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THM If $\{\vec{u}_1, \dots, \vec{u}_p\}$ is an orthonormal basis for $W \subset \mathbb{R}^n$, then
$\text{proj}_W \vec{y} = (y \cdot \vec{u}_1)\vec{u}_1 + \dots + (\vec{y} \cdot \vec{u}_p)\vec{u}_p$

If $U = [\vec{u}_1\ \vec{u}_2 \cdots \vec{u}_p]$, then $\boxed{\text{proj}_W \vec{y} = UU^T\vec{y}}$ for all $\vec{y} \in \mathbb{R}^n$.

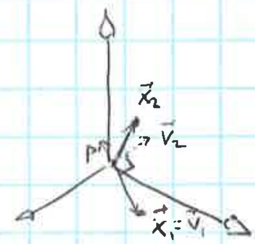
6.4 Gram-Schmidt process

problem: find an orthogonal basis $\{\vec{v}_1, \dots, \vec{v}_p\}$ for a given subspace $W \subset \mathbb{R}^n$

$\overset{\parallel}{\text{Span}\{\vec{x}_1, \dots, \vec{x}_p\}}$ not orthogonal.

Ex: $\vec{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $W = \text{Span}\{\vec{x}_1, \vec{x}_2\} \subset \mathbb{R}^3$

Q: find an orthogonal basis $\{\vec{v}_1, \vec{v}_2\}$ for W.



Sol: set $\vec{v}_1 = \vec{x}_1$. $\vec{x}_2 = \cancel{\text{proj}_{\text{Span}\{\vec{v}_1\}} \vec{x}_2}\ \cancel{+}\ \underset{\substack{\uparrow \\ \text{proj}_{\text{Span}\{\vec{v}_1\}} \vec{x}_2}}{\vec{p}} + \underbrace{\boxed{(\vec{x}_2 - \vec{p})}}_{\text{orthogonal to } \vec{v}_1} \leftarrow \vec{v}_2$

$\vec{p} = \dfrac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 = \dfrac{-1}{5} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/5 \\ -2/5 \\ 0 \end{bmatrix}$

$\vec{v}_2 = \vec{x}_2 - \vec{p} = \begin{bmatrix} -4/5 \\ 2/5 \\ 1 \end{bmatrix}$

So: $\left\{ \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -4/5 \\ 2/5 \\ 1 \end{bmatrix} \right\}$

- orthogonal set of 2 vectors in W
$\Rightarrow$ ~~basis~~ orthog basis for W.

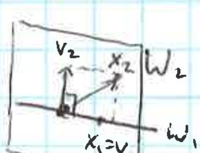
• note: $\left\{ \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \underset{\substack{\parallel \\ 5 \cdot \vec{v}_2}}{\vec{v}_2'} = \begin{bmatrix} -4 \\ 2 \\ 5 \end{bmatrix} \right\}$ - also an orthog. basis for W

Generally: let $W = \text{Span}\underbrace{\{\vec{x}_1, \dots, \vec{x}_p\}}_{\text{basis}} \subset \mathbb{R}^n$. Want to construct an orthog basis $\{\vec{v}_1, \dots, \vec{v}_p\}$ for W.

Step 1: set $\boxed{\vec{v}_1 = \vec{x}_1}$, $W_1 = \text{Span}\{\vec{x}_1\} = \text{Span}\{\vec{v}_1\}$

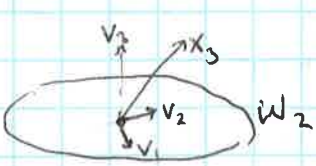
Step 2: $W_2 = \text{Span}\{\vec{x}_1, \vec{x}_2\}$. orthog. basis for W_2: $\vec{v}_1 = \vec{x}_1$, $\boxed{\vec{v}_2} = \vec{x}_2 - \text{proj}_{W_1} \vec{x}_2 = \boxed{\vec{x}_2 - \dfrac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1}$

~~orthog.~~ component of $\vec{x}_2$ orthog. to W_1



Step 3 $W_3 = \text{Span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ orthog. basis: $\underbrace{\vec{v}_1, \vec{v}_2}_{\text{already constructed}}$, $\boxed{\vec{v}_3} = \vec{x}_3 - \text{proj}_{W_2} \vec{x}_3$

$\boxed{= \vec{x}_3 - \dfrac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \dfrac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2}$



. . . .

