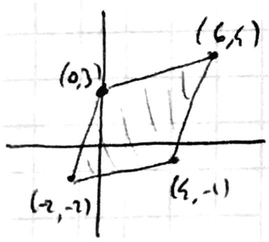
(ii) Area (parall. det. by $\vec{a}_2, \vec{a}_1$) = Area (parall. det. by $\vec{a}_1, \vec{a}_2$)

(i) Area (par. $\vec{a}_1, \vec{a}_2 + c\vec{a}_1$) = Area (par. $\vec{a}_1, \vec{a}_2$)



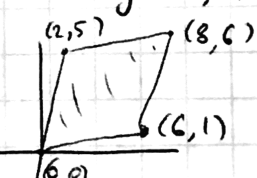

both areas = $(\underbrace{\text{base}}_{b}) \cdot (\text{height } h)$

Ex: find the area of parallelogram with vertices at $(-2,-2)$, $(0,3)$, $(4,-1)$, $(6,4)$




Sol: translate parall. by $(2,2)$, to one having $\vec{0}$ as a vertex.

new parall:




Area $= |\underbrace{\det \begin{bmatrix} 2 & 6 \\ 5 & 1 \end{bmatrix}}_{-28}| = 28$

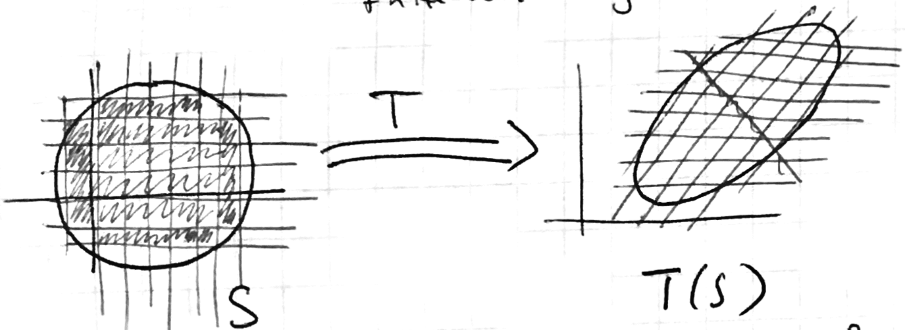
**How areas/volumes are changed by lin. transformations?**

**THM*** (a) Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a lin. transf. determined by a $2\times 2$ matrix $A$. If $S$ is a parallelogram in $\mathbb{R}^2$, then (Area of $T(S)$) $= |\det A| \cdot$ (area of $S$)

(b) If $T: \mathbb{R}^3 \to \mathbb{R}^3$ is determined by a $3\times 3$ matrix $A$ and $S$ a parallelipiped in $\mathbb{R}^3$, then

(Volume of $T(S)$) $= |\det A| \cdot$ (Volume of $S$)

THM* generalizes to finite area regions $S$ of $\mathbb{R}^2$ / finite volume regions of $\mathbb{R}^3$



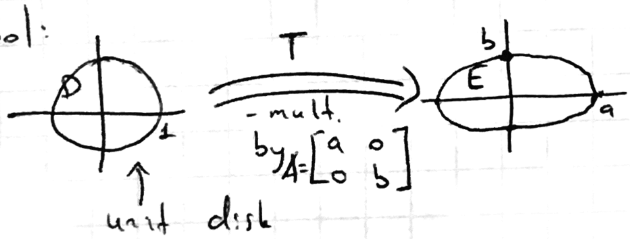

$$\text{Area}(T(S)) = |\det A| \cdot \text{Area}(S)$$

$S$ can be approximated by a union of little squares

$T(S)$ - union of little parallelograms $= T(\text{little squares})$

Ex: let $E$ be the region on $\mathbb{R}^2$ bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. $\text{Area}(E) = ?$

Sol:




indeed: $T\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \begin{array}{l} u_1 = \frac{x_1}{a} \\ u_2 = \frac{x_2}{b} \end{array}$

$\Rightarrow \vec{u}$ is in the unit disk $D$ iff $\vec{x}$ in $E$:

$u_1^2 + u_2^2 \leq 1 \qquad \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} \leq 1$

Thus, $\text{Area}(E) = \underbrace{|\det A|}_{ab} \cdot \underbrace{\text{Area}(D)}_{\pi \cdot 1^2} = \pi ab$