sol. of the init. val. prob. : $v(x) = \pm\sqrt{v_0^2 - 2gR + \frac{2gR^2}{R+x}}$ ($+$ rising, $-$ falling) — velocity as a function of altitude

(b) $v(A_{max}) = 0 \Rightarrow v_0^2 - 2gR + \frac{2gR^2}{R+A_{max}} = 0 \Rightarrow v_0^2 = \frac{2gR\,A_{max}}{R+A_{max}} \Rightarrow v_0 = \sqrt{\frac{2gR\,A_{max}}{R+A_{max}}}$

(c) taking $A_{max} \to \infty$, we get $v_{escape} = \sqrt{2gR} \approx 11.1\ \text{km/s}$

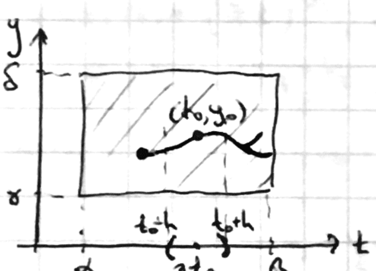
2.4 Differences between linear and non-linear diff. equations.

So far, every init. val. problem we considered had a unique solution (on some interval t)

THM 1 (existence & uniqueness of sol. for 1st order linear equations)
If p, g are continuous for $\alpha < t < \beta$ (on interval I) containing t_0, then there exists a unique solution $y = \varphi(t)$ on I for the init. val. prob. $\boxed{\begin{aligned} y' + p(t)y &= g(t) \\ y(t_0) &= y_0 \end{aligned}}$

(Idea of proof: can construct the explicit solution by the method of integrating factors)

THM 2 (existence & uniqueness for 1st order non-linear equations)
Consider init. val. prob. $\boxed{y' = f(t,y),\ y(t_0) = y_0}$ (*) Assume that f and $\frac{\partial f}{\partial y}$ are continuous in a rectangle $\boxed{\begin{aligned} \alpha < t < \beta \\ \gamma < y < \delta \end{aligned}}$ containing (t_0, y_0).
Then a solution of (*) exists and is unique in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$.



guaranteed interval of existence & uniqueness

Remark:
- THM 2 gives a sufficient but not necessary condition for existence and uniqueness
- existence of sol. follows just from continuity of f. (but not uniqueness)

Ex $ty' + 2y = 4t^2$, $y(1) = 2$. Find an interval in which the sol. exists and is unique.

Sol: $y' + \left(\frac{2}{t}\right)y = 4t$, with $p(t) = \frac{2}{t}$, $g(t) = 4t$

g continuous for all t; p continuous for $t \neq 0$

I.e. assumptions of THM 1 hold for two intervals: $I_1: t > 0$, $I_2: t < 0$

I_1 contains the init. time $t_0 = 1$, so the sol. exists and is unique for $t > 0$.

Note: if init. cond. were $y(-1) = 2$, interval of existence would have been $-\infty < t < 0$.

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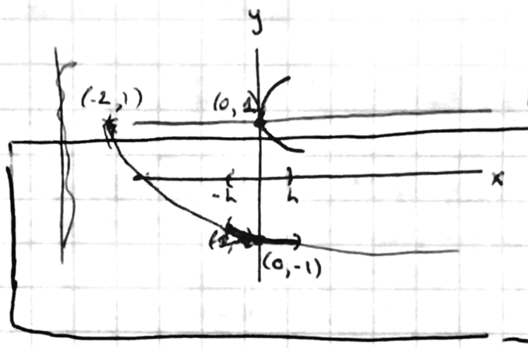
Ex (a) $\dfrac{dy}{dx} = \dfrac{3x^2+4x+2}{2(y-1)}$, $y(0) = -1$ ($f(x,y)$)

apply THM 2.

