4.1 **Vector spaces and subspaces** 02/19/2018 1

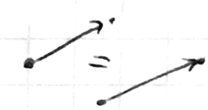
DEF
A **Vector space** - a nonempty set V, objects - "vectors", with two operations
- addition ① $\vec{u}+\vec{v} \in V$ for $\vec{u},\vec{v} \in V$
- ~~scalar~~ multiplication by scalars ⑥ $c\vec{u} \in V$ for $\vec{u} \in V$, $c \in \mathbb{R}$.

such that:
③ $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$, ② $\vec{u}+\vec{v} = \vec{v}+\vec{u}$,
⑦ $c(\vec{u}+\vec{v}) = c\vec{u}+c\vec{v}$, ⑧ $(c+d)\vec{u} = c\vec{u}+d\vec{u}$,
⑨ $c(d\vec{u}) = (cd)\vec{u}$

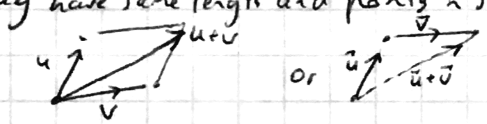
④ $\exists$ zero vector $\vec{0} \in V$ s.t. $\vec{u}+\vec{0} = \vec{u}$ $\forall \vec{u}$.
⑤ $\forall \vec{u}$ $\exists$ negative $-\vec{u}$ s.t. $\vec{u}+(-\vec{u}) = \vec{0}$
⑩ $1\cdot\vec{u} = \vec{u}$

Corollaries $\vec{0}$ is unique, $-\vec{u}$ is unique
$0\cdot\vec{u} = \vec{0}$, $c\vec{0} = \vec{0}$, $-\vec{u} = (-1)\cdot\vec{u}$.

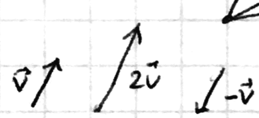
(Main example up to now)
Ex: Spaces $\mathbb{R}^n$, $n \geq 1$



Ex: set of arrows (directed line segments) in $\mathbb{R}^2$ (or $\mathbb{R}^3$).
two arrow are considered equal if they have same length and points in same direction
addition: by parallelogram rule



scalar multiplication:



Ex $\mathbb{S}$ - space of all doubly-infinite sequences of numbers $\{y_k\} = \{\dots y_{-2}, y_{-1}, y_0, y_1, y_2, \dots\}$ - "discrete-time signals"
$\{y_k\}+\{z_k\} = \{y_k+z_k\}$, $c\cdot\{y_k\} = \{cy_k\}$

Ex for $n \geq 0$, $\mathbb{P}_n$ - set of polynomials of degree $\leq n$, $\vec{p}(t) = \underbrace{a_0+a_1 t+\dots+a_n}_{\text{real coeffs.}} t^n$ (*)
degree of $\vec{p}$ - highest power of t whose coeff in (*) is $\neq 0$.
for $\vec{p}(t) = a_0 \neq 0$, degree $= 0$. If all $a_j = 0$, $\vec{p} \equiv 0$ is called the zero polynomial (its degree is not defined)

sum: for $\vec{q}(t) = b_0+b_1 t+\dots+b_n t^n$,
$\vec{p}+\vec{q}$ is defined by $(\vec{p}+\vec{q})(t) = \vec{p}(t)+\vec{q}(t) = (a_0+b_0)+(a_1+b_1)t+\dots+(a_n+b_n)t^n$.

$c\cdot\vec{p}$ is defined by $(c\vec{p})(t) = c\vec{p}(t) = ca_0+(ca_1)t+\dots+(ca_n)t^n$

$\vec{0}$ = zero polynomial
$(-1)\vec{p} = -\vec{p}$ negative

Ex V = set of all real-valued functions on a set $\mathbb{D}$

sums $(f+g)(t) = f(t) + g(t)$, $(cf)(t) = c\, f(t)$

E.g. $\mathbb{D} = \mathbb{R}$, $f = 1 + \sin 3t$ $g = 2 + 7t$

then: $(f+g)(t) = 3 + \sin 3t + 7t$, $(2g)(t) = 4 + 14t$.

zero vector: $f(t) = 0 \ \forall t$.

- each function f is a "point" of V.

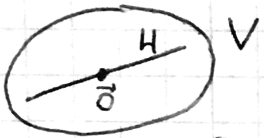
Def A <u>subspace</u> of a vector space V is a subset $H \subset V$ satisfying:

(a) $\underset{\text{zero vector of } V}{\vec{0}} \in H$

(b) for $\vec{u}, \vec{v} \in H$, $\vec{u} + \vec{v} \in H$
H closed under addition

(c) for $\vec{u} \in H$, $c \in \mathbb{R}$, $c\vec{u} \in H$
H closed under scalar multiplication

- a subspace $H \subset V$ is automatically itself a vector space.



