3.5 Nonhomogeneous equations. Method of undetermined coefficients

04/27/2018

(1) $y'' + p(t)y' + q(t)y = g(t)$ - nonhomog. eq.

same

(2) $y'' + p(t)y' + q(t)y = 0$ - corresponding homog. eq.

THM a) If Y_1, Y_2 - two sol. of nonhomog. eq. (1), then $Y_1 - Y_2$ - sol. of corresp. homog. eq. (2)

If y_1, y_2 - FSS for (2), then $Y_1 - Y_2 = c_1 y_1 + c_2 y_2$ for some c_1, c_2 constants

(b) The general sol. of (1) is $y = \underbrace{c_1 y_1(t) + c_2 y_2(t)}_{y_c \text{ - general sol of (2) - "complementary solution"}} + \underbrace{Y(t)}_{\text{any sol. of (1) - "particular solution"}}$

How to find a particular solution Y?

- method of undetermined coefficients - assume a form of Y keeping some coeff. undetermined; obtain their values from substituting into (1). (ansatz)

Ex 1: $y'' - 3y' - 4y = 3e^{2t}$. Find Y.

Sol: try looking for Y in the form $Y = \underbrace{A}_{\text{coeff. to be determined}} e^{2t} \rightsquigarrow Y' = 2Ae^{2t} \rightsquigarrow Y'' = 4Ae^{2t}$

$Y'' - 3Y' - 4Y = \underbrace{(4A - 6A - 4A)}_{-6A} e^{2t} = 3e^{2t} \rightsquigarrow A = -\frac{1}{2} \rightsquigarrow \boxed{Y = -\frac{1}{2}e^{2t}}$ - a solution!

Ex 2: $y'' - 3y' - 4y = 2\sin t$

Sol: Try $Y = A\sin t \rightsquigarrow Y' = A\cos t \rightsquigarrow Y'' = -A\sin t$ $\quad Y'' - 3Y' - 4Y = \underbrace{(-A - 4A)}_{-5A}\sin t - 3A\cos t \overset{\text{WANT}}{=} 2\sin t$

$t = 0 \rightsquigarrow -3A = 0$
$t = \frac{\pi}{2} \rightsquigarrow -5A = 2$ } inconsistent!! → cannot find A s.t. $Y = A\sin t$ is a solution!

Next try: $Y = A\sin t + B\cos t \rightarrow Y' = -B\sin t + A\cos t \rightarrow Y'' = -A\sin t - B\cos t$

$Y'' - 3Y' - 4Y = \underbrace{(-A + 3B - 4A)}_{-5A+3B}\sin t + \underbrace{(-B - 3A - 4B)}_{-3A-5B}\cos t \overset{\text{WANT}}{=} 2\sin t$

$-5A + 3B = 2$
$-3A - 5B = 0$ } $\rightsquigarrow A = -\frac{5}{17}, B = \frac{3}{17} \rightsquigarrow \boxed{Y = -\frac{5}{17}\sin t + \frac{3}{17}\cos t}$ a solution.

- for ~~when~~ $g(t)$ a polynomial, try Y a polynomial of same degree, e.g.

$y'' - 3y' - 4y = 4t^2 - 1 \rightsquigarrow$ try $Y = At^2 + Bt + C$

Thus: • for $g(t) = e^{\alpha t}$, try $Y = Ae^{\alpha t}$

• for $g(t) = \sin\beta t$ or $\cos\beta t$, try $Y = A\sin\beta t + B\cos\beta t$

• for $g(t)$ a polynomial of degree n, try Y a polynomial of deg. n.

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§ For g a product of two or three of these types of functions ~~same appl~~ take a product of these forms of Y.

