Free boson with values in S^1 (free boson "compactified" on a circle)

Classically: $S[\varphi] = \frac{1}{2}\int_\Sigma d\tau\, d\sigma \left((\partial_\tau\varphi)^2 + (\partial_\sigma\varphi)^2\right)$ $\qquad \varphi: \underset{\text{cylinder}}{\Sigma} \to \underbrace{S^1}_{\mathbb{R}/2\pi r}$ $\qquad \varphi(\sigma+2\pi,\tau) = \varphi(\sigma,\tau) + 2\pi r\, m$

$m\in\mathbb{Z}$ - winding number

Space of fields $F = \mathrm{Map}(\Sigma, S^1) = \coprod_{m\in\mathbb{Z}} \mathrm{Map}_m(\Sigma, S^1)$ ← maps with winding number m.

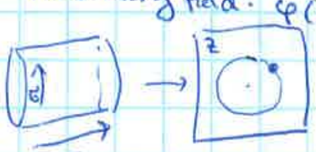
Configuration space: $X = \coprod_{m\in\mathbb{Z}} \underbrace{\mathrm{Map}_m(S^1,S^1)}_{X_m}$ $\qquad \varphi(\sigma) = \varphi_0 + m r\cdot\sigma + \sum_{n\neq 0}\varphi_n e^{in\sigma}$ in X_m sector

Hamiltonian: $H = \pi_0^2 + \underbrace{\left(\frac{mr}{2}\right)^2}_{} + \sum_{n\neq 0}\left(\pi_n\pi_{-n} + \frac{1}{4}n^2\varphi_n\varphi_{-n}\right)$

due to σ-dependence of 0-mode in φ

Canonical quantization $\quad \mathcal{H} = \bigoplus_{m\in\mathbb{Z}} \mathcal{H}_m$ ← states with winding number m

Heisenberg field: $\hat\varphi(z,\bar z) = \hat\varphi_0 - i\frac{\hat m r}{2}\log\frac{z}{\bar z} - i\hat\pi_0\log(z\bar z) + \sum_{n\neq 0}\frac{i}{n}\left(\hat a_n z^{-n} + \hat{\bar a}_n \bar z^{-n}\right)$



- since $\hat\varphi_0$ is defined mod $2\pi r\mathbb{Z}$, $\hat\pi_0$ has eigenvalue spectrum $\frac{1}{r}\mathbb{Z}$.

Indeed: in Schrödinger rep.,

$$\mathcal{H}_{z.m.} = L^2(S^1) = \{\Psi(\varphi_0) \text{ - } 2\pi r\text{-periodic: } \Psi(\varphi_0 + 2\pi r) = \Psi(\varphi_0)\}$$
$$\hat\varphi_0 : \Psi(\varphi_0) \mapsto \varphi_0\Psi(\varphi_0)$$
$$\hat\pi_0 : \Psi(\varphi_0) \mapsto -i\frac{\partial}{\partial\varphi_0}\Psi(\varphi_0)$$

in the basis $\{\Psi_e = e^{\frac{ie\varphi_0}{r}}\}_{e\in\mathbb{Z}}$, $\hat\pi_0$ is diagonal, with eigenvalues $\frac{e}{r}$, $e\in\mathbb{Z}$.

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Introduce $\hat e := r\hat\pi_0$. Then $\hat e$ has spectrum $\mathbb{Z}$.

Then: $\hat\varphi(z,\bar z) = \hat\varphi_0 - i\frac{\hat m r}{2}\log\frac{z}{\bar z} - i\frac{\hat e}{r}\log z\bar z + \sum_{n\neq 0}\frac{i}{n}\left(\hat a_n z^{-n} + \hat{\bar a}_n\bar z^{-n}\right)$

$i\partial\hat\varphi(z) = \sum_{n\in\mathbb{Z}}\hat a_n z^{-n-1}$, setting $\hat a_0 := \frac{\hat e}{r} + \frac{\hat m r}{2}$

$i\bar\partial\hat\varphi(\bar z) = \sum_{n\in\mathbb{Z}}\hat{\bar a}_n\bar z^{-n-1}$, setting $\hat{\bar a}_0 := \frac{\hat e}{r} - \frac{\hat m r}{2}$

$e^{\frac{i\hat\varphi_0}{r}}$ - single-valued operator (well-defined)

$[\hat\pi_0, e^{\frac{i\hat\varphi_0}{r}}] = \frac{1}{r}e^{\frac{i\hat\varphi_0}{r}}$ ← from "$[\hat\pi_0,\hat\varphi] = -i$" - mnemonic

$\to [\hat e, e^{\frac{i\hat\varphi_0}{r}}] = e^{\frac{i\hat\varphi_0}{r}}$

~~total~~ Hamiltonian $\hat H = \hat L_0 + \hat{\bar L}_0 = \frac{1}{2}\sum_{n\in\mathbb{Z}}\left(:\hat a_n\hat a_{-n}: + :\hat{\bar a}_n\hat{\bar a}_{-n}:\right)$

total momentum $\hat P = \hat L_0 - \hat{\bar L}_0 = \frac{1}{2}\sum_{n\in\mathbb{Z}}\left(:\hat a_n\hat a_{-n}: - :\hat{\bar a}_n\hat{\bar a}_{-n}:\right)$

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Space of states

$$\mathcal{H} = \bigoplus_{(e,m)\in\mathbb{Z}^2} V^{\mathrm{Heis}\oplus\overline{\mathrm{Heis}}}_{\left(\underbrace{\frac{e}{r}+\frac{mr}{2}}_{=:\alpha_{e,m}},\ \underbrace{\frac{e}{r}-\frac{mr}{2}}_{=:\bar\alpha_{e,m}}\right)} = \mathrm{Span}\left\{ \hat a_{-k_r}\cdots\hat a_{-k_1}\ \hat{\bar a}_{-\ell_s}\cdots\hat{\bar a}_{-\ell_1}|e,m\rangle \right\}_{\substack{e,m\in\mathbb{Z}\\ 1\le k_1\le\cdots\le k_r\\ 1\le \ell_1\le\cdots\le\ell_s}}$$

eigenvalues for $\hat a_0, \hat{\bar a}_0$

pseudo-vacuum

$$\hat H|e,m\rangle = \left(\frac12\left(\frac{e}{r}+\frac{mr}{2}\right)^2+\frac12\left(\frac{e}{r}-\frac{mr}{2}\right)^2\right)|e,m\rangle = \left(\left(\frac{e}{r}\right)^2+\left(\frac{mr}{2}\right)^2\right)|e,m\rangle$$

$$\hat P|e,m\rangle = \left(- \text{''} - - \text{''} - \right)|e,m\rangle = e\cdot m\,|e,m\rangle$$

also: $\hat L_0|e,m\rangle = \overbrace{\frac12\left(\frac{e}{r}+\frac{mr}{2}\right)^2}^{h_{e,m}}|e,m\rangle$

$\hat{\bar L}_0|e,m\rangle = \underbrace{\frac12\left(\frac{e}{r}-\frac{mr}{2}\right)^2}_{\bar h_{e,m}}|e,m\rangle$

Vertex operators

Introduce the operator $\hat\mu$ ("defined mod $2\pi\mathbb{Z}$") s.t. $[\hat\mu,\hat m]=i$ and commuting with everything else; then $[\hat m, e^{ik\hat\mu}] = k\,e^{ik\hat\mu}$ for $k\in\mathbb{Z}$

thus $e^{ik\hat\mu}: \mathcal{H}_{e,m}\to\mathcal{H}_{e,m+k}$ - increases m by $+k$

likewise $[\hat\varphi_0, \underbrace{\hat\pi_0}_{\text{``}\frac{\hat e}{r}\text{''}}]=i \Rightarrow e^{i\frac{\ell\hat\varphi_0}{r}}: \mathcal{H}_{e,m}\to\mathcal{H}_{e+\ell,m}$

Define the "chiral parts" of $\hat\varphi$:

