## 3.1 Homogeneous equations with constant coefficients.



$2^{nd}$ order ODE: $\frac{d^2y}{dt^2} = f(t, y, \frac{dy}{dt})$ $(*)$

linear case: $y'' + p(t)y' + q(t)y = g(t)$ or $P(t)y'' + Q(t)y' + R(t)y = G(t)$ $(**)$

Initial condition for $2^{nd}$ order ODE: $y(t_0) = y_0$, $y'(t_0) = y_0'$ — fixed numbers

Linear eq. $(*, **)$ is homogeneous if r.h.s. is zero: $g(t) = 0$, $G(t) = 0$

Today: coefficients are constants, i.e. $\boxed{ay'' + by' + cy = 0}$

Ex: $y'' - y = 0$, $y(0) = 2$, $y'(0) = -1$ - Solve the IVP

Sol: look for solutions of $y'' = y$. $y_1(t) = e^t$, $y_2(t) = e^{-t}$

also, $c_1 e^t$ and $c_2 e^{-t}$ are solutions.

Note: a sum of two solutions is again a solution

Thus $y = c_1 y_1(t) + c_2 y_2(t) = \boxed{c_1 e^t + c_2 e^{-t}}$ a solution

check: $y' = c_1 e^t - c_2 e^{-t}$, $y'' = c_1 e^t + c_2 e^{-t} = y$ ✓

we have a 2-parameter family of solutions. Fix $c_1, c_2$ from init. cond.:

$y(0) = c_1 + c_2 = 2$
$y'(0) = c_1 - c_2 = -1$ $\Rightarrow$ $c_1 = 1/2$, $c_2 = 3/2$ $\Rightarrow$ $y = \frac{1}{2}e^t + \frac{3}{2}e^{-t}$ solution of the IVP

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Generally: $ay'' + by' + cy = 0$ $(\#)$ — $a, b, c$ real constants

Idea: try to find solutions of form $y = e^{rt}$ — $r$ to be determined

$y' = re^{rt}$, $y'' = r^2 e^{rt}$. So, $(\#) \rightsquigarrow (ar^2 + br + c)e^{rt} = 0$.

So, $y = e^{rt}$ a solution of $(\#)$ iff $\boxed{ar^2 + br + c = 0}$ (@)

"characteristic equation" of the eq. $(\#)$

Char. eq. can have
(1) two different real roots $r_1 \neq r_2$ ← assume this for today.
(2) two complex conjugate roots $r_1$, $r_2 = \bar{r}_1$
(3) repeated real root, $r_1 = r_2$

$y_1(t) = e^{r_1 t}$, $y_2(t) = e^{r_2 t}$ solutions ($r_1, r_2$ roots of char. eq.). Thus $y(t) = c_1 y_1(t) + c_2 y_2(t) = \boxed{c_1 e^{r_1 t} + c_2 e^{r_2 t}}$ also a solution - general solution of $(\#)$

Given an init. cond. $y(t_0)=y_0$, $y'(t_0)=y_0'$,
can determine uniquely $c_1, c_2$.

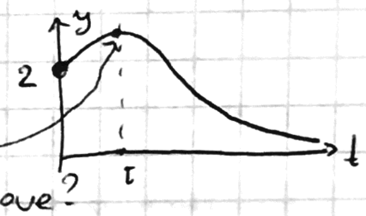
Ex $y''+5y'+6y=0$ (a) find the general sol. (b) solve the IVP $y(0)=2, y'(0)=3$

Sol ~~char. eq.~~ $y=e^{rt}$ a sol. iff $\underbrace{r^2+5r+6}_{(r+2)(r+3)}=0 \rightsquigarrow r_1=-2,\ r_2=-3$ roots

So, gen. sol.: (a) $y=c_1e^{-2t}+c_2e^{-3t}$.

(b) $y(0)=c_1+c_2=2$ $\qquad c_1=9$
$y'(0)=-2c_1-3c_2=3$ $\qquad c_2=-7$ $\Rightarrow \boxed{y=9e^{-2t}-7e^{-3t}}$ sol. of the IVP

$$\begin{bmatrix}1&1&2\\-2&-3&3\end{bmatrix}\sim\begin{bmatrix}1&1&2\\1&0&9\end{bmatrix}\sim\begin{bmatrix}1&0&9\\0&1&-7\end{bmatrix}$$




• find the location of the maximum point of the sol. above?

Sol: $y'(\tau)=0$ $\rightsquigarrow e^{\tau}=\frac{21}{18}=\frac{7}{6} \rightarrow \tau=\ln\frac{7}{6}\approx 0.15$
$\Leftrightarrow$ $-18e^{-2\tau}+21e^{-3\tau}$

$y(\tau)=9e^{-2\ln\frac{7}{6}}-7e^{-3\ln\frac{7}{6}}=9\cdot\left(\frac{7}{6}\right)^{-2}-7\left(\frac{7}{6}\right)^{-3}=\frac{9\cdot 6^2-6^3}{7^2}=\frac{108}{49}\approx 2.2$

thus, maximum is at $\left(t=\ln\frac{7}{6},\ y=\frac{108}{49}\right)$

Ex $4y''-8y'+3y=0$, $y(0)=2$, $y'(0)=\frac{1}{2}$

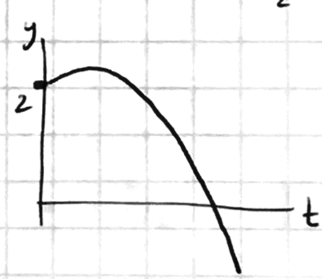
Sol: $4r^2-8r+3=0 \rightarrow r=\frac{8\pm\sqrt{16}}{8}$; $r_1=\frac{3}{2}$, $r_2=\frac{1}{2}$ - roots

$y=c_1e^{\frac{3}{2}t}+c_2e^{\frac{1}{2}t}$ - general solution.

$y(0)=c_1+c_2=2$
$y'(0)=\frac{3}{2}c_1+\frac{1}{2}c_2=\frac{1}{2}$

$$\begin{bmatrix}1&1&2\\\frac{3}{2}&\frac{1}{2}&\frac{1}{2}\end{bmatrix}\sim\begin{bmatrix}1&1&2\\3&1&1\end{bmatrix}\sim\begin{bmatrix}1&1&2\\2&0&-1\end{bmatrix}\sim\begin{bmatrix}0&1&\frac{5}{2}\\1&0&-\frac{1}{2}\end{bmatrix}\sim\begin{bmatrix}1&0&-\frac{1}{2}\\0&1&\frac{5}{2}\end{bmatrix}$$

$c_1=-1/2$ $\quad c_2=5/2$ $\qquad y=-\frac{1}{2}e^{3/2\,t}+\frac{5}{2}e^{\frac{1}{2}t}$

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Solution of $ay''+by'+cy=0$, as $t\to\infty$:

(a) $y\xrightarrow[t\to\infty]{}0$ if both exponents $r_{1,2}<0$

(b) $y\to\pm\infty$ if at least one exponent positive

(c) $y\to$ constant if $r_1=0$, $r_2<0$