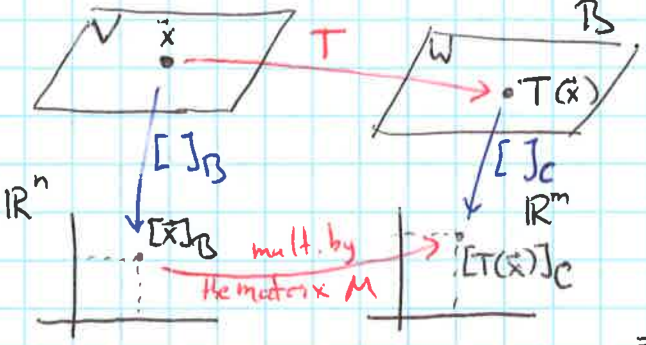
5.4 Eigenvectors and linear transformations

Idea: If $A = PDP^{-1}$, the lin. transf. $\vec{x} \mapsto A\vec{x}$ is "essentially the same" as a simple transformation $\vec{u} \mapsto D\vec{u}$.

- Matrix of a lin. transf. $T: \underset{\substack{\text{v.sp.}\\ \mathcal{B}}}{V} \longrightarrow \underset{\substack{\text{v.sp.}\\ \mathcal{C}}}{W}$ - bases



how to ~~compare~~ connect $[\vec{x}]_{\mathcal{B}}$ and $[T(\vec{x})]_{\mathcal{C}}$?

$$\vec{x} = r_1\vec{b}_1 + \dots + r_n\vec{b}_n \Rightarrow [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}$$

$$T(\vec{x}) = r_1 T(\vec{b}_1) + \dots + r_n T(\vec{b}_n)$$

$$\Rightarrow [T(\vec{x})]_{\mathcal{C}} = r_1 [T(\vec{b}_1)]_{\mathcal{C}} + \dots + r_n [T(\vec{b}_n)]_{\mathcal{C}}$$

Thus: $[T(\vec{x})]_{\mathcal{C}} = M[\vec{x}]_{\mathcal{B}}$ with $M = \left[[T(\vec{b}_1)]_{\mathcal{C}} \cdots [T(\vec{b}_n)]_{\mathcal{C}} \right]$ (*)

- matrix for T relative to the bases $\mathcal{B}, \mathcal{C}$
= matrix representation of T

Ex: $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ basis for V,
$\mathcal{C} = \{\vec{c}_1, \vec{c}_2, \vec{c}_3\}$ basis for W. Let $T: V \to W$ lin. transf. s.t.
$T(\vec{b}_1) = 3\vec{c}_1 - 2\vec{c}_2 + 5\vec{c}_3$, $T(\vec{b}_2) = 4\vec{c}_1 + 7\vec{c}_2 - \vec{c}_3$.

Q: find the matrix for T rel. to $\mathcal{B}, \mathcal{C}$

Sol:

$$M = \begin{bmatrix} 3 & 4 \\ -2 & 7 \\ 5 & -1 \end{bmatrix}$$

$[T(\vec{b}_1)]_{\mathcal{C}}$ $[T(\vec{b}_2)]_{\mathcal{C}}$

- If $V = W$ and $T(\vec{x}) = \vec{x}$ the identity transf., then the matrix M is just $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ - change-of-coordinates matrix.

- Lin. transformations $T: \underset{\mathcal{B}}{V} \longrightarrow \underset{\mathcal{B}}{V}$ - same space
  - same basis

In this case $M =: [T]_{\mathcal{B}}$ - "matrix of T relative to $\mathcal{B}$" or "$\mathcal{B}$-matrix of T"

we have $[T(\vec{x})]_{\mathcal{B}} = [T]_{\mathcal{B}} [\vec{x}]_{\mathcal{B}}$, for all $\vec{x} \in V$

$$\begin{array}{ccc} \vec{x} & \longrightarrow & T(\vec{x}) \\ \downarrow & & \downarrow \\ [\vec{x}]_{\mathcal{B}} & \xrightarrow[{[T]_{\mathcal{B}}}]{\text{mult. by}} & [T(\vec{x})]_{\mathcal{B}} \end{array}$$

Ex: $T: \mathbb{P}_2 \longrightarrow \mathbb{P}_2$ given by
$T(a_0 + a_1 t + a_2 t^2) = a_1 + 2a_2 t$ (differentiation in t)