$: :$ puts $\hat e,\hat m$ to the right, $\hat\varphi_0,\hat\mu$ to the left

$$\hat\chi(z) := \frac12\hat\varphi_0 + \frac{\hat\mu}{r} - i\underbrace{\left(\frac{\hat e}{r}+\frac{\hat m r}{2}\right)}_{\hat a_0}\log z + \sum_{n\neq0}\frac{i}{n}\hat a_n z^{-n}$$

$$\hat{\bar\chi}(\bar z) := \frac12\hat\varphi_0 - \frac{\hat\mu}{r} - i\underbrace{\left(\frac{\hat e}{r}-\frac{\hat m r}{2}\right)}_{\hat{\bar a}_0}\log\bar z + \sum_{n\neq0}\frac{i}{n}\hat{\bar a}_n\bar z^{-n}$$

so that $\hat\varphi(z,\bar z) = \hat\chi(z)+\hat{\bar\chi}(\bar z)$

Vertex operator: $$\hat V_{e,m}(z,\bar z) := \ :e^{i\left(\frac{e}{r}+\frac{mr}{2}\right)\hat\chi(z)}\, e^{i\left(\frac{e}{r}-\frac{mr}{2}\right)\hat{\bar\chi}(\bar z)}:$$

- $\hat V_{e,m}$ is primary, with conf. dimension $\left(h=\frac12\left(\frac{e}{r}+\frac{mr}{2}\right)^2,\ \bar h = \frac12\left(\frac{e}{r}-\frac{mr}{2}\right)\right)$

- $\lim_{z\to0}\hat V_{e,m}(z,\bar z)\underbrace{|\mathrm{vac}\rangle}_{|e=0,m=0\rangle} = |e,m\rangle$

this is a single-valued expression, despite real powers, since $\alpha_{e_im_i}\alpha_{e_jm_j}-\bar\alpha_{e_im_i}\bar\alpha_{e_jm_j} = e_im_j+m_ie_j \in\mathbb{Z}$

- Correlator: $$\left\langle\prod_{k=1}^n V_{e_k,m_k}(z_k,\bar z_k)\right\rangle = \begin{cases}\prod_{1\le i<j\le n}(z_i-z_j)^{\alpha_{e_im_i}\alpha_{e_jm_j}}(\bar z_i-\bar z_j)^{\bar\alpha_{e_im_i}\bar\alpha_{e_jm_j}}, & \text{if } \sum_i^n e_i = 0 = \sum_i^n m_i\\ 0 & \text{otherwise}\end{cases}$$

Ex: $n=2$ $$\langle V_{e,m}(z,\bar z)V_{-e,-m}(w,\bar w)\rangle = |z-w|^{-2\left(\left(\frac{e}{r}\right)^2+\left(\frac{mr}{2}\right)^2\right)}\cdot\left(\frac{z-w}{\bar z-\bar w}\right)^{-em}$$

field-state correspondence
$$\lim_{z\to0} :\prod_{j=1}^{r}\frac{i\partial^{p_j}\hat\varphi(z)}{(p_j-1)!}\prod_{k=1}^{s}\frac{i\bar\partial^{q_k}\hat\varphi(\bar z)}{(q_k-1)!}\hat V_{e,m}(z,\bar z):|\mathrm{vac}\rangle = \hat a_{-p_r}\cdots\hat a_{-p_1}\hat{\bar a}_{-q_s}\cdots\hat{\bar a}_{-q_1}|e,m\rangle$$

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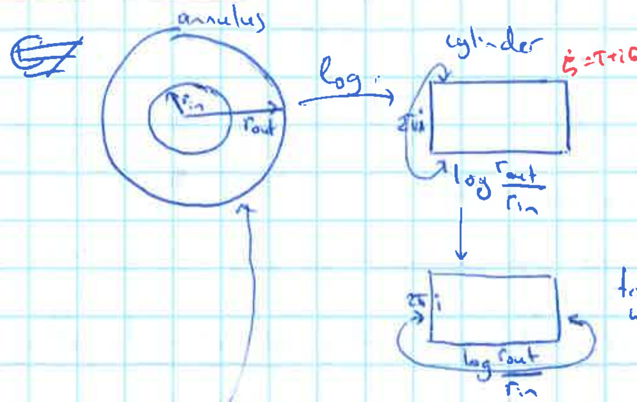
Torus partition function

~~torus~~



$$Z_{cyl} = e^{-\hat{H}\log\frac{r_{out}}{r_{in}}}$$

torus with modular parameter $\tau = \frac{i}{2\pi}\log\frac{r_{out}}{r_{in}}$

$$Z_{torus} = \mathrm{tr}_{\mathcal{H}}\, e^{-\hat{H}\log\frac{r_{out}}{r_{in}}} = \mathrm{tr}_{\mathcal{H}}\, e^{2\pi i\tau(\hat{L}_0 + \hat{\bar{L}}_0)}$$

• can allow relative rotation of the two circles by angle θ

$$Z\left(2\pi i,\ i\theta,\ \log\frac{r_{out}}{r_{in}}\right) = e^{-\hat{H}\log\frac{r_{out}}{r_{in}} - i\hat{P}\theta} = e^{2\pi i\tau\hat{L}_0 - 2\pi i\bar{\tau}\hat{\bar{L}}_0} = q^{\hat{L}_0}\bar{q}^{\hat{\bar{L}}_0}$$

$\tau = \frac{i}{2\pi}\left(\log\frac{r_{out}}{r_{in}} + i\theta\right)$, where $q := e^{2\pi i\tau}$

$$\to Z\left(\begin{array}{c}\text{torus}\\ \text{with modular}\\ \text{parameter } \tau\end{array}\right) = \mathrm{tr}_{\mathcal{H}}\, q^{\hat{L}_0}\bar{q}^{\hat{\bar{L}}_0}$$

Correction

$$\boxed{Z(\tau) = \mathrm{tr}_{\mathcal{H}}\, q^{\hat{L}_0 - \frac{c}{24}}\,\bar{q}^{\hat{\bar{L}}_0 - \frac{\bar{c}}{24}}}$$

• due to Schwarzian derivative correction in the transformation of $T, \bar{T}$ (under log) $T_{(plane)}(z)dz^2 = (T_{(cyl)}(\zeta) + \frac{c}{24})d\zeta^2$

• equivalently: this is the part. fn. relative to flat metric on the torus, not the pushforward of flat metric on annulus by log-map. So, it incorporates the "Liouville correction action".

• pragmatically: with this correction, the answer is ~~a modular fn~~ invariant under $\tau \to -1/\tau$, otherwise - not.

In our case (compactified free boson):

$c = \bar{c} = 1$,

$$Z(\tau) = \mathrm{tr}_{\mathcal{H}}\, q^{\hat{L}_0 - \frac{1}{24}}\,\bar{q}^{\hat{\bar{L}}_0 - \frac{1}{24}}$$

$$= \sum_{(e,m)\in\mathbb{Z}^2} q^{\frac{1}{2}\left(\frac{e}{r} + \frac{mr}{2}\right)^2}\bar{q}^{\frac{1}{2}\left(\frac{e}{r} - \frac{mr}{2}\right)^2}(q\bar{q})^{-\frac{1}{24}}\sum_{k,l\geq 0}P(k)P(l)q^k\bar{q}^l$$

$\mathcal{H} = \bigoplus_{e,m}\mathcal{H}_{e,m}$

no. of partitions, e.g. $4 = 1+1+1+1 = 2+1+1 = 2+2 = 3+1 = 4 \rightsquigarrow P(4) = 5$

Aside: $\sum_{k\geq 0}P(k)q^k = \prod_{n=1}^{\infty}\frac{1}{1-q^n} = \frac{q^{1/24}}{\eta(\tau)}$

$|q| < 1$

where $\eta(\tau) = q^{+1/24}\prod_{n\geq 1}(1-q^n)$ Dedekind eta-function $\Big\{$ - it satisfies modular equivariance properties

$\eta(\tau+1) = \eta(\tau)\, e^{i\pi/12}$

$\eta(-\frac{1}{\tau}) = \eta(\tau)\,(-i\tau)^{1/2}$

(from Euler's f-la $\prod_{n\geq 1}(1-q^n) = \sum_{j=-\infty}^{\infty}(-1)^j q^{\frac{3j^2-j}{2}}$ and Poisson summation in j)

Thus: $$Z(\tau) = \frac{1}{\eta(\tau)\overline{\eta(\tau)}}\sum_{(e,m)\in\mathbb{Z}^2} q^{\frac{1}{2}\left(\frac{e}{r} + \frac{mr}{2}\right)^2}\bar{q}^{\frac{1}{2}\left(\frac{e}{r} - \frac{mr}{2}\right)^2}$$

Properties * modular invariance: $Z(\tau+1) = Z(\tau)$

$Z(-1/\tau) = Z(\tau)$ - from Poisson summation in e, m. Relies on $q^{-\frac{1}{24}}$ factor!

