04/04/2018
1

1.3. Classification of differential equations

- ordinary and partial diff. eq. (ODE and PDE)

ODE: unknown fun. depends on a single independent variable t

Ex: falling object (1) $v(t)$, mice-owls (2) $p(t)$; $L\dfrac{d^2Q(t)}{dt^2} + R\dfrac{dQ(t)}{dt} + \dfrac{1}{C}Q(t) = E(t)$ (3)

- eq. on $Q(t)$ - charge on a capacitor in an electric circuit.

PDE: (4) $\alpha^2 \dfrac{\partial^2 u(x,t)}{\partial x^2} = \dfrac{\partial u(x,t)}{\partial t}$ heat equation

(5) $a^2 \dfrac{\partial^2 u(x,t)}{\partial x^2} = \dfrac{\partial^2 u(x,t)}{\partial t^2}$ wave equation

- ~~single diff eq. vs~~ systems of diff. eq.

if there are several unknown functions

Lotka-Volterra equations: (predator-prey)

(6) $\dfrac{dx}{dt} = ax - \alpha xy$, $\dfrac{dy}{dt} = -cy + \gamma xy$ - system of eq. for $x(t)$ - prey population, $y(t)$ - predator population

- Order

Order of an eq. - highest derivative that appears in the eq.

(*) $F(t, y, y', \dots, y^{(n)}) = 0$ ODE of order n.

we assume that (*) - can be solved for $y^{(n)}$, expressing it via $t, y, y', \dots, y^{(n-1)}$, (**) $y^{(n)} = f(t, y, y', \dots, y^{(n-1)})$

Ex: (7) $y''' + 2e^t y'' + yy' = t^4$ - 3rd order ODE on $y(t)$

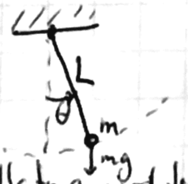
- Linear vs nonlinear diff. eq.

an ODE $F(t, y, y', \dots, y^{(n)}) = 0$ is linear if F is a linear function in $y, y', \dots, y^{(n)}$. (no linearity in t assumed!)

General linear n-th order ODE: $a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \dots + a_n(t)y = g(t)$

Ex: falling object, mice-owls, (3), (4,5) - lin. PDE

Ex: (7) is non-linear, because of yy' term. (6) are non-linear (each equation) due to xy terms.

Ex: 

oscillating pendulum

(8) $\dfrac{d^2\theta}{dt^2} + \dfrac{g}{L}\sin\theta = 0$ $\theta(t)$ - unknown function - non-linear ODE! due to $\sin\theta$ term.

04/04/2018

linear eq. - easier, well-developed theory
non-linear eq. - much harder, less satisfactory methods of solution.
non-linear can (sometimes) be approximated by linear:
for θ small, $\sin\theta \simeq \theta$ and (8) can be approximated by $\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$ - linear eq.
- "linearization" of (8).

<u>Solutions</u> a <u>solution</u> of the n^{th} order ODE (**)
on the interval $\alpha < t < \beta$ is a fun. φ s.t. $\varphi', \varphi'', \dots, \varphi^{(n)}$ exist and satisfy
$\varphi^{(n)}(t) = f(t, \varphi(t), \varphi'(t), \dots, \varphi^{(n-1)}(t))$ for every $\alpha < t < \beta$.

Ex: $\frac{dp}{dt} = \frac{p}{2} - 450$ has the sol. $p(t) = 900 + c\, e^{t/2}$ (***), c arbitrary const.

given an eq., it is generally not easy to find a sol.; given a fun. - easy to verify whether it is a sol. (by substitution).

<u>Ex:</u> $y'' + y = 0$, Q: $y_1(t) = \cos t$ a solution?
Sol: $y_1'(t) = -\sin t$, $y_1''(t) = -\cos t \Rightarrow y_1'' + y_1 = 0$ ✓

Questions
<u>Existence:</u> (**) does not ~~automatically~~ always have solutions. (but for some classes of eq., it does)
<u>Uniqueness:</u> usually, solutions come in a family like (***),
but one may ask about uniqueness for the init. value problem.
<u>Determining actual solutions (explicitly)</u> - not always possible; sometimes can do only numerically.

