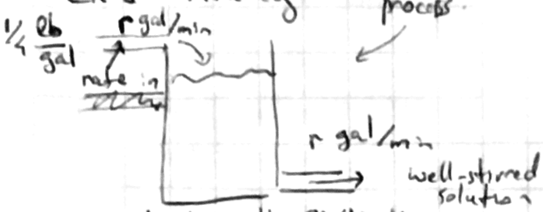
2.3 Modeling with 1^{st} order diff. equations

04/09/2018

1

Ex 1: mixing

process:



tank with $Q(t)$ lb of salt dissolved in 100 gal of water

at $t=0$, $Q=Q_0$

model:

$$\underbrace{\frac{dQ}{dt}}_{\text{rate of change of amount of salt in the tank}} = \underbrace{\text{rate (of salt) in}}_{\frac{1}{4}\cdot r} - \underbrace{\text{rate out}}_{r\cdot\frac{Q(t)}{100}} \quad \leftarrow \text{conservation of matter law}$$

$\frac{Q(t)}{100}$ concentration of salt in the tank

$$\boxed{\frac{dQ}{dt} = \frac{1}{4}r - \frac{rQ}{100}} \quad \text{diff. eq.} \quad (*)$$

$$\boxed{Q(0) = Q_0} \quad \text{- init. condition.} \quad (**)$$

physical expectation: at $t \to \infty$, concentration in the tank tends to $\frac{Q_L}{100} = \frac{1}{4}\ \text{lb/gal} \Rightarrow \underbrace{Q_L = 25\ \text{lb}}$

can find it from setting rhs of $(*)$ to 0.

analytical solution: $\frac{dQ}{dt} + \frac{r}{100}Q = \frac{1}{4}r$ - linear, 1^{st} order

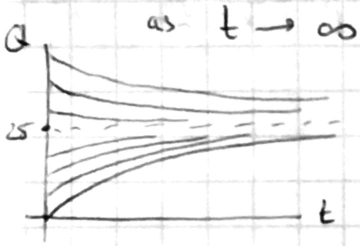
integrating factor $\mu(t) = e^{\frac{r}{100}t}$, general solution

$$Q(t) = \frac{1}{\mu(t)}\left(\int \tfrac{1}{4}r\,\mu(t)\,dt + c\right) = e^{-\frac{r}{100}t}\left(25e^{\frac{r}{100}t} + c\right) = 25 + c\,e^{-\frac{r}{100}t}$$

to satisfy init. condition, $\underset{25+c}{Q(0)} = Q_0 \Rightarrow c = (Q_0 - 25) \Rightarrow Q(t) = 25 + (Q_0-25)e^{-\frac{r}{100}t}$ $(***)$

or $Q(t) = 25\left(1-e^{-\frac{r}{100}t}\right) + Q_0 e^{-\frac{r}{100}t}$

So: $Q(t) \to 25 = Q_L$ as $t \to \infty$ - confirms the physical prediction!



question: Let $r=3$, $Q_0 = 2Q_L$. find, after what time T, salt level will be within 2% of Q_L

Sol: $(***)$: $Q(t) = 25\left(1 + e^{-\frac{3}{100}t}\right)$

$$Q(T) = \underbrace{25\cdot 1.02}_{25 + 2\%\text{ of }25} \iff e^{-\frac{3}{100}T} = \frac{1}{50} \to \frac{3T}{100} = \ln 50$$

$$\rightsquigarrow T = \frac{\ln 50}{0.03} \approx 130.4\ \text{min}$$

Question: what should be value of r so that $T = 45$ min?