* "T-duality": $Z(\tau, r) = Z(\tau, 2/r)$

radius of target S^1

* large radius asymptotics: $Z(\tau, r) \underset{r\to\infty}{\sim} r\,\boxed{\frac{1}{\sqrt{\mathrm{Im}\,\tau}\,\eta(\tau)\overline{\eta(\tau)}}}$

one might call it the regularized torus partition function of the non-compactified free boson. it is modular invariant

Aside: Poisson summation formula $\sum_{n\in\mathbb{Z}} f(n) = \sum_{p\in\mathbb{Z}} \tilde{f}(p)$, with $\tilde{f}(p) = \int_{-\infty}^{\infty} dx\, e^{2\pi i p x} f(x)$ - Fourier transform

E.g. $Z(\tau,r) = \frac{1}{\eta(\tau)\eta(\bar\tau)} \sum_{(e,m)\in\mathbb{Z}^2} q^{\frac{1}{2}\alpha_{e,m}^2}\, \bar{q}^{\frac{1}{2}\bar\alpha_{e,m}^2} \underset{\substack{\uparrow \\ \text{Poisson summation in } e}}{=} \frac{1}{\eta(\tau)\eta(\bar\tau)} \sum_{(p,m)\in\mathbb{Z}^2} \frac{r}{\sqrt{\operatorname{Im}\tau}}\, e^{-\frac{\pi}{2} r^2 \left(\frac{(p+m\operatorname{Re}\tau)^2}{\operatorname{Im}\tau} + m^2\right)}$ (*)

- asymptotics $r\to\infty$ is given by the ~~constant~~ term $p=m=0$; all others behave as $e^{-(\dots) r^2}$

Torus partition function in the path integral formalism [ref: K. Gawedzki "Lectures on CFT" at IAS]

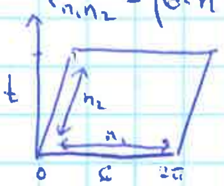
www.math.ias.edu/QFT/fall/NewGaw.ps

$$Z = \int_{\substack{\text{Maps}(\Sigma,\, S^1) \\ \mathbb{C}/\mathbb{Z}\oplus\tau\mathbb{Z} \text{ - torus}}} D\varphi\; e^{-S[\varphi]} = \sum_{(n_1,n_2)\in\mathbb{Z}^2} \int_{\text{Maps}_{n_1,n_2}(\Sigma, S^1)} D\varphi\; e^{-S[\varphi]}$$

(n_1,n_2) - winding numbers around two cycles in Σ

$\varphi = \underset{\substack{\mathbb{R}/2\pi r\mathbb{Z} \\ \text{- constant}}}{\varphi_0} + \underset{\substack{\uparrow \\ \text{class. solution of EL eq. with given winding numbers}}}{\varphi^{cl}_{n_1 n_2}} + \underset{\substack{\uparrow \\ \Sigma\text{-periodic, with } \int_\Sigma \tilde\varphi = 0}}{\tilde\varphi}$

$\varphi^{cl}_{n_1 n_2} = \left(\sigma\cdot n_1 + t\cdot\frac{n_2 - n_1\operatorname{Re}\tau}{\operatorname{Im}\tau}\right) r$



$S[\varphi^{cl}_{n_1 n_2}] = i\pi r^2 \frac{(n_2-\tau n_1)(n_2-\bar\tau n_1)}{\tau-\bar\tau}$

So: $Z = \sum_{(n_1,n_2)\in\mathbb{Z}^2} \underbrace{\oint_{S^1} d\varphi_0}_{2\pi r} \underbrace{\int_{\text{Maps}'(\Sigma,\mathbb{R})} D\tilde\varphi\; e^{-S[\tilde\varphi]}}_{=|\det'(-\Delta_\Sigma)|^{-\frac{1}{2}}} \cdot e^{-S[\varphi^{cl}_{n_1 n_2}]} =$

int. over space of sol. to EL

$= \sum_{n_1,n_2} 2\pi r \frac{1}{|\eta(\tau)|^2\sqrt{\operatorname{Im}\tau}\cdot 2\pi}\cdot e^{i\pi r^2\frac{(n_2-\tau n_1)(n_2-\bar\tau n_1)}{\tau-\bar\tau}} = (*)$

under identification $m=n_1$, $p=n_2$

$\boxed{\begin{array}{l} * \det'(-\Delta_\Sigma) = |\eta(\tau)|^4 \cdot \operatorname{Im}\tau\cdot(2\pi)^2 \\ \quad\uparrow \\ \text{on functions on } \Sigma \end{array}}$ - Explained in detail in Gawedzki.

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Rem: primary fields (in compactified free boson theory)

$\mathcal{H} = \bigoplus_{(e,m)\in\mathbb{Z}^2} \underbrace{V_{e,m}^{\text{Heis}\oplus\overline{\text{Heis}}}}$ splits ~~into~~ into $\text{Vir}\oplus\overline{\text{Vir}}$ - irred. submodules with highest weight vectors $|e,m\rangle$, $a_{-1}|e,m\rangle$, $\bar{a}_{-1}|e,m\rangle$

$V_{0,0}^{H\oplus\bar H}$ splits as a sum of 4 $\text{Vir}\oplus\overline{\text{Vir}}$ irred submodules, with highest vectors $\underset{\substack{\updownarrow \\ \mathbb{1}}}{|vac\rangle}$, $\underset{\substack{\updownarrow \\ i\partial\varphi}}{a_{-1}|vac\rangle}$, $\underset{\substack{\updownarrow \\ i\bar\partial\varphi}}{\bar{a}_{-1}|vac\rangle}$, $\underset{\substack{\updownarrow \\ -\partial\varphi\bar\partial\varphi}}{a_{-1}\bar{a}_{-1}|vac\rangle}$ ← primary fields

$V_{e,m}^{H\oplus\bar H}$: if $\alpha_{e,m}=0$, $\bar\alpha_{e,m}\neq 0$ - two $\text{Vir}\oplus\overline{\text{Vir}}$ irr. submod., with highest vectors $\underset{\substack{\updownarrow \\ V_{e,m}}}{|e,m\rangle}$, $\underset{\substack{\updownarrow \\ i\partial\varphi V_{e,m}}}{a_{-1}|e,m\rangle}$ ← primary fields

if $\alpha_{e,m}\neq 0$, $\bar\alpha_{e,m}=0$ — " — $\underset{\substack{\updownarrow \\ V_{e,m}}}{|e,m\rangle}$, $\underset{\substack{\updownarrow \\ i\bar\partial\varphi V_{e,m}}}{\bar{a}_{-1}|e,m\rangle}$

if $\alpha_{e,m}\neq 0$, $\bar\alpha_{e,m}\neq 0$ - single $\text{Vir}\oplus\overline{\text{Vir}}$ module, highest vector $|e,m\rangle \sim V_{e,m}$

- in a general CFT, correlators factorize

depends on $z_1 \dots z_n$ holomorphically for $z_i \neq z_j$ but can have monodromy

$$(*)\quad \underbrace{\langle \Phi_1(z_1,\bar z_1) \dots \Phi_n(z_n,\bar z_n)\rangle}_{\substack{\text{single-valued,}\\ \text{smooth on open conf. space}}} = \sum_\rho F_\rho(z_1,\dots,z_n)\, \bar F_\rho(\bar z_1,\dots,\bar z_n)$$

similarly here

runs over an indexing set; finite in nice cases

F_ρ, $\bar F_\rho$ - "conformal blocks" (for the correlation function (*))

similarly, genus 1 partition function $Z(\tau,\bar\tau) = \sum_\rho \chi_\rho(\tau)\bar\chi_\rho(\bar\tau)$

conf. blocks for genus 1 part. fun.

