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Ex: $H = \{\text{vectors of form } (a-3b, b-a, a, b) \mid a,b \in \mathbb{R}\}$
show that $H \subset \mathbb{R}^4$ subspace

Sol: $\begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} = a\underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}}_{\vec{v}_1} + b\underbrace{\begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v}_2}$ Thus $H = \text{Span}\{\vec{v}_1, \vec{v}_2\}$ - subspace of $\mathbb{R}^4$

4.2 <u>Null spaces, column spaces and lin. transformations</u>

<u>Recall</u>: for A $m\times n$ mat., $\text{Nul}\,A = \{\vec{x} \in \mathbb{R}^n \text{ s.t. } A\vec{x} = \vec{0}\}$ - subspace of $\mathbb{R}^n$

$\text{Col}\,A = \text{Span}\{\vec{a}_1, \dots, \vec{a}_n\} = \{\vec{b} \in \mathbb{R}^m \text{ s.t. } \vec{b} = A\vec{x} \text{ for some } \vec{x} \in \mathbb{R}^n\}$
- subspace of $\mathbb{R}^m$.

can describe as $\text{Span}\{\vec{v}_1, \dots, \vec{v}_p\}$
with $\vec{v}_1 \dots \vec{v}_p$ from the parametric vector solution of $A\vec{x} = \vec{0}$

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<u>def</u> A linear transf. T from a v.sp. V into a v.sp. W is a rule assigning to each vector $\vec{x}$ in V a unique vector $T(\vec{x})$ in W, s.t.

(i) $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ (ii) $T(c\vec{u}) = cT(\vec{u})$
any $\vec{u}, \vec{v} \in V$

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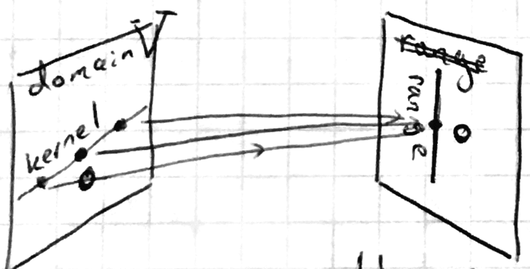
<u>kernel</u> (or null space) of T : set of $\vec{u}$ in V s.t. $T(\vec{u}) = 0$ ← subspace of V

<u>range</u> of T = all vectors of form $T(\vec{x})$ in W ← subspace of W



W - codomain

Ex: $T : V = \mathbb{R}^n \longrightarrow W = \mathbb{R}^m$ $\ker T = \text{Nul}\,A$
$\vec{x} \longmapsto \underbrace{A}_{m\times n \text{ matrix}}\vec{x}$ $\text{range}\,T = \text{Col}\,A$

<u>Ex:</u> V = functions on $[a,b]$ which have continuous derivatives
W = continuous functions on $[a,b]$

$D : V \longrightarrow W$ - linear transf., $\ker D = \{\text{constant functions on } [a,b]\}$
$f \longmapsto f'$ range $= W$.

4.3 Linearly independent sets ; bases

Vectors $\vec{v}_1, \dots, \vec{v}_p$ in a vect.sp. V are <u>linearly independent</u> iff the vect. eq. $(*)$ $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_p\vec{v}_p = \vec{0}$ has only the trivial solution $c_1 = \dots = c_p = 0$

otherwise, set $\{\vec{v}_1, \dots, \vec{v}_p\}$ is <u>lin. dep.</u> and $(*)$ is an <u>lin. dep. relation</u> among $\vec{v}_1, \dots, \vec{v}_p$ (with not all weights zero)

- as in $\mathbb{R}^n$; $\{\vec{v}\}$ is lin. indep iff $\vec{v} \neq \vec{0}$
  - $\{\vec{u}, \vec{v}\}$ lin. dep. iff $\vec{v} = c\,\vec{u}$ or $\vec{u} = d\,\vec{v}$
  - $\{\vec{0}, \vec{v}_1, \dots, \vec{v}_p\}$ is lin. dep.

<u>THM</u> set $\{\vec{v}_1, \dots, \vec{v}_p\}$ of ≥ 2 vectors in V with $\vec{v}_1 \neq \vec{0}$ is lin. dep. iff some $\vec{v}_j$ $(j > 1)$ is a lin. comb. of $\vec{v}_1, \dots, \vec{v}_{j-1}$.

<u>Note</u>: for $V \neq \mathbb{R}^n$, $(*)$ cannot be cast as matrix eq. $A\vec{x} = \vec{0}$; we must rely on def. and THM for lin. (in)dependence.

<u>Ex</u>: $V = \mathbb{P}$, $p_1(t) = 1$ $p_2(t) = t$ $p_3(t) = 2 - 3t$

Then $\{p_1, p_2, p_3\}$ is <u>lin. dep.</u> in $\mathbb{P}$, because <u>$p_3 = 2p_1 - 3p_2$</u>

<u>Ex</u>: - set $\{\sin t, \cos t\}$ is <u>lin. indep</u> in $C[0,1]$ (space of continuous functions on $0 \leq t \leq 1$)

I.e. there is no scalar c s.t. $\cos t = c \cdot \sin t$ for all $t \in [0,1]$.

