04/04/2018
3

Ex: $\frac{dy}{dt} + \frac{1}{2} y = \frac{1}{2} e^{t/3}$ (2) find the general sol.

Sol: $\underbrace{\mu(t)}_{}\cdot (2)$: $\underbrace{\mu(t)\frac{dy}{dt} + \frac{1}{2}\mu(t) y}_{} = \frac{1}{2}\mu(t) e^{t/3}$ (4)

(yet) undetermined function

WANT IT to be $\frac{d}{dt}(\mu(t) y) = \mu(t)\frac{dy}{dt} + \frac{d\mu(t)}{dt} y$ - this is true if $\frac{d\mu(t)}{dt} = \frac{1}{2}\mu(t)$ (3)

Solve (3): $\frac{1}{\mu(t)}\frac{d\mu(t)}{dt}\cancel{\mu(t)} = \frac{1}{2} \rightarrow \frac{d}{dt}\ln|\mu(t)| = \frac{1}{2}$

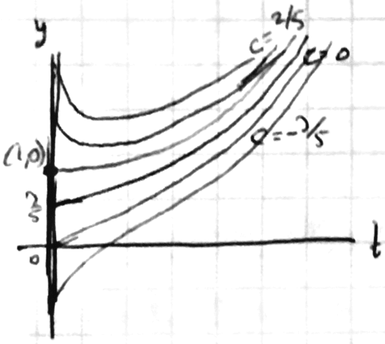
$\rightarrow \ln|\mu(t)| = \frac{t}{2} + C \rightarrow \mu(t) = c e^{t/2}$ We do not need the most general $\mu(t)$, just need one, $\mu \neq 0$. Choose $c=1$.

5.3 (1M) $\rightarrow \mu(t) = e^{t/2}$

04/06/2018
1

So: (4) is $\underbrace{e^{t/2}\frac{dy}{dt} + \frac{1}{2}e^{t/2} y}_{\frac{d}{dt}(e^{t/2} y)} = \frac{1}{2}e^{5t/6} \xrightarrow{\text{integrate}} e^{t/2} y = \frac{3}{5}e^{5t/6} + c$

$\xrightarrow{\text{solve for } y} y = \frac{3}{5}e^{t/3} + c e^{-t/2}$ (5)



Ex: (2) + init. condition $y(0) = 1$

$y(0) = \frac{3}{5} + c = 1 \Rightarrow c = \frac{2}{5} \Rightarrow \boxed{y(t) = \frac{3}{5}e^{t/3} + \frac{2}{5}e^{-t/5}}$

from gen. sol. (5)

General case: $\frac{dy}{dt} + p(t) y = g(t)$ (*)

$\xrightarrow{\cdot\mu(t)}$ $\underbrace{\mu(t)\frac{dy}{dt} + p(t)\mu(t) y}_{=} = \mu(t) g(t)$ (**)

yet undetermined - integrating factor

$\frac{d}{dt}(\mu(t) y)$ if $\frac{d\mu(t)}{dt} = p(t)\mu(t) \rightarrow \boxed{\frac{1}{\mu(t)}\frac{d\mu(t)}{dt} = p(t)}$ (***) $\xrightarrow{\text{integrate}}$

$\rightarrow \ln|\mu(t)| = \int p(t)\,dt + k \rightarrow \mu(t) = C e^{\int p(t)\,dt} \xrightarrow{\text{choose } C=1} \boxed{\mu(t) = e^{\int p(t)\,dt}}$ ← integrating factor

-simplest solution -sol. of (***)

Thus (**) becomes: $\frac{d}{dt}(\mu(t) y) = \mu(t) g(t)$

$\rightarrow \mu(t) y = \int \mu(t) g(t)\,dt + c \rightarrow \boxed{y = \frac{1}{\mu(t)}\left(\int_{t_0}^{t}\mu(s) g(s)\,ds + c\right)}$

Note: solution involves two integrations (for μ and for y)

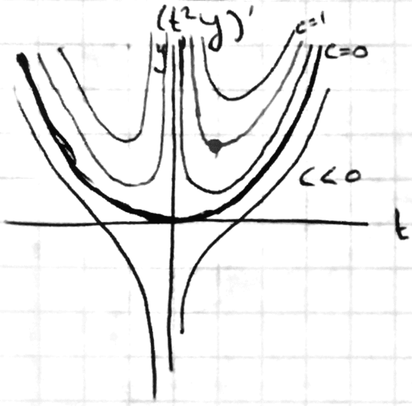
04/06/2018
2

Ex: $ty' + 2y = 4t^2$ (#)

$y(1) = 2$ - init condition

Sol $\xrightarrow{\frac{1}{t}}$ $y' + \underbrace{\frac{2}{t}}_{p(t)} y = \underbrace{4t}_{g(t)}$ (##) Integrating factor: $\mu(t) = e^{\int \frac{2}{t} dt} = e^{2\ln|t|} = t^2$.