- chiral (holomorphic) free boson (with values in S^1):

$$\mathcal{H} = \bigoplus_{e,m} V^{\mathrm{Heis}}_{e,m} = \mathrm{Span}\{\hat a_{-k_r}\cdots \hat a_{-k_1}|e,m\rangle\}_{1\le k_1\le \dots \le k_r}$$

only one Heis algebra!

$V^{\mathrm{chiral}}_{e,m}(z) = :e^{i\alpha_{e,m}\hat\varphi(z)}:$ - chiral vertex operator

$\langle \prod_{i=1}^n V^{\mathrm{chiral}}_{e_i,m_i}(z_i)\rangle$ is multivalued (has monodromies). If $r^2\in\mathbb{Q}$, monodromies are rational

$$= \prod_{1\le i<j\le n}(z_i-z_j)^{\alpha_{e_i m_i}\alpha_{e_j m_j}} \text{ if } \sum e_i = 0 = \sum m_i$$

- $$\langle \prod_{i=1}^n V^{\text{non-chiral}}_{e_i,m_i}(z_i,\bar z_i)\rangle = \langle \prod_{i=1}^n V^{\mathrm{chiral}}_{e_i,m_i}(z_i)\rangle \cdot \langle \prod_{i=1}^n V^{\mathrm{chiral}}_{e_i,m_i}(\bar z_i)\rangle$$

conf. block

- For $r^2\in\mathbb{Q}$, $Z^{\text{non-chiral}}(\tau,\bar\tau;r) = \sum_{\rho\in\text{finite set}} \chi_\rho(\tau)\bar\chi_\rho(\bar\tau)$ (for some functions $\chi,\bar\chi$)

[For $r^2\notin\mathbb{Q}$ one cannot split Z as a finite sum of products of hol/anti-hol functions]

E.g. for $r=\sqrt2$,
$$Z(\tau,\bar\tau) = \left(\sum_{k\in\mathbb{Z}}\frac{q^{k^2}}{\eta(\tau)}\right)\left(\sum_{\ell\in\mathbb{Z}}\frac{\bar q^{\ell^2}}{\eta(\bar\tau)}\right) + \left(\sum_{k\in\mathbb{Z}+\frac12}\frac{q^{k^2}}{\eta(\tau)}\right)\left(\sum_{\ell\in\mathbb{Z}+\frac12}\frac{\bar q^{\ell^2}}{\eta(\bar\tau)}\right)$$

Free Fermion

$$S = \frac{i}{8\pi}\int_\Sigma d^2z\, \psi\bar\partial\psi + \bar\psi\partial\bar\psi \qquad \text{field: } \psi(dz)^{1/2} \in \Gamma(\Sigma, K^{\otimes\frac{1}{2}}), \quad \bar\psi(d\bar z)^{1/2} \in \Gamma(\Sigma, \bar K^{\otimes \frac12})$$

need a spin structure to determine the sign of ~~...~~ of transition function

fields are treated as anti-commuting variables

EL eq: $\bar\partial\psi = 0 = \partial\bar\psi$

As a Hamiltonian system on a cylinder: phase space $\Phi = C^\infty(S^1)\otimes \mathbb{C}^{0|2}$ symp form $\omega = \frac{i}{4\pi}\oint d\sigma\, \delta\psi\wedge\delta\psi + \delta\bar\psi\wedge\delta\bar\psi$

$\to$ Poisson (anti)brackets $\{\psi^a(\sigma), \psi^b(\sigma')\} = 2\pi i\, \delta^{ab}\delta(\sigma-\sigma')$

$\psi^1 := \psi,\ \psi^2 := \bar\psi$

Hamiltonian: $H = \frac{1}{4\pi}\oint d\sigma(\psi\partial_\sigma\psi - \bar\psi\partial_\sigma\bar\psi)$

Fourier modes: $\psi(\sigma) = \sum_n e^{-in\sigma} b_n$, $\bar\psi(\sigma) = \sum_n e^{-in\sigma}\bar b_n$, $\{b_n, b_m\} = i\delta_{n,-m}$, $\{\bar b_n, \bar b_m\} = i\delta_{n,-m}$

Canonical quantization

Heisenberg field $\hat\psi(\xi) = \sum e^{-n\xi}\hat b_n$ ($\xi = \tau + i\sigma$), $\hat{\bar\psi}(\bar\xi) = \sum e^{-n\bar\xi}\hat{\bar b}_n$ with anti-commutation relations $[\hat b_n, \hat b_m]_+ = \delta_{n,-m}\mathbb{1}$ ← Cliff, $[\hat{\bar b}_n, \hat{\bar b}_m]_+ = \delta_{n,-m}\mathbb{1}$ ← $\overline{\text{Cliff}}$

- cylinder $\to$ plane, $\xi \mapsto z = e^\xi$

$$\psi_{cyl}(\xi) \longmapsto \psi_{plane}(z) = \overset{\boxed{\left(\frac{dz}{d\xi}\right)^{-\frac12}}}{z^{-\frac12}}\psi_{cyl}(\xi)$$
$$\bar\psi_{cyl}(\bar\xi) \longmapsto \bar\psi_{plane}(\bar z) = \bar z^{-1/2}\bar\psi_{cyl}(\bar\xi)$$

I.e., in terms of modes: $\hat\psi(z) = \sum_n \hat b_n z^{-n-\frac12}$

- periodic boundary condition (P) on the cylinder $\longleftrightarrow$ anti-periodic on the plane
  $\psi_{cyl}(\sigma+2\pi) = \psi_{cyl}(\sigma)$ $\qquad$ $\psi_{plane}(e^{2\pi i}\cdot z) = -\psi_{plane}(z)$
  - "Ramond" sector / spin structure
  - here $\hat\psi_P(z) = \sum_{\boxed{n\in\mathbb{Z}}}\hat b_n z^{-n-\frac12}$ - summation over integers n.
- anti-periodic condition (A) on the cylinder $\longleftrightarrow$ periodic on the plane - "Neveu-Schwarz sector"
  $\hat\psi_A(z) = \sum_{\boxed{n\in\mathbb{Z}+\frac12}}\hat b_n z^{-n-\frac12}$

- Space of states (for a chiral fermion)

$\mathcal{H} = \mathcal{H}_P \oplus \mathcal{H}_A$ $\qquad$ $\hat b_{>0}|vac_P\rangle = 0$, $\hat b_{>0}|vac_A\rangle = 0$

$$\mathcal{H}_P = \mathrm{Span}\{\cdots \hat b_{-2}^{n_2}\hat b_{-1}^{n_1}\hat b_0^{n_0}|vac_P\rangle\}_{n_0,n_1,n_2,\ldots\in\{0,1\}}$$
only fin. many nonzero

$$\mathcal{H}_A = \mathrm{Span}\{\cdots \hat b_{-\frac52}^{n_{5/2}}\hat b_{-\frac32}^{n_{3/2}}\hat b_{-\frac12}^{n_{1/2}}|vac_A\rangle\}_{n_{1/2},n_{3/2},n_{5/2},\ldots\in\{0,1\}}$$

fermionic occupation numbers (in $\{0,1\}$ since $\hat b_n^2 = 0$, $n\neq 0$)

