3.6 Variation of parameters

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finding a particular sol. Y of a 2^{nd} order ODE
linear, nonhomogeneous

→ undetermined coeff [need to assume a form of Y based on $g(t)$]

→ variation of parameters - general method [but requires potentially complicated integrals]

Ex: Find the general sol. of $y''+4y=8\tan t$ $(*)$, $-\frac{\pi}{2}<t<\frac{\pi}{2}$

- rhs - not a sum or product of sin, cos
  → undetermined coeff cannot be applied

homog. eq.: $y''+4y=0 \rightsquigarrow y_c(t)=c_1\cos 2t+c_2\sin 2t$.

Idea: try to find a sol. of $(*)$ in the form $y=u_1(t)\cos 2t+u_2(t)\sin 2t$

$$y'=-2u_1(t)\sin 2t+2u_2(t)\cos 2t+u_1'(t)\cos 2t+u_2'(t)\sin 2t$$

Idea (Lagrange): to have two eq. on u_1, u_2, impose an auxiliary eq. $\boxed{u_1'(t)\cos 2t+u_2'(t)\sin 2t=0}$

then:

$$y'=-2u_1\sin 2t+2u_2\cos 2t$$

$$y''=-4u_1\cos 2t-4u_2\sin 2t-2u_1'\sin 2t+2u_2'\cos 2t$$

$y''+4y=$ $\begin{cases}-2u_1'(t)\sin 2t+2u_2'(t)\cos 2t \overset{\text{WANT}}{=} 8\tan t\\ u_1'(t)\cos 2t+u_2'(t)\sin 2t=0\end{cases}$. system of two linear eq. on $u_1'(t), u_2'(t)$

auxiliary

$$u_1'(t)=\frac{\begin{vmatrix}8\tan t & 2\cos 2t\\ 0 & \sin 2t\end{vmatrix}}{\begin{vmatrix}-2\sin 2t & 2\cos 2t\\ \cos 2t & \sin 2t\end{vmatrix}}=\frac{8\overset{\frac{\sin t}{\cos t}}{\tan t}\,\overset{2\sin t\cdot\cos t}{\sin 2t}}{-2}=-8\sin^2 t$$

$$u_2'(t)=\frac{\begin{vmatrix}-2\sin 2t & 8\tan t\\ \cos 2t & 0\end{vmatrix}}{|\dots|}=\frac{-8\overset{\frac{\sin t}{\cos t}}{\tan t}\,\overset{2\cos^2 t-1}{\cos 2t}}{-2}=4\sin t\left(2\cos t-\frac{1}{\cos t}\right)$$

~~$\sin^2 t = 1-\cos^2$~~
$\cos 2t=2\cos^2 t-1$
$=1-2\sin^2 t$

$$u_1'(t)=\underbrace{-8\sin^2 t}_{-4+4\cos 2t}\ \xrightarrow[\text{integrate}]{u_1(t)=}\ \underbrace{4\sin t\cos t}_{2\sin 2t}-4t+c_1$$

$$u_2'(t)=\underbrace{4\sin t\left(2\cos t-\frac{1}{\cos t}\right)}_{4\sin 2t-4\frac{\sin t}{\cos t}}\ \xrightarrow[\text{integrate}]{u_2(t)=}\ 4\ln\cos t\ \underbrace{-4\cos^2 t}_{2(\cos 2t+1)}+c_2$$

Thus: $y=u_1\cos 2t+u_2\sin 2t=2\sin 2t\,\cancel{\cos 2t}-4t\cos 2t+4\ln(\cos t)\sin 2t-2(\cos 2t+1)\sin 2t+c_1\cos 2t+c_2\sin 2t$

$$=\underbrace{-2\sin 2t-4t\cos 2t+4\ln(\cos t)\sin 2t}_{Y}\ \underbrace{+c_1\cos 2t+c_2\sin 2t}_{y_c\text{ - sol. of homog. eq.}}$$

← General solution of $(*)$

Generally: $y''+p(t)y'+q(t)y=g(t)$ (@). Assume we know the gen. sol. of ~~y_c~~

$y_c=c_1y_1(t)+c_2y_2(t)$ of $y''+py'+qy=0$ (homog. eq.)

try: $y=u_1(t)y_1(t)+u_2(t)y_2(t)$ for (@):

$$y'=u_1y_1'+u_2y_2'+\underbrace{u_1'y_1+u_2'y_2}_{=0\ \text{- auxiliary eq.}}$$

$$y''=u_1y_1''+u_2y_2''+u_1'y_1'+u_2'y_2'$$

$$y'' + py' + qy = u_1\cancel{(y_1'' + py_1' + qy_1)} + u_2\cancel{(y_2'' + py_2' + qy_2)}$$

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$$+ y_1' u_1' + y_2' u_2' = g(t)$$

$$y_1 u_1' + y_2 u_2' = 0$$

$$\rightsquigarrow u_1'(t) = -\frac{y_2(t)\, g(t)}{W[y_1, y_2](t)} \rightsquigarrow u_1(t) = -\int \frac{y_2 g}{W}\, dt + c_1$$

$$u_2'(t) = \frac{y_1(t)\, g(t)}{W[y_1, y_2](t)} \rightsquigarrow u_2(t) = \int \frac{y_1 g}{W}\, dt + c_2$$

THM Consider eq. $y'' + p(t)y' + q(t)y = g(t)$ (@), p, q, g continuous on an interval I

Assume y_1, y_2 are a FSS of the homog. eq. $y'' + p(t)y' + q(t) = 0$

Then a particular sol. of (@) is:

$$Y(t) = -y_1(t)\int_{t_0}^{t} \frac{y_2(s)\, g(s)}{W[y_1, y_2](s)}\, ds + y_2(t)\int_{t_0}^{t} \frac{y_1(s)\, g(s)}{W[y_1, y_2](s)}\, ds$$

t_0 - a chosen point in I.

The gen. sol of (@) is: $y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$