$\cdot\mu(t)$: $\underbrace{t^2 y' + 2ty}_{(t^2y)'} = 4t^3 \xrightarrow[\text{integrate}]{} t^2 y = t^4 + C \xrightarrow[\text{for } t>0]{\text{solve for } y} \boxed{y = t^2 + \frac{C}{t^2}}$



to satisfy init. cond. $\underset{1+C}{y(1)} = 2 \Rightarrow C = 1$

$\Rightarrow \boxed{y = t^2 + \frac{1}{t^2}}, t > 0$

- sol. of the init. val. problem

Note: solution becomes unbounded at $t \to 0$
(due to discontinuity of $p(t)$ at $t \to 0$)

$y = t^2 + \frac{1}{t^2}, t < 0$ part of general solution of (#) but not part of the solution of the init. val. prob.

2.2 Separable differential equations

First order ODE $\frac{dy}{dt} = f(t,y)$ is separable if it can be written as $\underset{\text{dep. only on } t}{M(t)} + \underset{\text{depends only on } y}{N(y)} \frac{dy}{dt} = 0$

or in differential form: $M(t)\,dt + N(y)\,dy = 0$

- solved by integrating M and N.

Ex: $\frac{dy}{dt} = \frac{t^2}{1-y^2}$ (@)

rewrite: $\underbrace{-t^2}_{M(t)} + \underbrace{(1-y^2)}_{N(y)} \frac{dy}{dt} = 0$ - separable

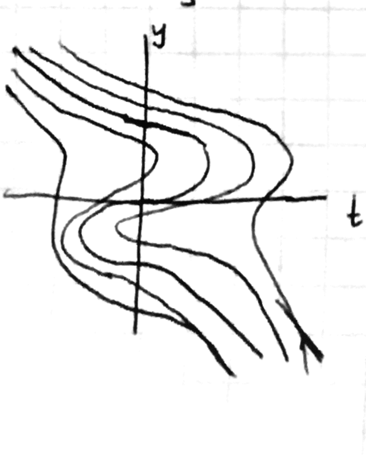
Recall the chain rule:
$$\frac{d}{dt} f(y) = \frac{d}{dy} f(y) \cdot \frac{dy}{dt} = f'(y) \frac{dy}{dt}$$

$\frac{d}{dt}\left(y - \frac{y^3}{3}\right) = (1-y^2)\frac{dy}{dt}$

So, eq. (@) is: $\frac{d}{dt}\left(-\frac{t^3}{3} + y - \frac{y^3}{3}\right) = 0$

$\Leftrightarrow \boxed{-t^3 + 3y - y^3 = C}$ - equation for integral curves

any differentiable $\varphi(t)$ satisfying is a solution of (@)



Generally, if H_1, H_2 - anti-derivatives for M, N: $H_1'(t) = M(t)$, $H_2'(y) = N(y)$,

eq. $M(t) + N(y)\frac{dy}{dt} = 0$ becomes $H_1'(t) + \underbrace{H_2'(y)\frac{dy}{dt}}_{\frac{d}{dt}H_2(y)} = 0$

$\rightsquigarrow \frac{d}{dt}\left(H_1(t) + H_2(y)\right) = 0 \rightarrow \boxed{H_1(t) + H_2(y) = C}$ implicit description of solutions if init. cond. $y(t_0)=y_0$ is given, find c from $H_1(t_0)+H_2(y_0)=C$.

04/06/2018

Ex: $\dfrac{dy}{dt} = \dfrac{3t^2+4t+2}{2(y-1)}$, $y(0)=-1$. - init. val prob.

Q: - find the solution
- determine the interval on which the sol. exists.

Sol: $2(y-1)\,dy = (3t^2+4t+2)\,dt$

$\underset{\text{integrate}}{\sim}$ $y^2-2y = t^3+2t^2+2t+C$

to satisfy init. cond. $y(0)=-1$: $(-1)^2-2(-1)=C$
$\Rightarrow C=3$

$\Rightarrow y^2-2y = t^3+2t^2+2t+3$

$\Rightarrow y = 1 \pm \sqrt{1+(t^3+2t^2+2t+3)} = 1-\underbrace{\sqrt{t^3+2t^2+2t+4}}_{(t+2)(t^2+2)}$

only $-$ ~~gives~~ satisfies the init. cond.

solution exists for $t>-2$

at $t=-2$ expression under $\sqrt{\cdots}$ vanishes.