- $\psi\psi$ propagator

P sector: $\langle\psi(z)\psi(w)\rangle_P \overset{\text{assume } |z|>|w|}{=} \langle vac_P|\hat\psi(z)\hat\psi(w)|vac_P\rangle = \sum_{n,m\in\mathbb{Z}}\langle vac_P|\hat b_n\hat b_m|vac_P\rangle z^{-n-\frac12}w^{-m-\frac12} =$

$$= \underbrace{\langle vac_P|\hat b_0\hat b_0|vac_P\rangle}_{1/2} z^{-1/2}w^{-1/2} + \sum_{\substack{n>0\\ n\in\mathbb{Z}}}\underbrace{\langle vac_P|\hat b_n\hat b_{-n}|vac_P\rangle}_{1} z^{-n-\frac12}w^{n-\frac12} = (zw)^{-1/2}\left(\frac12 + \sum_{n=1}^\infty\left(\frac{w}{z}\right)^n\right) = \boxed{\frac12\,\frac{\left(\frac{z}{w}\right)^{\frac12} + \left(\frac{w}{z}\right)^{\frac12}}{z-w}}$$

A sector: $\langle\psi(z)\psi(w)\rangle_A = \sum_{n\in\mathbb{Z}+\frac12,\, n>0}\langle vac_A|\hat b_n\hat b_{-n}|vac_A\rangle z^{-n-\frac12}w^{n-\frac12} = \boxed{\frac{1}{z-w}}$

Note: $\langle\psi\psi\rangle$ is translation-invariant in A-sector, suggesting that $|vac_A\rangle$ is the true vacuum while P is not.

- in both sectors we have $\langle\psi(z)\psi(w)\rangle = -\langle\psi(w)\psi(z)\rangle$

Stress-energy tensor

classically: $T(z) = -\frac{1}{2}\psi(z)\partial\psi(z)$; for chiral fermion, $\bar{T}\equiv 0$

A-sector: $\hat{T}(z) = -\frac{1}{2}:\hat{\psi}(z)\partial\hat{\psi}(z):$ satisfies the standard OPE (from Wick's lemma)

$$T(z)T(w) \sim \frac{c/2\,\mathbb{1}}{(z-w)^4} + \frac{2T(z)}{(z-w)^2} + \frac{\partial T(z)}{z-w} + \text{reg} \quad (*)$$

with $c=\frac{1}{2}$ (and $\bar{c}=0$ as $\bar{T}\bar{T}\equiv 0$)

P-sector try: $\hat{T}^{naive}(z) = -\frac{1}{2}:\hat{\psi}(z)\partial\hat{\psi}(z):$ — has wrong $T^{naive}T^{naive}$ OPE!!

The good definition is: $\hat{T}(z) := \lim_{w\to z}\left(-\frac{1}{2}\hat{\psi}(z)\partial\hat{\psi}(w) + \frac{1}{2}\frac{\hat{\mathbb{1}}}{(z-w)^2}\right)$

then: $\hat{T}(z) = \hat{T}^{naive}(z) + \underbrace{\frac{1}{16z^2}}_{\text{shift}}$ — has correct OPE $(*)$ with $c=\frac{1}{2}$; $\langle\hat{T}(z)\rangle_P = \frac{1}{16z^2}$ - nonzero vacuum energy in P sector

Virasoro generators

- can be obtained from $\hat{T}(z) = \sum_{n\in\mathbb{Z}} z^{-n-\frac{1}{2}}\hat{L}_n$

A sector: $\hat{L}_n = \sum_{m\in\mathbb{Z}+\frac{1}{2}}\left(\frac{m}{2}+\frac{1}{4}\right):\hat{b}_{n-m}\hat{b}_m:$

P sector: $\hat{L}_n = \sum_{m\in\mathbb{Z}}\left(\frac{m}{2}+\frac{1}{4}\right):\hat{b}_{n-m}\hat{b}_m: + \delta_{n,0}\frac{\hat{\mathbb{1}}}{16}$

In particular, $\hat{L}_0|vac_A\rangle = 0$

$\hat{L}_0|vac_P\rangle = \frac{1}{16}|vac_P\rangle$

$[\hat{L}_0, \hat{b}_n] = -n\hat{b}_n$ (in A and P sectors)

A-states

L_0-eigenvalue	states
0	$\lvert vac_A\rangle$
1/2	$b_{-1/2}\lvert vac_A\rangle$
1	$\emptyset$
3/2	$b_{-3/2}\lvert vac_A\rangle$
2	$b_{-3/2}b_{-1/2}\lvert vac_A\rangle$
5/2	$b_{-5/2}\lvert vac_A\rangle$
3	$b_{-5/2}b_{-1/2}\lvert vac_A\rangle$

P-states

L_0-eigenvalue	states	
1/16	$\lvert vac_P\rangle$	$b_0\lvert vac_P\rangle$
1+1/16	$b_{-1}\lvert vac_P\rangle$	$b_{-1}b_0\lvert vac_P\rangle$
2+1/16	$b_{-2}\lvert vac_P\rangle$	$b_{-2}b_0\lvert vac_P\rangle$
3+1/16	$b_{-3}\lvert vac_P\rangle$	$b_{-3}b_0\lvert vac_P\rangle$
	$b_{-2}b_{-1}\lvert vac_P\rangle$	$b_{-2}b_{-1}b_0\lvert vac_P\rangle$

States $|vac_A\rangle$, $b_{-\frac{1}{2}}|vac_A\rangle$, $|vac_P\rangle$, $b_0|vac_P\rangle$ are Virasoro-primary (killed by $\hat{L}_{>0}$) (pseudo-vacua)

$\quad\quad\quad\quad \| \quad\quad \| \quad\quad \| \quad\quad \|$

$\quad\quad |0\rangle$, $|\frac{1}{2}\rangle$, $|\frac{1}{16}\rangle_+$, $|\frac{1}{16}\rangle_-$

$\mathbb{Z}_2$-grading $(-1)^F$	even	odd	even	odd

space of states $\mathcal{H} = \underbrace{V_0^{Vir}\oplus V_{\frac{1}{2}}^{Vir}}_{\mathcal{H}_A} \oplus \underbrace{V_{\frac{1}{16}}^{Vir}\oplus V_{\frac{1}{16}}^{Vir}}_{\mathcal{H}_P}$ - splitting of the space of states into highest-weight Virasoro modules (some quotients of Verma modules)

corresponding primary fields: $\mathbb{1}$ — (0,0)-primary; $\psi(z)$ — $(\frac{1}{2},0)$-primary; $\underbrace{\sigma(z) \quad \mu(z)}$ "twist fields", $(\frac{1}{16},0)$-primary

$$\psi(z)\sigma(w) \sim (z-w)^{-1/2}\mu(w) + \ldots$$

5/24/19 ~~4/17/19~~

Non-chiral fermion (pairing left- and right-chiral fermions)

- require that P/A boundary condition is the same for ψ and $\bar\psi$
- impose $\hat{b}_0 \equiv \hat{\bar{b}}_0$

6 conf. families

$$\mathcal{H}_{\text{non-chiral}} = \underbrace{V^{Vir\oplus\overline{Vir}}_{(0,0)} \oplus V^{Vir\oplus\overline{Vir}}_{(\frac12,0)} \oplus V^{Vir\oplus\overline{Vir}}_{(0,\frac12)} \oplus V^{Vir\oplus\overline{Vir}}_{(\frac12,\frac12)}}_{A\text{ sector}} \oplus \underbrace{V^{Vir\oplus\overline{Vir}}_{(\frac1{16},\frac1{16})_+} \oplus V^{Vir\oplus\overline{Vir}}_{(\frac1{16},\frac1{16})_-}}_{P\text{ sector}}$$