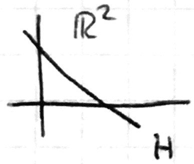
Ex $H = \{\vec{0}\} \subset V$ is a subspace - zero subspace of V.

Ex: let $\mathbb{P}$ = all polynomials in t.
- $\mathbb{P} \subset \{$ functions on $\mathbb{R}\}$ - subspace
- $\mathbb{P}_n \subset \mathbb{P}$, $n \geq 0$ - subspace

Ex: $\mathbb{R}^2$ is <u>not a subspace</u> of $\mathbb{R}^3$, (not even a subset).

But $H = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} \middle| \, s, t \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^3$ which "looks and acts" like $\mathbb{R}^2$.

Ex



a line not through $\vec{0}$ in $\mathbb{R}^2$ - <u>not a subspace</u>

a subspace spanned by a set

Ex: for $\vec{v}_1, \vec{v}_2 \in V$, let $H = \text{Span}\{\vec{v}_1, \vec{v}_2\}$. Then $H \subset V$ is a subspace.

Indeed, $\vec{0} = 0 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2 \in H$; $\begin{matrix} \vec{u} = s_1\vec{v}_1 + s_2\vec{v}_2 \\ \vec{w} = t_1\vec{v}_1 + t_2\vec{v}_2 \end{matrix} \Rightarrow \vec{u} + \vec{w} = (s_1 + t_1)\vec{v}_1 + (s_2 + t_2)\vec{v}_2 \in H$; $c \cdot \vec{u} = (cs_1)\vec{v}_1 + (cs_2)\vec{v}_2 \in H$.

THM If $\vec{v}_1, \ldots, \vec{v}_p$ are in V then $H = \text{Span}\{\vec{v}_1, \ldots, \vec{v}_p\}$ is a subspace of V.

H - subspace spanned / generated by $\{\vec{v}_1, \ldots, \vec{v}_p\}$

$\{\vec{v}_1, \ldots, \vec{v}_p\}$ - spanning / generating set for H.

Ex: $H = \{$vectors of form $(a-3b, b-a, a, b) \mid a, b \in \mathbb{R}\}$
show that $H \subset \mathbb{R}^4$ subspace

Sol: $\begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} = a\underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}}_{\vec{v}_1} + b\underbrace{\begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v}_2}$ Thus $H = \mathrm{Span}\{\vec{v}_1, \vec{v}_2\}$ - subspace of $\mathbb{R}$

4.2 Null spaces, column spaces and lin. transformations

Recall: for A $m \times n$ mat., $\mathrm{Nul}\, A = \{\vec{x} \in \mathbb{R}^n \text{ s.t. } A\vec{x} = \vec{0}\}$ - subspace of $\mathbb{R}^n$

$\mathrm{Col}\, A = \mathrm{Span}\{\vec{a}_1, \dots, \vec{a}_n\} = \{\vec{b} \in \mathbb{R}^m \text{ s.t. } \vec{b} = A\vec{x} \text{ for some } \vec{x} \in \mathbb{R}^n\}$
- subspace of $\mathbb{R}^m$.

can describe as $\mathrm{Span}\{\vec{v}_1, \dots, \vec{v}_p\}$
with $\vec{v}_1 \cdots \vec{v}_p$ from the parametric vector solution of $A\vec{x} = \vec{0}$

3
02/21/2018
1

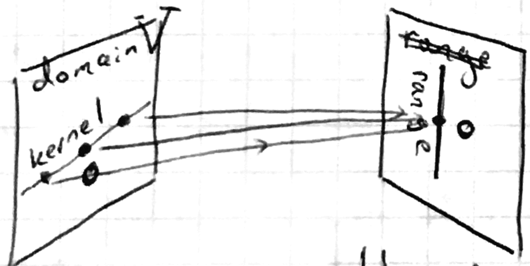
def A linear transf. T from a v.sp. V into a v.sp. W is a rule assigning to each vector $\vec{x}$ in V a unique vector $T(\vec{x})$ in W, s.t.

(i) $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ any $\vec{u}, \vec{v} \in V$ (ii) $T(c\vec{u}) = cT(\vec{u})$

5

kernel (or null space) of T : set of $\vec{u}$ in V s.t. $T(\vec{u}) = 0$ ← subspace of V

range of T = all vectors of form $T(\vec{x})$ in W ← subspace of W



W- codomain

Ex: $T: V = \mathbb{R}^n \longrightarrow W = \mathbb{R}^m$
$\vec{x} \longmapsto \underbrace{A}_{m \times n \text{ matrix}}\vec{x}$

$\ker T = \mathrm{Nul}\, A$
$\mathrm{range}\, T = \mathrm{Col}\, A$

Ex: V = functions on $[a,b]$ which have continuous derivatives
W = continuous functions on $[a,b]$

$D: V \longrightarrow W$ - linear transf.,
$f \longmapsto f'$

$\ker D = \{$constant functions on $[a,b]\}$
range $= W$.