4.6. Rank

Row space A $m \times n$ matrix. Rows have n entries, so each row is in $\mathbb{R}^n$ (a vector).

Row space, $\text{Row } A = \text{Span}\{\text{rows of } A\}$ - subspace of $\mathbb{R}^n$. $\boxed{\text{Row } A = \text{Col } A^T}$

Ex:(*) $A = \begin{bmatrix} 1 & -2 & 3 & 1 \\ 2 & -4 & 7 & 7 \\ 3 & -6 & 8 & -2 \end{bmatrix}$ $\begin{matrix} \leftarrow \vec{r}_1 = (1,-2,3,1) \\ \leftarrow \vec{r}_2 = (2,-4,7,7) \\ \leftarrow \vec{r}_3 = (3,-6,8,-2) \end{matrix}$ $\text{Row } A = \text{Span}\{\vec{r}_1, \vec{r}_2, \vec{r}_3\} \subset \mathbb{R}^4$.

Warning: row operations on A change lin. dependence relations of rows!
$\Rightarrow$ cannot figure out which rows to exclude from REF.

THM If $A \sim B$, then $\text{Row } A = \text{Row } B$.
If B is REF, then nonzero rows of B form a basis for $\text{Row } B = \text{Row } A$.

Ex: $A \sim \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B$ basis for Row A: $\{(1,-2,3,1), (0,0,1,5)\}$

basis for Col A: $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 8 \end{bmatrix} \right\}$ pivot cols of A

basis for Nul A: $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 14 \\ 0 \\ -5 \\ 1 \end{bmatrix} \right\}$
- from RREF $\rightarrow$ param. vector solution of homog. eq.

~~Rank~~ **Recall** $\text{rank } A = \dim \text{Col } A = \#\text{pivots} = \dim \text{Row } A$
$= \dim \text{Col } A^T$

Note: $\text{rank } A = \text{rank } A^T$

Rank theorem: for A $m \times n$ matrix, $\boxed{\text{rank } A + \dim \text{Nul } A = n}$

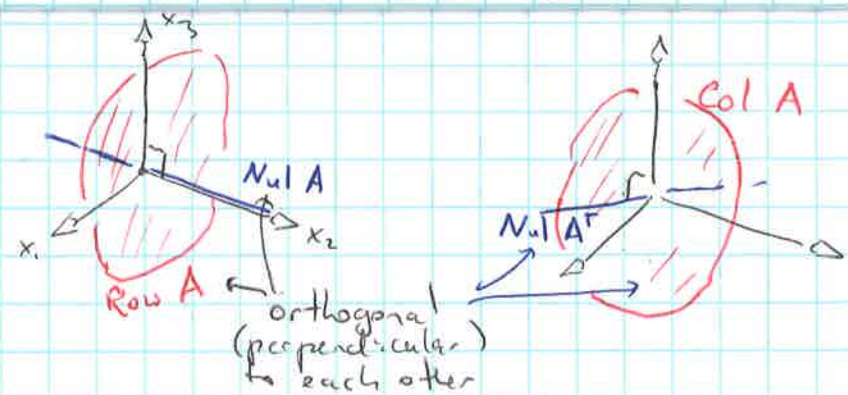
Ex: can a 3×7 matrix have a 2-dimensional null space?
Sol: $\underbrace{\text{rank } A}_{\leq 3} + \dim \text{Nul } A = 7$ $\Rightarrow$ NO!

Ex: A 40×42, $\dim \text{Nul } A = 2$. Q: Is it true that $A\vec{x} = \vec{b}$ has a solution for any $\vec{b} \in \mathbb{R}^{40}$?
Sol: $\underset{\dim \text{Col } A}{\text{rank } A} = 42 - 2 = 40$. So, Col A - 40-dim. subspace of $\mathbb{R}^{40}$
$\Rightarrow \text{Col } A = \underbrace{\mathbb{R}^{40}}_{\text{entire}}$ $\Rightarrow$ YES

Ex: A 5×7, $\dim \text{Nul } A = 4$
Q: $\dim \text{Nul } A^T = ?$

Sol: $\text{rank} = 7 - 4 = 3$
Rank Thm for $A^T \Rightarrow 3 + \dim \text{Nul } A^T = 5 \Rightarrow \dim \text{Nul } A^T = 2$

Ex: (1) $A = \begin{bmatrix} 3 & 0 & -1 \\ 3 & 0 & -1 \\ 4 & 0 & 5 \end{bmatrix}$



