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- $y'' + p(t)y' + q(t)y = 0$ p, q continuous real ~~sol~~ fun.

If $y = u(t) + iv(t)$ - complex-valued solution, then u and v are also solutions.

Abel's THM

$y'' + p(t)y' + q(t)y = 0$; let p, q be continuous ~~for~~ $\alpha < t < \beta$ and y_1, y_2 be solutions.

then $W[y_1, y_2](t) = c\exp\left(-\int p(t)\,dt\right)$. (#) c - const., depends on y_1, y_2, but not on t.

$W[y_1, y_2](t)$ is either zero for all t (if $c=0$), or else nonzero everywhere in $\alpha < t < \beta$.

Idea:
$$\begin{aligned} &y_1'' + p y_1' + q y_1 = 0 \mid \cdot(-y_2) \\ + \; &y_2'' + p y_2' + q y_2 = 0 \mid \cdot y_1 \\ \hline &\underbrace{y_1 y_2'' - y_1'' y_2}_{(y_1 y_2' - y_2 y_1')'} + p(y_1 y_2' - y_2 y_1') = 0 \end{aligned}$$

So: $W' + p(t)W = 0 \longrightarrow W = ce^{-\int p(t)\,dt}$

Ex: $2t^2 y'' + 3t y' - y = 0$, $t > 0$ $\quad y_1 = t^{1/2}$, $y_2 = t^{-1}$. Verify that W is given by Abel's f-la (#).

Sol: we found $W = -\frac{3}{2} t^{-\frac{3}{2}}$

$\hookrightarrow y'' + \underbrace{\frac{3}{2}t^{-1}}_{p} y' \underbrace{- \frac{1}{2t^2}}_{q} y = 0 \rightarrow W = ce^{-\int \frac{3}{2}t^{-1}dt} = ce^{-\frac{3}{2}\ln t} = c \cdot t^{-3/2}$ ✓ $c = -\frac{3}{2}$

3.3. Complex roots of char. eq.

$ay'' + by' + cy = 0$ $\quad a, b, c$ real $\quad y = e^{rt}$ is a sol. iff $\underbrace{ar^2 + br + c = 0}_{\text{char. eq.}}$

$\underbrace{r_1, r_2 \text{ real}, r_1 \neq r_2}_{\text{if } b^2 - 4ac > 0} \rightarrow$ general sol.: $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$.

Suppose $b^2 - 4ac < 0 \rightarrow \begin{matrix} r_1 = \lambda + i\mu \\ r_2 = \lambda - i\mu \end{matrix} \rightarrow \begin{matrix} y_1 = e^{(\lambda + i\mu)t} \\ y_2 = e^{(\lambda - i\mu)t} \end{matrix}$

- Euler's formula: $e^{it} = \cos t + i \sin t$ $\rightarrow e^{s+it} = e^s(\cos t + i\sin t)$, $e^{s+it} = e^s e^{it}$

Then $e^z e^{z'} = e^{z+z'}$ for z, z' complex

$e^{(\lambda + i\mu)t} = e^{\lambda t}(\cos \mu t + i \sin \mu t)$

One has: $\frac{d}{dt} e^{rt} = re^{rt}$ (r complex)

Ex: $y'' + y' + 9.25y = 0$, $y(0) = 2$, $y'(0) = 8$

Sol: $r^2 + r + 9.25 = 0$ $\quad r = \frac{-1 \pm \sqrt{1 - 37}}{2}$ $\quad r_1 = -\frac{1}{2} + 3i$, $r_2 = -\frac{1}{2} - 3i$

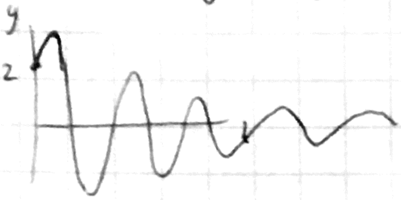
$\left.\begin{aligned} y_1 &= e^{(-\frac{1}{2} + 3i)t} = e^{-\frac{t}{2}}(\cos 3t + i \sin 3t) \\ y_2 &= e^{(-\frac{1}{2} - 3i)t} = e^{-\frac{t}{2}}(\cos 3t - i \sin 3t) \end{aligned}\right\} \xrightarrow[\text{real solutions}]{}$ $\begin{aligned} u(t) &= \operatorname{Re} y_1 = e^{-t/2}\cos 3t \\ v(t) &= \operatorname{Im} y_1 = e^{-t/2}\sin 3t \end{aligned}$

Wronskian: $W[u,v] = \begin{vmatrix} e^{-t/2}\cos 3t & e^{-\frac{t}{2}}\sin 3t \\ e^{-\frac{t}{2}}(-\frac{1}{2}\cos 3t - 3\sin 3t) & e^{-\frac{t}{2}}(3\cos 3t - \frac{1}{2}\sin 3t) \end{vmatrix}$

$= 3e^{-t} \neq 0$ so, u,v - FSS

$y = c_1 e^{-\frac{t}{2}}\cos 3t + c_2 e^{-\frac{t}{2}}\sin 3t = e^{-\frac{t}{2}}(c_1\cos 3t + c_2 \sin 3t)$ - general solution

Init cond. $y(0) = c_1 = 2$, $y'(0) = -\frac{1}{2}c_1 + 3c_2 = 8$ $\rightarrow$ $c_1 = 2$, $c_2 = 3$ $\rightarrow$ $y = e^{-\frac{t}{2}}(2\cos 3t + 3\sin 3t)$ - sol of the IVP



- the sol. oscillates with period $2\pi/3$, with decaying amplitude.

Generally: $ay'' + by' + cy = 0$, $r = \lambda \pm i\mu$ $(\mu \neq 0)$ $\rightarrow$ $y_1 = e^{(\lambda + i\mu)t}$, $y_2 = e^{(\lambda - i\mu)t}$ - complex solutions (FSS) $\rightarrow$ $u(t) = e^{\lambda t}\cos\mu t$, $v(t) = e^{\lambda t}\sin\mu t$ Real FSS

$W[u,v](t) = \mu e^{2\lambda t} \neq 0$

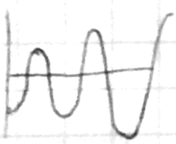
General sol: $\boxed{y = e^{\lambda t}(c_1\cos\mu t + c_2 \sin\mu t)}$

Ex: $16y'' - 8y' + 145y = 0$ $y(0) = -2$ $y'(0) = 1$

Sol: char. eq. $16r^2 - 8r + 145 = 0$ $r = \frac{8 \pm \sqrt{64(1-145)}}{32} = \frac{1 \pm i12}{4} = \frac{1}{4} \pm 3i$

gen sol.: $y = e^{t/4}(c_1\cos 3t + c_2\sin 3t)$

$y(0) = c_1 = -2$ $y'(0) = \frac{1}{4}c_1 + 3c_2 = 1$ $\rightarrow c_2 = \frac{1}{2}$ $\rightarrow y = e^{t/4}(-2\cos 3t + \frac{1}{2}\sin 3t)$



- growing oscillation. (since exponent is positive).

Ex $y'' + 9y = 0$. Find gen. sol.

Sol $r^2 + 9 = 0$ $r = \pm 3i$ $\rightarrow$ gen. sol: $y = c_1\cos 3t + c_2 \sin 3t$

pure oscillations of constant amplitude, period $2\pi/3$ (amplitude & phase shift determined by init. cond.)