Q: find the $\mathcal{B}$-matrix for T, for $\mathcal{B} = \{1, t, t^2\}$

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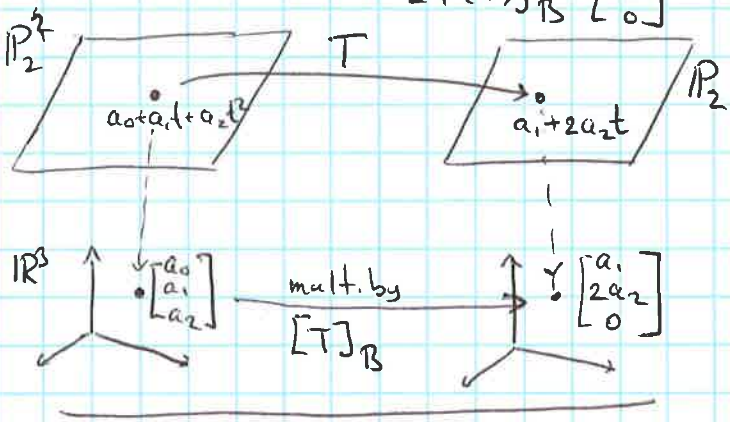
Sol: $T(1) = 0 \quad [T(1)]_{\mathcal{B}} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$T(t) = 1 \quad [T(t)]_{\mathcal{B}} = \begin{bmatrix}1\\0\\0\end{bmatrix}$

$T(t^2) = 2t \quad [T(t^2)]_{\mathcal{B}} = \begin{bmatrix}0\\2\\0\end{bmatrix}$

$\Rightarrow [T]_{\mathcal{B}} = \begin{bmatrix}0&1&0\\0&0&2\\0&0&0\end{bmatrix}$



Lin. transformations of $\mathbb{R}^n$

$T: \mathbb{R}^n \to \mathbb{R}^n$
$\vec{x} \mapsto A\vec{x}$

if A diagonalizable, $\mathcal{B}$-matrix of A is diagonal, for $\mathcal{B}$ - basis of eigenvectors.

THM (Diagonal matrix representation)
Suppose $A = PDP^{-1}$ with D a diagonal $n \times n$ mat. If $\mathcal{B}$ is the basis of $\mathbb{R}^n$ formed from the columns of P, then D is the $\mathcal{B}$-matrix of the transf. $\vec{x} \mapsto A\vec{x}$.

Ex: $T: \mathbb{R}^2 \to \mathbb{R}^2$, $\vec{x} \mapsto A\vec{x}$, $A = \begin{bmatrix}7&2\\-4&1\end{bmatrix}$ find a basis $\mathcal{B}$ for $\mathbb{R}^2$ s.t. $[T]_{\mathcal{B}}$ is diagonal.

Sol: $A = PDP^{-1}$, $P = \begin{bmatrix}1&1\\-1&-2\end{bmatrix}$, $\underbrace{}_{\vec{b}_1}\ \underbrace{}_{\vec{b}_2}$, $D = \begin{bmatrix}5&0\\0&3\end{bmatrix}$

Thus for $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$,
$[T]_{\mathcal{B}} = D$

I.e. mappings $\vec{x} \mapsto A\vec{x}$ and $\vec{u} \mapsto D\vec{u}$ describe the same lin. transf. rel. to different bases.

Similarity of matrix representations

THM above does not in fact require D to be diagonal:
if $\underset{\text{similar}}{A \approx C}$, i.e. $A = PCP^{-1}$ and $T: \underset{\mathbb{R}^n \to \mathbb{R}^n}{\vec{x} \mapsto A\vec{x}}$, then $[T]_{\mathcal{B}} = C$

basis of columns of P.

$$\begin{tikzcd}
\vec{x} \arrow[r, "\text{mult. by } A"] \arrow[d, "\text{mult. by } P^{-1}"'] & A\vec{x} \\
{[\vec{x}]_{\mathcal{B}}} \arrow[r, "\text{mult. by } C"'] & {[A\vec{x}]_{\mathcal{B}}} \arrow[u, "\text{mult. by } P"']
\end{tikzcd}$$

Conversely, matrix of T rel. to any basis $\mathcal{B}$ is similar to A.