$\mathbb{Z}_2$-parity	even	odd	odd	even	even	odd
highest vector	$\lvert vac_A\rangle$	$b_{-\frac12}\lvert vac_A\rangle$	$\bar b_{-\frac12}\lvert vac_A\rangle$	$b_{-\frac12}\bar b_{-\frac12}\lvert vac_A\rangle$	$\lvert vac_P\rangle$	$b_0\lvert vac_P\rangle$
primary field	$\mathbb{1}$ $(0,0)$	$\psi(z)$ $(\frac12,0)$	$\bar\psi(\bar z)$ $(0,\frac12)$	$\varepsilon(z,\bar z)=\psi(z)\bar\psi(\bar z)$ $(\frac12,\frac12)$	$\sigma(z,\bar z)$ $(\frac1{16},\frac1{16})$	$\mu(z,\bar z)$ $(\frac1{16},\frac1{16})$

plays the role of a vertex operator

Examples of correlators

- $\langle\psi(z)\psi(w)\rangle = \frac{1}{z-w}$ • $\langle\psi(z_1)\cdots\psi(z_{2n})\rangle = \mathrm{Pf}\left(\frac{1}{z_i-z_j}\right)$
  - Ex: $\langle\psi(z_1)\cdots\psi(z_4)\rangle = \frac{1}{z_1-z_2}\cdot\frac{1}{z_3-z_4} - \frac{1}{z_1-z_3}\cdot\frac{1}{z_2-z_4} + \frac{1}{z_1-z_4}\cdot\frac{1}{z_2-z_3}$
- $\langle\sigma(z,\bar z)\,\sigma(w,\bar w)\rangle = \frac{1}{|z-w|^{1/4}}$ - spin-spin correlator in Ising model at $T=T_{crit}$
  - from global conformal invariance
- $\langle\sigma(z_1,\bar z_1)\cdots\sigma(z_4,\bar z_4)\rangle = \left|\frac{z_{13}z_{24}}{z_{12}z_{23}z_{34}z_{41}}\right|^{1/4} F(\lambda,\bar\lambda)$ - from global conf. invariance, $\lambda = \frac{z_{12}z_{34}}{z_{13}z_{24}}$

$$\underbrace{\left(L_{-2} - \frac{3}{2(2h+1)}L_{-1}^2\right)\lvert vac_P\rangle}_{\text{vector of zero length in } V^{Vir}_{c=\frac12,h=\frac1{16}}} = 0 \qquad h=\frac1{16}, \quad \frac{3}{2(2h+1)} = \frac43$$

Exercise: check in terms of the Cliff-module.

$\to \left(L^{(z)}_{-2} - \frac{3}{2(2h+1)}L^{(z)\,2}_{-1}\right)\sigma(z,\bar z) = 0 \rightsquigarrow$ diff. equation on $\langle\sigma\cdots\sigma\rangle$ correlator (Ward identity)

$\to$ (hypergeometric) ODE on $F(\lambda,\bar\lambda)$

$$\left(\lambda(1-\lambda)\frac{\partial^2}{\partial\lambda^2} + \left(\frac12-\lambda\right)\frac{\partial}{\partial\lambda} + \frac1{16}\right)F(\lambda,\bar\lambda)$$

Solutions: $f_{1,2}(\lambda) = (1\pm\sqrt{1-\lambda})^{1/2}$ - conf. blocks for $\langle\sigma\sigma\sigma\sigma\rangle$ correlator

$F(\lambda,\bar\lambda) = a\left(f_1(\lambda)f_1(\bar\lambda) + f_2(\lambda)f_2(\bar\lambda)\right)$ — from ~~invariance of $\langle\sigma\cdots\sigma\rangle$ under permutations of $z_1\ldots z_4$~~ single-valuedness under $\lambda\leftrightarrow\bar\lambda$ + no monodromy condition

$a=\frac12$ from $\sigma(z_1,\bar z_1)\,\sigma(z_2,\bar z_2) \sim \frac{\mathbb{1}}{z_{12}^{1/8}\,\bar z_{12}^{1/8}} + \ldots$ OPE

So: $\langle\sigma(z_1,\bar z_1)\cdots\sigma(z_4,\bar z_4)\rangle = \frac12\left|\frac{z_{13}z_{24}}{z_{12}z_{23}z_{34}z_{41}}\right|^{1/4}\left(|1+\sqrt{1-\lambda}| + |1-\sqrt{1-\lambda}|\right)$

Aside: Kac' determinant formula

$V_{c,h} = \bigoplus V_{c,h}^{(N)}$ ← level-N states, $\leftarrow n_1+\dots+n_r = N$

Let $V_{c,h}$ be the Virasoro Verma module with highest weight $|h\rangle$, central charge c.
$= \mathrm{Span}\{L_{-n_r}\cdots L_{-n_1}|h\rangle\}_{1\le n_1\le\dots\le n_r}$ i.e. $L_0|h\rangle = h|h\rangle$

- there is a (unique) inner product on $V_{c,h}$ s.t. $\langle h|h\rangle = 1$, $(L_n)^\dagger = L_{-n}$, $n\in\mathbb{Z}$ — not necessarily positive-definite!

$|\chi\rangle \overset{\neq|h\rangle}{\in} V_{c,h}$ is a null-vector (or singular vector) if $L_{>0}|\chi\rangle = 0$

$\Leftrightarrow$ null-vector $|\chi\rangle$ is orthogonal to the whole $V_{c,h}$:
$\langle h|L_{n_1}\cdots L_{n_p}|\chi\rangle = 0$, $n_i>0$; in particular, $\boxed{\langle\chi|\chi\rangle = 0}$

- $|\chi\rangle$ generates its own Verma module $\underbrace{\mathrm{Span}\{L_{-n_1}\cdots L_{-n_r}|\chi\rangle\}}_{\simeq V_{c,h+N}} \subset V_{c,h}$ orthogonal to $V_{c,h}$. (N = level of $|\chi\rangle$)

- null-vector at level $N=1$:
$|\chi\rangle = L_{-1}|h\rangle$ is null iff $\underbrace{L_1 L_{-1}}_{2L_0 - L_{-1}L_1}|h\rangle = 2h|h\rangle = 0 \Leftrightarrow \underline{h=0}$

- null-vector at level $N=2$: $|\chi\rangle = (L_{-2}+\beta L_{-1}^2)|h\rangle$ is null if $\begin{cases} 0 = L_1|\chi\rangle = \dots \\ 0 = L_2|\chi\rangle = \dots \end{cases}$ → solvable for α,β if nontrivial condition on h:
$$\begin{vmatrix} 3 & 4h+2 \\ 4h+\frac{c}{2} & 6h \end{vmatrix} = 0$$

Let $M^{(N)} = (\langle i|j\rangle)_{i,j}$ run over basis in $V_{c,h}^N$ — $P(N)\times P(N)$ matrix
Gram matrix
– matrix of inner products

Kac det. formula: $\det M^{(N)} = \alpha_N \prod_{\substack{p,q\ge1 \\ pq\le N}} (h-h_{p,q}(c))^{P(N-pq)}$

here: • $\alpha_N = \prod_{\substack{p,q\ge1 \\ pq\le N}} \left((2p)^q q!\right)^{P(N-pq)-P(N-p(q+1))}$ — numerical factor,

• $h_{p,q} = \dfrac{((m+1)p-mq)^2-1}{4m(m+1)}$

where $m = -\frac12 \pm \frac12\sqrt{\frac{25-c}{1-c}}$ $\Leftrightarrow$ $c = 1-\dfrac{6}{m(m+1)}$

- $h = h_{p,q}$ implies that $V_{c,h}$ contains a null-vector for some $p,q\ge1$ at the level pq.