- set $\{\sin t \cos t, \sin 2t\}$ is <u>lin. dep.</u> in $C[0,1]$, since $\sin 2t = 2\sin t \cos t$ $\forall t$.

<u>def</u> let H be a <u>subspace</u> of v.sp. V. A set of vectors $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_p\}$ in V is a <u>basis</u> for H if

(i) $\mathcal{B}$ is a lin. indep. set,

(ii) subspace spanned by $\mathcal{B}$ is H, $H = \text{Span}\{\vec{b}_1, \dots, \vec{b}_p\}$.

<u>Ex</u>: let A - invertible $n \times n$ matrix, $A = [\vec{a}_1 \dots \vec{a}_n]$. Then the columns of A form a <u>basis</u> for $\mathbb{R}^n$ - they are LI and span $\mathbb{R}^n$ by the Inv. Mat. THM.

<u>Ex</u>: Let $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ $\dots$ $\vec{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$ be columns of I_n.

$\{\vec{e}_1, \dots, \vec{e}_n\}$ - <u>stand. basis</u> for $\mathbb{R}^n$

Ex: $\vec{v}_1 = \begin{bmatrix}3\\0\\6\end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix}-4\\1\\7\end{bmatrix}$ $\vec{v}_3 = \begin{bmatrix}-2\\1\\7\end{bmatrix}$. Q: Is $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ a basis for $\mathbb{R}^3$?

Sol: $A[\vec{v}_1 \vec{v}_2 \vec{v}_3] \sim \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ So, A invertible $\Rightarrow$ YES

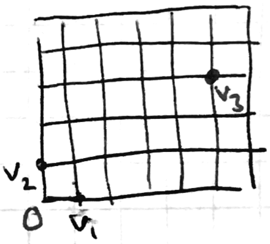
Ex: let $S = \{1, t, t^2, \dots, t^n\}$. S is a basis for $\mathbb{P}_n$ - the standard basis.

Indeed: S spans $\mathbb{P}_n$. Linear independence: assume $c_0 \cdot 1 + c_1 t + \dots + c_n t^n = \underbrace{\vec{0}(t)}_{\text{zero poly.}}$
then $c_0 = c_1 = \dots = c_n = 0$. ▨

The spaning set theorem

Ex: $\vec{v}_1 = \begin{bmatrix}0\\2\\-1\end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix}2\\2\\0\end{bmatrix}$ $\vec{v}_3 = \begin{bmatrix}6\\16\\-5\end{bmatrix}$, $H = \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

Note: $\vec{v}_3 = 5\vec{v}_1 + 3\vec{v}_2$. Q: (a) show that $H = \text{Span}\{\vec{v}_1, \vec{v}_2\}$
(b) find a basis for H



Sol (a): a vector in $\text{Span}\{\vec{v}_1, \vec{v}_2\}$ is $c_1\vec{v}_1 + c_2\vec{v}_2 = c_1\vec{v}_1 + c_2\vec{v}_2 + 0\vec{v}_3$ - in H.

a vector in H, $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3(5\vec{v}_1 + 3\vec{v}_2)$
$= (c_1 + 5c_3)\vec{v}_1 + (c_2 + 3c_3)\vec{v}_2$ - in $\text{Span}\{\vec{v}_1, \vec{v}_2\}$.

(a) ✓ (b): $\vec{v}_1, \vec{v}_2$ not multiple of each other $\Rightarrow \{\vec{v}_1, \vec{v}_2\}$ LI

and, by (a), spans H $\Rightarrow \{\vec{v}_1, \vec{v}_2\}$ a basis for H.

THM (Spaning set THM)

Let $S = \{\vec{v}_1, \dots, \vec{v}_p\}$ be a set in H and let $H = \text{Span}\{\vec{v}_1, \dots, \vec{v}_p\}$

(a) If a vector $\vec{v}_k$ from S is a lin. comb. of others vectors in S, then the set formed from S by removing $\vec{v}_k$ still spans H.

(b) If $H \neq \{\vec{0}\}$, some subset of S is a basis for H.

A basis for H is
- the smallest spanning set for H
- the largest lin. indep. set in H

✓ can delete lin. dep. vectors from a spanning set H; when we arriv to a LI subset, if we delete one more, the result will not spa H any more.

Ex: $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\3\\0\end{bmatrix}\right\} \overset{\xleftarrow{\text{further shrinking}}}{\xrightarrow[\text{enlarge to a basis}]{}} \left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\3\\0\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}\right\} \overset{\xleftarrow{\text{shrink to a basis}}}{\xrightarrow[\text{further enlargement}]{}} \left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\3\\0\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}7\\8\\9\end{bmatrix}\right\}$

LI set, does not span $\mathbb{R}^3$ — basis in $\mathbb{R}^3$ — spans $\mathbb{R}^3$, not LI