2.1 Integrating factors

general 1st. order linear ODE $\boxed{\frac{dy}{dt} + p(t)\,y = g(t)}$ or $\boxed{P(t)\frac{dy}{dt} + Q(t)\,y = G(t)}$ (*)

divide by $P(t)$ if $P(t) \neq 0$

<u>Ex:</u> $\boxed{(4+t^2)\frac{dy}{dt} + 2ty = 4t}$ (1)
$\underbrace{\phantom{(4+t^2)\frac{dy}{dt} + 2ty}}$ $= \frac{d}{dt}((4+t^2)y)$

derivative of a product

Thus: eq. (1) $\Leftrightarrow \frac{d}{dt}((4+t^2)y) = 4t$
$\xrightarrow[\text{w.r.t. } t]{\text{integrate}} (4+t^2)y = 2t^2 + \underbrace{c}_{\text{arbitrary const of integration}} \xrightarrow{\text{solve for } y} \boxed{y = \frac{2t^2}{4+t^2} + \frac{c}{4+t^2}}$

- general sol. for (1).

Generally lhs of (*) is not a derivative of a product
Idea (Leibniz): find a function $\mu(t)$ s.t. once we multiply (*) by $\mu(t)$, lhs becomes a der. of a product.

Ex: $\frac{dy}{dt} + \frac{1}{2}y = \frac{1}{2}e^{t/3}$ (2)   find the general sol.

Sol: $\underbrace{\mu(t)}_{}\cdot(2)$: $\underbrace{\mu(t)\frac{dy}{dt} + \frac{1}{2}\mu(t)y}_{} = \frac{1}{2}\mu(t)e^{t/3}$ (4)

(yet undetermined function)

WANT IT to be $\frac{d}{dt}(\mu(t)y) = \mu(t)\frac{dy}{dt} + \frac{d\mu(t)}{dt}y$ - this is true if

$\frac{d\mu(t)}{dt} = \frac{1}{2}\mu(t)$ (3)

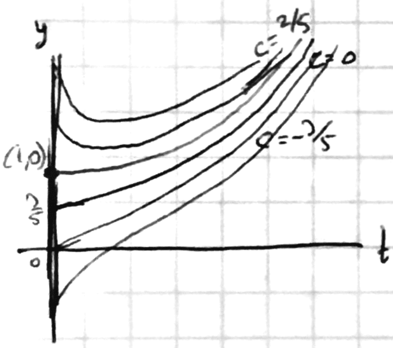
Solve (3): $\frac{1}{\mu(t)}\frac{d\mu(t)}{dt}\cancel{\mu(t)dt} = \frac{1}{2} \rightsquigarrow \frac{d}{dt}\ln|\mu(t)| = \frac{1}{2}$

$\rightsquigarrow \ln|\mu(t)| = \frac{t}{2} + C \to \mu(t) = c\,e^{t/2}$. We do not need the most general $\mu(t)$, just need one, $\mu \neq 0$. Choose $c=1$.

$\rightsquigarrow \mu(t) = e^{t/2}$

So: (4) is $\underbrace{e^{t/2}\frac{dy}{dt} + \frac{1}{2}e^{t/2}y}_{\frac{d}{dt}(e^{t/2}y)} = \frac{1}{2}e^{5t/6} \xrightarrow{\text{integrate}} e^{t/2}y = \frac{3}{5}e^{5t/6} + c$

$\xrightarrow{\text{solve for } y} y = \frac{3}{5}e^{t/3} + c\,e^{-t/2}$ (5)



Ex: (2) + init. condition $y(0) = 1$

$y(0) = \frac{3}{5} + c = 1 \Rightarrow c = \frac{2}{5} \Rightarrow \boxed{y(t) = \frac{3}{5}e^{t/3} + \frac{2}{5}e^{-t/5}}$

↑ from gen. sol. (5)