(b) — " — , $y(0) = 1$



Sol: $f = \dfrac{3x^2+4x+2}{2(y-1)}$, $\dfrac{\partial f}{\partial y} = -\dfrac{3x^2+4x+2}{2(y-1)^2}$

continuous away from $y = 1$
the line

(a) THM 2 guarantees that sol. exists and is unique for $-h<t<h$ for some h

(in fact, from explicit sol: it exists & is unique for $-2<t<\infty$)

(b) cannot draw a rectangle around $(0,1)$ s.t. $f, \frac{\partial f}{\partial y}$ are continuous in the rectangle. So, THM 2 does not say anything!

(in fact, from seperation of variables; $y = 1 \pm \sqrt{x^3+2x^2+2x}$ for $x>0$)

– two solutions, exist only to one side of init. cond.

Ex: $\boxed{y' = y^{1/3} , \; y(0) = 0 \text{ for } t \geq 0}$ $(**)$ (f)

apply THM 2; solve the init. val. prob.

Sol: $f = y^{1/3}$, $\dfrac{\partial f}{\partial y} = \dfrac{1}{3} y^{-2/3}$

continuous   discontinuous at $y = 0$ — ours $y_0 = 0$ is here

By Remark, sol. exists in an interval around $t=0$, but possibly not unique.

Solve (separation of var.) $y^{-\frac{1}{3}} dy = dt \rightsquigarrow \frac{3}{2} y^{2/3} = t + c \rightarrow y = \left(\frac{2}{3}(t+c)\right)^{3/2}$

$c = 0$ – from init. cond.

$y = \varphi_1(t) = \left(\frac{2}{3}t\right)^{3/2}$, $t \geq 0$

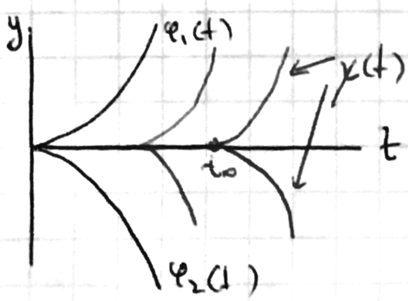
$y = \varphi_2(t) = -\left(\frac{2}{3}t\right)^{3/2}$, $t \geq 0$ – two solutions of $(**)$

There are even more!

$$y = \chi(t) = \begin{cases} 0, & 0 \leq t \leq t_0 \\ \pm\left(\frac{2}{3}(t-t_0)\right)^{3/2}, & t \geq t_0 \end{cases}$$

– differentiable everywhere, including t_0

for any $t_0 \geq 0$



– infinitely many solutions of the init. value problem $(**)$.

Interval of existence: $y' + p(t)y = g(t)$, $y(t_0) = y_0$ – solution exists and is unique on $\alpha < t < \beta$

solution can become singular only at values of t for which p or g is sing.

α: nearest singularity of p or g to the left of t_0; β: ... to the right of t_0

Solutions sometimes may remain continuous even at the point of discontinuity of coefficients:

$y' + \frac{2}{t}y = 4t$, $y(1) = 1 \rightsquigarrow y = t^2$

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For a non-linear eq., ~~the~~ interval of existence - difficult to determine

Ex: $y' = y^2$, $y(0) = 1$ determine the interval ~~of~~ on which the sol. exists.

Sol: $y^{-2}\,dy = dt \rightsquigarrow -y^{-1} = t + c \rightsquigarrow y = -\frac{1}{t+c}$ to satisfy init. cond., $c = -1$

So: $y = \frac{1}{1-t}$, interval of existence: $-\infty < t < 1$. Note: point $t=1$ does not seem remarkable in any way from the eq!

If the init. cond. is $y(0) = y_0$, then $c = -1/y_0$ and $y = \frac{y_0}{1 - y_0 t}$
$y_0 \neq 0$

interval of existence $-\infty < t < 1/y_0$ if $y_0 > 0$
and $1/y_0 < t < \infty$ if $y_0 < 0$

- singularities of solutions depend on the init. conditions.

linear eq $y' + p(t)y = g(t)$	non-linear eq.
there is a general sol., depending on c - a constant	- might be exceptional sol. (e.g. $y=0$ in Ex above)
sol. is given ~~by~~ explicitly, $y = \dots$	implicit sol. $F(t,y) = 0$
possible points of discontinuity of the sol. can be identified by finding the points of disc. of the coefficients	