2.2 The inverse of a matrix

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A $n\times n$ mat. is **invertible** if there is an $n\times n$ mat. C s.t. $CA=I$ and $AC=I$. ($I = I_n$)

Then C is called the inverse of A.

It is unique (if exists), notation: A^{-1}. Thus $A^{-1}A=I$, $AA^{-1}=I$.

a non-invertible A is called "singular".

Ex: $A=\begin{bmatrix}2&5\\-3&-7\end{bmatrix}$ $C=\begin{bmatrix}-7&-5\\3&2\end{bmatrix}$ $AC=\begin{bmatrix}1&0\\0&1\end{bmatrix}$, $CA=\begin{bmatrix}1&0\\0&1\end{bmatrix}$

Thus, $C=A^{-1}$.

THM Let $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$. If $ad-bc\neq 0$, then A is invertible and

$$A^{-1}=\frac{1}{ad-bc}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}.$$

If $\boxed{ad-bc}=0$, then A is non-invertible

"determinant", det A

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Ex: $A=\begin{bmatrix}2&5\\-3&-7\end{bmatrix}$ $\det A = 2(-7)-5(-3)=1$, $A^{-1}=\begin{bmatrix}-7&-5\\3&2\end{bmatrix}$ cf. prev. Ex.

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Ex: $\begin{bmatrix}1&2\\3&4\end{bmatrix}^{-1}=\frac{1}{-2}\begin{bmatrix}4&-2\\-3&1\end{bmatrix}=\begin{bmatrix}-2&1\\\frac{3}{2}&-\frac{1}{2}\end{bmatrix}$

- If A is an invertible $n\times n$ matrix, then for each $\vec{b}\in\mathbb{R}^n$, eq. $A\vec{x}=\vec{b}$ has the unique sol. $\boxed{\vec{x}=A^{-1}\vec{b}}$
- properties: • $(A^{-1})^{-1}=A$ • $\boxed{(AB)^{-1}=\underbrace{B^{-1}A^{-1}}_{\text{reverse order!}}}$ • $(A^T)^{-1}=(A^{-1})^T$

Elementary matrices

an elem. matrix is the result of a single elem. row operation on the identity matrix.

Ex: $\underset{r_2\mapsto r_2-3r_1}{E_1=\begin{bmatrix}1&0\\-3&1\end{bmatrix}}$ $\underset{r_1\leftrightarrow r_2}{E_2=\begin{bmatrix}0&1\\1&0\end{bmatrix}}$ $\underset{r_2\mapsto 5r_2}{E_3=\begin{bmatrix}1&0\\0&5\end{bmatrix}}$

for $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$, $E_1A=\begin{bmatrix}a&b\\c-3a&d-3b\end{bmatrix}$ $E_2A=\begin{bmatrix}c&d\\a&b\end{bmatrix}$ $E_3A=\begin{bmatrix}a&b\\5c&5d\end{bmatrix}$

- If an elem row op. is performed on $m\times n$ mat. A, the resulting matrix is EA, here E is the $m\times m$ elem mat created by doing same row op. on I_m

$A \underset{\rho}{\sim} EA$, where $I \underset{\rho}{\sim} E$ ~~02/02/2018 4~~ 02/05/2018 2

- each E is invertible; the inverse is an elem. matrix of same type.

THM (a) An $n\times n$ matrix is invertible iff $A \sim I_n$. (b) In this case, any sequence of row op. that reduces A to I_n, also transforms I_n to A^{-1}!

Argument: (b)
$A \sim E_1 A \sim E_2(E_1 A) \sim \cdots \sim \underbrace{E_p \cdots E_1}_{A^{-1}} A = I$
$I \sim E_1 I = E_1 \sim E_2 E_1 \sim \cdots \sim E_p \cdots E_1$

Algorithm for finding A^{-1}

Row reduce the augmented matrix $[A \; I]$ $n \times 2n$ - matrix

If A is invertible, ~~then A~~ the RREF is $[I \; A^{-1}]$

Ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $[A \; I] = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{bmatrix} \sim \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right]$ with $\underbrace{}_{I}$ $\underbrace{}_{A^{-1}}$

Argument (a): A invertible iff $A\vec{x} = \vec{b}$ $\begin{cases} \text{has a sol. } \forall \vec{b} \iff \text{each row contains a pivot} \\ \text{sol. is unique} \iff \text{each column contains a pivot} \end{cases}$

2.3 Characterizations of invertible matrices

THM "The invertible matrix theorem"

Let A be a square, $n\times n$, matrix. The following are equivalent:

(a) A invertible

(b) $A \sim I_n$ ← most useful!