- $h_{1,1} = 0$ $\forall c$ • for $c=\frac12$ $\Leftrightarrow m=3$:
$h_{1,1} = 0$ ← 1 — correlators satisfy a 1st order ODE – are indep. of z
$h_{1,2} = \frac{1}{16}$, $= h_{2,2}$ ← σ
$h_{2,1} = \frac12$, $= h_{1,3}$ ← ψ
conformal dimensions of primary fields in the free fermion theory
correlators satisfy a hypergeom. equation!

q		
3	1/2	0
2	1/16	1/16
1	0	1/2
	1	2 → p

$c=\frac12$ ($m=3$) Kac table

Thm [Kac / Feigin–Fuchs] "Verma modules over Virasoro alg."
- $V_{c,h}$ is reducible iff $h=h_{p,q}$ for some $p,q\ge1$ (Kac)
- Quotient $M_{c,h}$ of $V_{c,h}$ by the max proper submodule is irreducible
- Any proper submodule of $V_{c,h}$ is generated by null-vectors

Ex: $V_{1,h}$ is reducible iff $h=\frac{n^2}{4}$, $n\in\mathbb{Z}$.

Thm $M_{c,h}$ is unitary ($\langle,\rangle$ is positive-definite)
iff either $c\ge1$, $h\ge0$
or $c = 1-\dfrac{6}{m(m+1)}$, $m=2,3,4,\dots$
and $h = h_{p,q}$ with $1\le p\le m-1$, $1\le q\le m$ / $h_{p,q} = h_{m-p,m+1-q}$

"minimal model of CFT" "$\mathcal{M}(m,m+1)$"
with primary fields $\Phi_{p,q}$ with $(h=h_{p,q},\bar h = h_{p,q})$, $c = 1-\frac{6}{m(m+1)}$
$\varphi_{p,q}(z)\varphi_{p,q}(\bar z)$

$m=3$ minimal model (= Ising model): $\Phi_{1,1} = \mathbb{1}$, $\Phi_{1,2} = \varepsilon$ ($=\psi\bar\psi$) "energy field", $\Phi_{2,1} = \sigma$ – "spin field"

$$\mathcal{H}_{\mathcal{M}(m,m+1)} = \bigoplus_{\substack{1\le p\le m-1 \\ 1\le q\le m}/\mathbb{Z}_2} M_{c,h_{p,q}} \left[\otimes M_{c,h_{p,q}}\right]$$
↑ non-chiral version

L_{2R} $\langle h | L_n L_{-n} | h \rangle = 2nh + c\,\frac{n^3-n}{12}$

conf. family
"fusion rules" in $m=3$ min. model:

$[\sigma][\sigma] = [\mathbb{1}] + [\varepsilon]$

$[\sigma][\varepsilon] = [\sigma]$

$[\varepsilon][\varepsilon] = [\mathbb{1}]$

Affine Lie algebras [ref: Kohno "CFT and topology"]

G compact simple Lie group → $\mathfrak{g}$ - assoc. Lie alg.

$LG = \mathrm{Maps}(S^1, G)$ with pointwise multiplication — loop group

$\mathbb{C}\otimes$ assoc. Lie alg

$L\mathfrak{g} = \mathrm{Maps}(S^1, \mathfrak{g}_{\mathbb{C}}) \simeq \mathfrak{g}\otimes\mathbb{C}[[t,t^{-1}]]$ - loop (Lie) algebra

with pointwise bracket $[X\otimes f, Y\otimes g] = [X,Y]\otimes fg$

$L\mathfrak{g}$ admits a unique ($H^2(L\mathfrak{g},\mathbb{C}) = \mathbb{C}$) central extension "$\hat{\mathfrak{g}}$" - "affine Lie algebra"

$0 \to \mathbb{C} \to \hat{\mathfrak{g}} \to L\mathfrak{g} \to 0$ with $[X\otimes f, Y\otimes g]_{\hat{\mathfrak{g}}} = \cancel{[X\otimes f, Y\otimes g]}\ [X,Y]\otimes fg + \underbrace{\mathbb{K}}_{\text{central elt}} \underbrace{\langle X,Y\rangle}_{\text{Killing form on } \mathfrak{g}} \underbrace{\mathrm{Res}_{t=0}}_{\text{coeff of } t^{-1}dt}(df\cdot g)$

more explicitly: $[X\otimes t^m, Y\otimes t^n] = [X,Y]\otimes t^{m+n} + \mathbb{K}\langle X,Y\rangle m\,\delta_{m,-n}$

- specialization $\mathbb{K} = k = 1,2,3,\ldots$ "level" corresponds to a central extension of the loop group

$$1 \to \mathbb{C}^* \to \widehat{LG}^k_{\mathbb{C}} \to LG_{\mathbb{C}} \to 1$$

Highest weight representations

$$\hat{\mathfrak{g}} = \underbrace{(\mathfrak{g}\otimes t\,\mathbb{C}[[t]] \oplus \mathfrak{g}_+)}_{N_+} \oplus \underbrace{(\mathbb{C}\mathbb{K} \oplus \overset{\text{Cartan for } \mathfrak{g}}{\mathfrak{h}})}_{N_0} \oplus \underbrace{(\mathfrak{g}\otimes t^{-1}\mathbb{C}[[t^{-1}]] \oplus \mathfrak{g}_-)}_{N_-}$$

Verma modules: $V^{\hat{\mathfrak{g}}}_{k,\lambda} = \mathrm{Ind}^{\hat{\mathfrak{g}}}_{N_0\oplus N_+} \underbrace{\mathbb{C}_{k,\lambda}}_{} = U(\hat{\mathfrak{g}}) \otimes_{N_0\oplus N_+} \mathbb{C}$

level; highest weight for $\mathfrak{g}$

1-dim rep of $N_0\oplus N_+$ where N_+ acts trivially, $\mathbb{K}$ acts as mult. by k, h - as mult. by λ (or λ_i if rank$(\mathfrak{g})>1$)

Irreducible h.w. modules:

$M^{\hat{\mathfrak{g}}}_{k,\lambda} = V^{\hat{\mathfrak{g}}}_{k,\lambda} / $ maximal proper submodule

A distinguished set of h.w. modules: integrable modules (those that induce a fin. dim. rep. for each $su(2) \hookrightarrow \hat{\mathfrak{g}}$)

those that satisfy $(e_\alpha\otimes t^{-j})^N u = 0$ $\forall u\in M\ \exists N$; e_α - root of $\mathfrak{g}$, $j\geq 0$ ("local nilpotency" condition)

- there are finitely many integrable reps at any given k.

Case $G = SU(2)$

$E = \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$F = \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$H = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$[H,E] = 2E$, $[H,F] = -2F$

$[E,F] = H$

For $k\in\mathbb{N}$, $0\leq\lambda\leq k$ (integer), there is an integrable rep. $H_{k,\lambda}$ of $\widehat{su(2)}_k$

- it induces a $(\lambda+1)$-dimensional ~~rep.~~ ("spin $= \frac{\lambda}{2}$") irrep $M^{\mathfrak{g}}_\lambda$ of $su(2)$

$\chi = E_{-1}^{k-\lambda+1} v$ - null-vector

$\psi = $ (also $E_0^{\lambda+1} v$)

$H_{k,\lambda} = (V^{\hat{\mathfrak{g}}}_{k,\lambda} / U(N_-)\psi) / U(N_-)\chi$

5/1/19 1   4/29/19 2

Sugawara construction.

One constructs a Virasoro rep. on $H_{k,\lambda}$ with

- $$L_n = \frac{1/2}{k + c^\vee} \sum_{j\in\mathbb{Z}} \sum_{a=1}^{\dim G} :\underbrace{(T^a\otimes t^{+j})}\cdot\underbrace{(T^a\otimes t^{n-j})}:$$

$c^\vee$ — dual Coxeter number for $\mathfrak{g}$, e.g. $c^\vee = 2$ for $su(2)$, $=N$ for $su(N)$

in representation $H_{k,\lambda}$

- $\{L_n\}$ satisfy Virasoro comm. rel. with $c = \dfrac{k\cdot\dim G}{k+c^\vee}$, e.g. for $\mathfrak{g}=su(2)$, $c = \dfrac{3k}{k+2}$

- for the highest vector $v\in H_{k,\lambda}$, $L_0 v = \underbrace{\boxed{\dfrac{C_\lambda}{k+c^\vee}}}_{=:\Delta_{k,\lambda}} v$ ← value of quadratic Casimir in $M^{\mathfrak{g}}_\lambda$, e.g. for $su(2)$: $C_\lambda = j(j+1)$, $j=\frac{\lambda}{2}$.