Sol: $\frac{r\overset{45\text{ min}}{T}}{100} = \ln 50 \to r = \frac{100\cdot\ln 50}{45} \approx 8.69\ \text{gal/min}$

Ex 3 (chemicals in a pond) 04/04/2018

2

water with chemicals $\rightarrow$ pond $\rightarrow$ mixed water

5 mln gal/yr, 10^7 gal of water, fresh (initially), 5 mln gal/yr

concentration of chemicals

$\gamma(t) = 2+\sin 2t$ g/gal - varies periodically

· construct a model

· determine the amount of chemical at any time in the pond

$$\frac{dQ}{dt} = \text{rate in} - \text{rate out} = 5\cdot 10^6\,(2+\sin 2t) - \frac{Q}{10^7}\cdot 5\cdot 10^6$$

let $q(t) = \frac{Q(t)}{10^6}$. Then: $\boxed{\frac{dq}{dt} + \frac{q}{2} = 10+5\sin 2t}$, $q(0)=0$ (initially water was fresh)

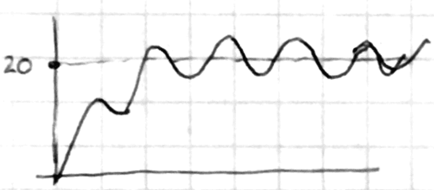
integr. factor $\mu(t) = e^{t/2}$

$$q(t) = e^{-\frac{t}{2}}\left(\int e^{t/2}(10+5\sin 2t)\,dt + c\right) = e^{-\frac{t}{2}}\left(20e^{\frac{t}{2}} + 5e^{\frac{t}{2}}\left(\underbrace{\frac{1/2}{\frac{1}{4}+4}}_{\frac{2}{17}}\sin 2t - \underbrace{\frac{2}{\frac{1}{4}+4}}_{\frac{8}{17}}\cos 2t\right) + c\right)$$

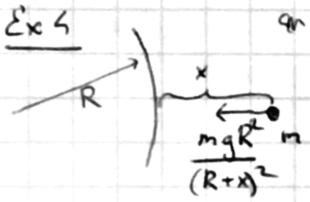
$$= \boxed{20 - \frac{40}{17}\cos 2t + \frac{10}{17}\sin 2t + c\,e^{-\frac{t}{2}}}$$

to satisfy the init. condition $q(0)=0$, $\underbrace{20-\frac{40}{17}}_{\frac{300}{17}} + c = 0 \Rightarrow c = -\frac{300}{17}$

$$\Rightarrow \boxed{q(t) = 20 - \frac{40}{17}\cos 2t + \frac{10}{17}\sin 2t - \frac{300}{17}e^{-\frac{t}{2}}}$$



Ex 4



an object is projected from Earth (perp. to the surface), with init. velocity v_0 assuming no air resistence, but taking into account the variation of Earth's grav. field with distance (a) find velocity ~~v(t)~~ during the motion

(b) find v_0 necessary to the reach maximum altitude A_{max}

(c) find smallest v_0 for which the object will not return to Earth (escape velocity).

grav. pull: $w(x) = \frac{k}{(x+R)^2}$ (k ← some constant; x ← altitude), $w(0) = -mg$ (at the surface level) $\Rightarrow k = -mgR^2$ and $w(x) = -\frac{mgR^2}{(R+x)^2}$

So: $\begin{cases} m\frac{dv}{dt} = -\frac{mgR^2}{(R+x)^2} & (*) \\ v(0) = v_0 \end{cases}$ ← too many variables (t, x and v)!

<u>Remedy</u>: let $\underline{\underline{x}}$ be the independent var., instead of v!

then: $\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$ (chain rule) and $(*)$ becomes $\boxed{v\frac{dv}{dx} = -\frac{gR^2}{(R+x)^2}}$ · separable!

general solution: $\boxed{\frac{v^2}{2} = \frac{gR^2}{R+x} + c}$

to satisfy init. cond. $v(0) = v_0$,

$c = \frac{v_0^2}{2} - gR$

04/09/2018
3

sol. of the init. val. prob. : $v(x) = \pm \sqrt{v_0^2 - 2gR + \frac{2gR^2}{R+x}}$ velocity as a function of altitude

rising falling

(b) $v(A_{max}) = 0 \Rightarrow V_0^2 - 2gR + \frac{2gR^2}{R+A_{max}} = 0 \Rightarrow V_0^2 = \frac{2gR\,A_{max}}{R+A_{max}} \Rightarrow V_0 = \sqrt{\frac{2gR\,A_{max}}{R+A_{max}}}$

(c) taking $A_{max} \to \infty$, we get $V_{escape} = \sqrt{2gR} \approx 11.1\ km/s$