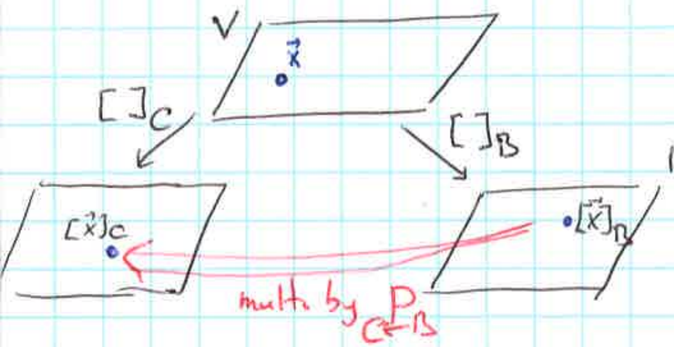
4.7 Change of basis

Ex V - v.sp. with two bases $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$, $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$ s.t.
$\vec{b}_1 = 4\vec{c}_1 + \vec{c}_2$, $\vec{b}_2 = -6\vec{c}_1 + \vec{c}_2$. Suppose $\vec{x} = 3\vec{b}_1 + \vec{b}_2 \; (*)$, i.e., $[\vec{x}]_\mathcal{B} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Q: Find $[\vec{x}]_\mathcal{C}$

Sol: Apply coord. mapping defined by $\mathcal{C}$ to $(*)$:
$[\vec{x}]_\mathcal{C} = 3[\vec{b}_1]_\mathcal{C} + [\vec{b}_2]_\mathcal{C}$ i.e. $[\vec{x}]_\mathcal{C} = \big[[\vec{b}_1]_\mathcal{C} \; [\vec{b}_2]_\mathcal{C} \big] \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$

THM Let $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$, $\mathcal{C} = \{\vec{c}_1, \dots, \vec{c}_n\}$ be bases of V. Then there is a unique $n \times n$ mat. $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ such that $[\vec{x}]_\mathcal{C} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} [\vec{x}]_\mathcal{B}$. $(**)$

Explicitly: $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \big[[\vec{b}_1]_\mathcal{C} \cdots [\vec{b}_n]_\mathcal{C} \big]$ ← change-of-coord. mat. from $\mathcal{B}$ to $\mathcal{C}$



Also: $(**)$ implies
$$\left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right)^{-1} [\vec{x}]_\mathcal{C} = [\vec{x}]_\mathcal{B}$$
hence: $\boxed{\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right)^{-1}}$

Change of basis in $\mathbb{R}^n$

Recall
If $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$, $\mathcal{E} = \{\vec{e}_1, \dots, \vec{e}_n\}$ stand. basis in $\mathbb{R}^n$, then $[\vec{b}_i]_\mathcal{E} = \vec{b}_i$
and $\underset{\mathcal{E}\leftarrow\mathcal{B}}{P} = P_\mathcal{B} = [\vec{b}_1 \cdots \vec{b}_n]$

- Change between two nonstandard bases in $\mathbb{R}^n$:

Ex: (2) $\underbrace{\vec{b}_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}}_{\mathcal{B}}$; $\underbrace{\vec{c}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}, \vec{c}_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}}_{\mathcal{C}}$ - two bases in $\mathbb{R}^2$. Q: Find $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$

We need $[\vec{b}_1]_\mathcal{C} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $[\vec{b}_2] = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. By def., $[\vec{c}_1 \; \vec{c}_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{b}_1$, $[\vec{c}_1 \; \vec{c}_2] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \vec{b}_2$

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To solve two mat. eq. simultaneously, augment coeff. mat. with $\vec{b}_1$ and $\vec{b}_2$:

$$[\vec{c}_1\ \vec{c}_2 \mid \vec{b}_1\ \vec{b}_2] = \left[\begin{array}{cc|cc} 1 & 3 & -9 & -5 \\ -4 & -5 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 6 & 4 \\ 0 & 1 & -5 & -3 \end{array}\right]$$

Thus: $[\vec{b}_1]_C = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$, $[\vec{b}_2]_C = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$ and $\underset{C \leftarrow B}{P} = \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix}$ ⑩ 5

Observe: $[\vec{c}_1\ \vec{c}_2 \mid \vec{b}_1\ \vec{b}_2] \sim [I \mid \underset{C \leftarrow B}{P}]$ ← works analogously for any two bases in $\mathbb{R}^n$

Another description of $\underset{C \leftarrow B}{P}$:

- $\underset{C \leftarrow B}{P} = \underset{C \leftarrow \mathcal{E}}{P} \cdot \underset{\mathcal{E} \leftarrow B}{P} = (P_C)^{-1} P_B$

or:

$$\vec{x} = P_B [\vec{x}]_B$$

$$\vec{x} = P_C [\vec{x}]_C \Rightarrow [\vec{x}]_C = P_C^{-1} \vec{x}$$

$$\Rightarrow [\vec{x}]_C = \underbrace{P_C^{-1} P_B}_{\underset{C \leftarrow B}{P}} [\vec{x}]_B$$