Step p: $W = W_p = \text{Span}\{x_1, \dots, x_p\}$,
orthog. basis: $\vec{v}_1, \dots, \vec{v}_{p-1}$, $\boxed{\vec{v}_p} = \vec{x}_p - \text{proj}_{W_{p-1}} \vec{x}_p$

$= \vec{x}_p - \dfrac{\vec{x}_p \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \dots - \dfrac{\vec{x}_p \cdot \vec{v}_{p-1}}{\vec{v}_{p-1} \cdot \vec{v}_{p-1}} \vec{v}_{p-1}$

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Ex: $\vec{x}_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix}0\\1\\1\\0\end{bmatrix}$, $\vec{x}_3 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$ - basis for $W = \mathrm{Span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\} \subset \mathbb{R}^4$

Q: find an orthog. basis for W

Sol: $\vec{v}_1 = \vec{x}_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix}$

$$\vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_2\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 = \begin{bmatrix}0\\1\\1\\0\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\1\\0\\0\end{bmatrix} = \begin{bmatrix}-1/2\\1/2\\1\\0\end{bmatrix} \xrightarrow[\text{rescaling}]{} \vec{v}_2' = \begin{bmatrix}-1\\1\\2\\0\end{bmatrix}$$

$$\vec{v}_3 = \vec{x}_3 - \frac{\vec{x}_3\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 - \frac{\vec{x}_3\cdot\vec{v}_2'}{\vec{v}_2'\cdot\vec{v}_2'}\vec{v}_2' = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} - \frac{2}{2}\begin{bmatrix}1\\1\\0\\0\end{bmatrix} - \frac{2}{6}\begin{bmatrix}-1\\1\\2\\0\end{bmatrix} = \begin{bmatrix}1/3\\-1/3\\1/3\\1\end{bmatrix} \xrightarrow[\text{rescaling}]{} \vec{v}_3' = \begin{bmatrix}1\\-1\\1\\3\end{bmatrix}$$

Thus: $\left\{ \vec{v}_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \vec{v}_2' = \begin{bmatrix}-1\\1\\2\\0\end{bmatrix}, \vec{v}_3' = \begin{bmatrix}1\\-1\\1\\3\end{bmatrix} \right\}$ - orthog. basis for W.

Q: find an orthonormal basis for W from above

Sol: normalize $\vec{v}_1, \vec{v}_2', \vec{v}_3'$ to unit length: $\left\{ \vec{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \vec{u}_2 = \frac{1}{\sqrt{6}}\begin{bmatrix}-1\\1\\2\\0\end{bmatrix}, \vec{u}_3 = \frac{1}{\sqrt{12}}\begin{bmatrix}1\\-1\\1\\3\end{bmatrix} \right\}$

- o/n basis for W.

QR factorization

THM (the QR factorization)
if A is mxn mat. with LI columns, then A can be factored as $A = QR$
where Q is an mxn mat. whose columns form an o/n basis for Col A and
R is an nxn upper triangular invertible mat. with positive diagonal entries.

Idea: $A = [\vec{x}_1 \cdots \vec{x}_n]$, $W = \underbrace{\mathrm{Span}\{\vec{x}_1, \cdots \vec{x}_n\}}_{=\,\mathrm{Col}\,A} \subset \mathbb{R}^m \xrightarrow[\text{Gram-Schmidt}]{} \{\vec{u}_1, \dots, \vec{u}_n\}$ - o/n basis for W

$$\vec{x}_k = r_{1k}\vec{u}_1 + \dots + r_{k-1\,k}\vec{u}_{k-1} + \underbrace{r_{kk}}_{\|\vec{v}_k\| > 0}\vec{u}_k + 0\cdot\vec{u}_{k+1} + \dots + 0\cdot\vec{u}_n$$

$$\Rightarrow A = \underbrace{[\vec{u}_1 \cdots \vec{u}_n]}_{Q} \underbrace{\begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ 0 & r_{22} & \cdots & r_{2n} \\ \vdots & 0 & \ddots & r \\ 0 & 0 & \cdots & r_{nn} \end{bmatrix}}_{R}$$

Ex: $A = \begin{bmatrix}1&0&1\\1&1&1\\0&1&1\\0&0&1\end{bmatrix}$ find the QR decomposition

Sol: $Q = [\vec{u}_1\, \vec{u}_2\, \vec{u}_3] = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{12} \\ 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{12} \\ 0 & 2/\sqrt{6} & 1/\sqrt{12} \\ 0 & 0 & 3/\sqrt{12} \end{bmatrix}$

$A = QR \Rightarrow Q^T A = \underbrace{Q^T Q}_{I} R = R$

$$R = Q^T A = \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} & \sqrt{2} \\ 0 & 3/\sqrt{6} & 2/\sqrt{6} \\ 0 & 0 & 4/\dots \end{bmatrix}$$