*(c) A has n pivots

(d) $A\vec{x} = \vec{0}$ has only the triv. sol.

(e) columns of A are lin. indep.

(f) lin. transf. $\vec{x} \mapsto A\vec{x}$ is one-to-one

(g) $A\vec{x} = \vec{b}$ has a sol. for each $\vec{b} \in \mathbb{R}^n$ $\rightsquigarrow$ (g') $A\vec{x} = \vec{b}$ has a unique sol. $\forall \vec{b} \in \mathbb{R}^n$

(h) Columns of A span $\mathbb{R}^n$

(i) lin. transf. $\vec{x} \mapsto A\vec{x}$ is onto $\mathbb{R}^n$

(j) there is a C s.t. $CA = I$

(k) there is a D s.t. $AD = I$

(l) A^T invertible

- If A, B $n \times n$ mat. and $AB = I$, then A, B both invertible, with $B = A^{-1}$, $A = B^{-1}$.

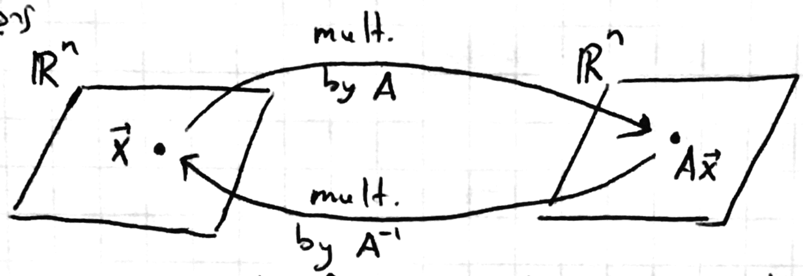
Ex: $A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix}$ invertible?

Sol: $A \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix} \underset{\text{(c) of THM}}{\Longrightarrow} A$ invertible

<u>Invertible lin. transformations</u>

for A invertible, we have

$A^{-1}A\vec{x} = \vec{x}$



- transforms $A\vec{x}$ back into $\vec{x}$!

- a lin. transf. $T: \mathbb{R}^n \to \mathbb{R}^n$ is <u>invertible</u> iff there exists $S: \mathbb{R}^n \to \mathbb{R}^n$ s.t.
  $S(T(\vec{x})) = \vec{x}$ for all $\vec{x} \in \mathbb{R}^n$
  $T(S(\vec{x})) = \vec{x}$ for all $\vec{x} \in \mathbb{R}^n$
- if such S exists, it is <u>unique</u> and is "the inverse of T", $S = T^{-1}$.
- if T has stand. matrix A, then T is invertible iff A invertible. In this case, S is given by $S(\vec{x}) = A^{-1}\vec{x}$.

Ex: let $T: \mathbb{R}^n \to \mathbb{R}^n$ one-to-one. By THM$^{(f)}$, A invertible $\Rightarrow$ T invertible.

<u>Practice questions</u>:

(I) $A = \begin{bmatrix} 2 & 3 & 0 \\ 3 & 5 & 0 \\ -4 & -6 & 1 \end{bmatrix}$ Is it invertible? If yes, find the inverse

(II) Is $\begin{bmatrix} 1 & 2 & 7 \\ 1 & 2 & 7 \\ 1 & 2 & 7 \end{bmatrix}$ Invertible? Is $\begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}$ Invertible? (if yes, find the inverse)

(III) Assume A, B $n \times n$, invertible is $(AB)^T$ invertible? what is the inverse? (in terms of A^{-1}, B^{-1})