2.9. Dimension and rank

Coordinate systems H - subspace, $\mathcal{B}=\{\vec{b}_1,\dots,\vec{b}_p\}$ a basis for H

any vector $\vec{x}\in H$ can be written as a lin. comb. of basis vectors in a unique way!

- if ① $\vec{x}=c_1\vec{b}_1+\dots+c_p\vec{b}_p$
② $\vec{x}=d_1\vec{b}_1+\dots+d_p\vec{b}_p$ - two presentations of $\vec{x}$ as a lin. comb. $\Rightarrow$ $\vec{0}=(c_1-d_1)\vec{b}_1+\dots+(c_p-d_p)\vec{b}_p$ ~~subtract~~ ①−②

since $\mathcal{B}$ lin. indep, we have ~~eq~~ $c_1=d_1,\dots,c_p=d_p$. I.e. ①=②.

DEF Let $\mathcal{B}=\{\vec{b}_1,\dots,\vec{b}_p\}$ a basis for H. For each $\vec{x}\in H$,
$\vec{x}=c_1\vec{b}_1+\dots+c_p\vec{b}_p$; $\boxed{c_1,\dots,c_p}$ - coordinates of $\vec{x}$ relative to the basis $\mathcal{B}$.

$[\vec{x}]_{\mathcal{B}}=\begin{bmatrix}c_1\\ \vdots\\ c_p\end{bmatrix}$ - coordinate vector of $\vec{x}$ (rel. to $\mathcal{B}$), or $\mathcal{B}$-coordinate vector of $\vec{x}$.

Ex: $\underbrace{\vec{v}_1=\begin{bmatrix}2\\5\\1\end{bmatrix},\ \vec{v}_2=\begin{bmatrix}-1\\0\\3\end{bmatrix}}$, $\vec{x}=\begin{bmatrix}1\\10\\11\end{bmatrix}$

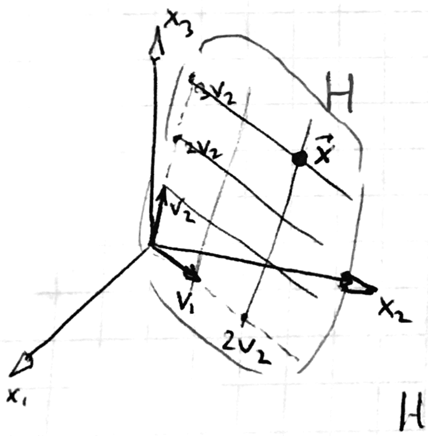
$\mathcal{B}=\{\vec{v}_1,\vec{v}_2\}$ - basis for $H=\text{Span}\{\vec{v}_1,\vec{v}_2\}$

Q: (a) Is $\vec{x}$ in H?
(b) If yes, find $[\vec{x}]_{\mathcal{B}}$.

Sol: $\vec{x}$ in H iff eq. $c_1\vec{v}_1+c_2\vec{v}_2=\vec{x}$ (*) consistent.

Aug Mat $\begin{bmatrix}2&-1&1\\5&0&10\\1&3&11\end{bmatrix}\sim\begin{bmatrix}1&0&2\\0&1&3\\0&0&0\end{bmatrix}$ Hence, eq. (*) consistent, $c_1=2, c_2=3 \Rightarrow [\vec{x}]_{\mathcal{B}}=\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}2\\3\end{bmatrix}$

$\vec{x}=2\vec{v}_1+3\vec{v}_2$



basis $\mathcal{B}$ determines a coordinate system on H.

points in H are in $\mathbb{R}^3$ but are determined by $[\vec{x}]_{\mathcal{B}}\in\mathbb{R}^2$

Correspondence $\vec{x}\mapsto[\vec{x}]_{\mathcal{B}}$ is a one-to-one correspondence between H and $\mathbb{R}^2$ preserving linear combinations. - "isomorphism"

H isomorphic to $\mathbb{R}^2$.

Generally, if $\mathcal{B}=\{\vec{b}_1,\dots,\vec{b}_p\}$ a basis for H, mapping $\vec{x}\mapsto[\vec{x}]_{\mathcal{B}}$, $H\to\mathbb{R}^p$ is a 1-1 correspondence which makes H "look and act" like $\mathbb{R}^p$.

Claim: If H has a basis of p vectors, then each basis in H has exactly p vectors.

def The dimension of a nonzero subspace H, $\dim H$, is the number of vectors in any basis for H. Also, $\dim\{\vec{0}\} = 0$ (convention)

Ex: • $\dim \mathbb{R}^n = n$, every basis in $\mathbb{R}^n$ consists of n vectors.

• $\dim$(plane in $\mathbb{R}^3$ through 0) $= 2$ • $\dim$(line through 0 in $\mathbb{R}^3$) $= 1$.

Ex: Nul A : basis vectors correspond to free variables of $A\vec{x} = \vec{0}$.

So, $\dim \text{Nul } A$ = # non-pivotal columns.
= # free variables.

Def The rank of a mat. A, rank A, is $\dim \text{Col } A$.

Thus, rank A = # pivot columns

Ex:
$$A = \begin{bmatrix} 1 & 2 & 1 & 7 & 1 \\ -2 & -4 & -1 & -10 & -2 \\ 3 & 6 & 0 & 9 & 1 \\ 1 & 2 & -2 & -5 & 7 \end{bmatrix}$$
Q: What is rank A?

Sol:
$$A \sim \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
3 pivot columns $\Rightarrow$ rank = 3

Note: $\dim \text{Nul } A$ = # non-pivot col. = 2

THM (The rank theorem)

If a matrix A has n columns, then rank A + $\dim \text{Nul } A = n$

Thm (basis theorem)

Let H be a p-dimensional subspace of $\mathbb{R}^n$. Any lin. indep. set of p vectors in H is automatically a basis for H. Also, any set of p vectors in H spanning H, is a basis for H.

Invertible mat. THM (continued)

A $n \times n$ mat. Following are equivalent to A being invertible:

(m) Columns of A form a basis for $\mathbb{R}^n$

(n) Col $A = \mathbb{R}^n$

(o) $\dim \text{Col } A = n$

(p) rank $A = n$

(q) Nul $A = \{\vec{0}\}$

(r) $\dim \text{Nul } A = 0$.