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Ex 3 $y''-3y'-4y=-8e^t\cos 2t$

Sol: Try $Y=Ae^t\cos 2t+Be^t\sin 2t \rightsquigarrow Y'=(A+2B)e^t\cos 2t+(-2A+B)e^t\sin 2t$

$\rightsquigarrow Y''=(\underbrace{A+2B-4A+2B}_{-3A+4B})e^t\cos 2t+(\underbrace{-2A+B-2A-4B}_{-4A-3B})e^t\sin 2t$

$Y''-3Y'-4Y=(\underbrace{(-3A+4B)-3(A+2B)-4A}_{-10A+2B})e^t\cos 2t+(\underbrace{(-4A-3B)-3(-2A+B)-4B}_{2A-10B})e^t\sin 2t \overset{\text{WANT}}{=} -8e^t\cos 2t$

$\left.\begin{aligned}-10A-2B&=-8\\ 2A-10B&=0\end{aligned}\right\}$ $A=\frac{80}{104}=\frac{10}{13}$, $B=\frac{16}{104}=\frac{2}{13}$ $\rightsquigarrow \boxed{Y=\frac{10}{13}e^t\cos 2t+\frac{2}{13}e^t\sin 2t}$

- If $g(t)=g_1(t)+g_2(t)$ and Y_1, Y_2 are solutions of $ay''+by'+cy=g_1(t)$, $ay''+by'+cy=g_2(t)$ then $Y=Y_1+Y_2$ is a sol. of $ay''+by'+cy=g(t)$.

Ex 4: $y''-3y'-4y=3e^{2t}+2\sin t$ ~~$+8e^t\cos 2t$~~

Sol: $y''-3y'-4y=3e^{2t} \rightarrow Y_1=-\frac{1}{2}e^{2t}$

$y''-3y'-4y=2\sin t \rightarrow Y_2=-\frac{5}{17}\sin t+\frac{3}{17}\cos t$

~~given~~ $\rightsquigarrow Y=Y_1+Y_2=-\frac{1}{2}e^{2t}-\frac{5}{17}\sin t+\frac{3}{17}\cos t$ - a sol.

Ex 5 $y''-3y'-4y=2e^{-t}$ $(*)$

Sol: try $Y=Ae^{-t} \rightarrow Y'=-Ae^{-t} \rightarrow Y''=Ae^{-t} \rightarrow Y''-3Y'-4Y=\underbrace{(A+3A-4A)}_{0}e^{-t} \overset{\text{WANT}}{=} 2e^{-t}$

- does not work! because e^{-t} is a sol. of homog. eq.

Toy problem:
Consider $y'+y=2e^{-t}$ $Y=Ae^{-t}$ fails for same reason: e^{-t} is a sol. of $y'+y=0$
Known by integr. factor $\mu(t)=e^t \rightsquigarrow y=\underbrace{2te^{-t}}_{Y}+\underbrace{ce^{-t}}_{\text{sol. of homog. eq.}}$

back to $(*)$: try $Y=Ate^{-t} \rightarrow Y'=A(1-t)e^{-t} \rightarrow Y''=A(t-2)e^{-t}$

$Y''-3Y'-4Y=A(\underbrace{(t-2)-3(1-t)-4t}_{0\cdot t-5})e^{-t}=2e^{-t} \Rightarrow A=-\frac{2}{5} \rightarrow \boxed{Y=-\frac{2}{5}te^{-t}}$

- If assumed form of the solution (ansatz) duplicates a sol. of the corresp. homog. eq., then modify then modify it by multiplying by t. In case it is again a sol. of homog. eq., multiply by t a second t (for second-order eq., further iterations cannot happen).

So: ~~for $g=g_1+\dots+g_N$ with each $g_i=\begin{cases}P_n(t)=a_nt^n\\ P_n(t)\end{cases}$~~

for $\boxed{ay''+by'+cy=g(t)}$:

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$g(t)=g_1(t)+\dots+g_N(t)$

$$g_i(t): \begin{cases} P_n(t)=a_0t^n+a_1t^{n-1}+\dots+a_n \\ P_n(t)e^{\alpha t} \\ P_n(t)e^{\alpha t}\begin{cases}\sin\beta t\\ \cos\beta t\end{cases} \end{cases}$$

$$Y_i=t^{s}(A_0t^n+A_1t^{n-1}+\dots+A_n)$$

$$Y_i=t^s(A_0t^n+\dots+A_n)e^{\alpha t}$$

$$Y_i=t^s((A_0t^n+\dots+A_n)e^{\alpha t}\cos\beta t+$$
$$+(B_0t^n+\dots+B_n)e^{\alpha t}\sin\beta t)$$

$s\in\{0,1,2\}$ smallest number that ensures that no term in $Y_i(t)$ is a sol. of homog eq.

$\rightarrow\ Y=Y_1+\dots+Y_N$.