$[L_0, T^a\otimes t^{-j}] = j\, T^a\otimes t^{j}$

$$\Rightarrow\ H_{k,\lambda} = \bigoplus_{d=0}^{\infty} \underbrace{H_{k,\lambda}(d)}_{L_0\text{-eigenspace with eigenvalue } \Delta_{k,\lambda}+d}$$

- spaces $H_{k,\lambda}(d)$ are finite-dimensional
- $H_{k,\lambda}(0) \simeq M^{\mathfrak{g}}_\lambda$ - the "multiplet" of Virasoro-primary ~~fields~~ states

Wess-Zumino-Witten model

fix $G = SU(2)$ [for simplicity; can extend to any compact simple Lie group]

$\sigma = \frac{1}{24\pi^2}\, \mathrm{tr}\big((X^{-1}dX)\wedge(X^{-1}dX)\wedge(X^{-1}dX)\big) \in \Omega^3(SU(2))$

- invariant (left & right) volume form on $SU(2)$, representing the generator of $H^3(SU(2),\mathbb{Z})$

. for Σ a closed Riemannian surface,

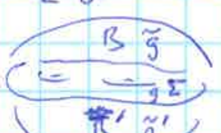
$$S_\Sigma(g) = \frac{-i}{4\pi}\int_\Sigma \mathrm{tr}(g^{-1}\partial g\wedge g^{-1}\bar\partial g) + \underbrace{\frac{-i}{12\pi}\int_B \mathrm{tr}(\tilde g^{-1}d\tilde g)^{\wedge 3}}_{(-i)2\pi\int_B \tilde g^*\sigma\ =:\ WZ(g)}$$

field $g\in \mathrm{Maps}(\Sigma, G)$

B - cpt oriented 3-mfd with $\partial B = \Sigma$

$\tilde g: B\to G$ - a smooth extension of $g:\Sigma\to G$

- $S_\Sigma(g)$ mod $2\pi i\mathbb{Z}$ does not depend on B and choice of $\tilde g$:



~~$\frac{1}{12\pi}\int_B \mathrm{tr}(\tilde g^{-1}d\tilde g)^3$~~

$$S_\Sigma(g) - S_\Sigma(g)' = \frac{-i}{12\pi}\left(\int_B \mathrm{tr}(\tilde g^{-1}d\tilde g)^3 - \int_{B'} \mathrm{tr}(\tilde g'^{-1}d\tilde g')^3\right)$$

$$= -i\, 2\pi \int_M G^*\sigma \in 2\pi i\mathbb{Z}.$$

$M = B\cup\bar B'$ - closed 3-mfd

$\Rightarrow e^{-k S_\Sigma(g)}$ does not depend on $B, \tilde g$

Rem: $WZ(g)$ term is non-local but its variation is $\delta WZ(g) = \int_\Sigma \delta g(\ldots)$ (Σ not B)

- Equation of motion: $\boxed{\partial(g^{-1}\bar\partial g) = 0}$ solution: $g(z,\bar z) = h(z)\cdot h(\bar z)$

- "Gauge invariance" $S_\Sigma(g)$ is invariant under $g(z,\bar z)\mapsto \Omega(z)\, g(z,\bar z)\, \bar\Omega^{-1}(\bar z)$

$\Omega\in \mathrm{Maps}_{\text{holom}}(\Sigma, G)$
$\bar\Omega \in \mathrm{Maps}_{\text{anti-holom}}(\Sigma, G)$

assoc. Noether currents: $J = g^{-1}\partial g$, $\bar J = g^{-1}\bar\partial g$ conservation: $\bar\partial J \sim 0$, $\partial\bar J\sim 0$ mod EL

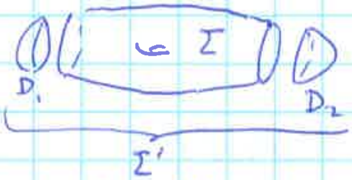
Polyakov-Wiegmann f-la $\qquad$ 5/1/19 (2) $\qquad$ ~~7/29/19~~ (3)

$$e^{-\xi k S_\Sigma(f\cdot g)} = e^{-\xi k S_\Sigma(f) + \xi k S_\Sigma(g) - \frac{ik}{2\pi}\overbrace{\int_\Sigma tr\left(f^{-1}\partial f \wedge g^{-1}\bar\partial g\right)}^{\Gamma_\Sigma(f,g)}}$$

$f\cdot g$: pointwise multiplication

$\Gamma_\Sigma(f,g)$: 2-cocycle for the group Maps(Σ, G)

* Case with boundary (not straightforward to generalize S_Σ to Σ with bdry due to non-local WZ term in the action)



construct Σ' closed, $\Sigma'\setminus\Sigma = \coprod_{i=1} D_i$ - disks

Idea: set $e^{-\xi k S_\Sigma(g)} := e^{-\xi k S_{\Sigma'}(g')}$ ← some extension of g to Σ'

ambiguity in the choice of extension g' →

→ $e^{-\xi k S_\Sigma(g)}$ takes values in the fiber of $\mathcal{L}_k^{(\pm)} \boxtimes \dots \boxtimes \mathcal{L}_k^{(\pm)}$ over $g|_{\partial\Sigma} \in$ Maps$(\partial\Sigma, G)$

$\mathcal{L}_k$ is a complex line bundle over LG constructed as:

$$\{(f_D : \underset{\text{unit disk}}{D} \to G,\ u\in\mathbb{C})\}\Big/ (f_D,u)\sim(g_D,v) \text{ iff } f|_{\partial D} = g|_{\partial D},$$

and $v = u\cdot e^{-\xi k S_{\mathbb{CP}^1}(h) - \frac{ik}{2\pi}\Gamma_D(f_D, h_D)}$

where h_D is defined by $g_D = f_D\cdot h_D$ and h is the extension by $\mathbb{1}$ to $\mathbb{CP}^1\setminus D$.

Rem $\mathcal{L}_k = \mathcal{L}^{\otimes k}$ where $\begin{matrix}\mathcal{L}\\ \downarrow\\ LG\end{matrix}$ is the Hermitian line bundle with 1st Chern class $c_1 = [\omega]$ $\quad \omega = \oint_{S^1} \varphi^* \underset{H^3(G,\mathbb{Z})}{\underset{\uparrow}{\sigma}} \in H^2(LG,\mathbb{Z})$

$$\begin{matrix} LG\times S^1 \xrightarrow{\varphi} G \\ \downarrow \\ LG \end{matrix}$$

Quantization

Space of states $\mathcal{H}_k = \bigoplus_{\lambda=0}^{k} H_{k,\lambda}\otimes H^*_{k,\lambda}$ $\hookrightarrow \Gamma(LG, \mathcal{L}_k)$ space of sections

- carries the action of $\hat{\mathfrak{g}}_k \oplus \hat{\mathfrak{g}}_k$ and therefore (by Sugawara), $Vir\oplus\overline{Vir}$ - action with $c = \bar c = \frac{3k}{k+2}$

$$\varphi_0 \in End(H_{k,\lambda}) \longmapsto tr_{R_\lambda}(\varphi_0 \cdot R_\lambda(g))\cdot e^{+k S_D(g_+ g g_-^*)} \quad \left(\text{“}\varphi(g_+ g g_-^*)\text{”}\right)$$

· current: $J(z) = g^{-1}\partial g \quad \underset{\text{in quantum world}}{\rightsquigarrow} \quad \hat{J}^a(z) = \sum_{n\in\mathbb{Z}} \hat{J}^a_n z^{-n-1}$ (operator acting on $\mathcal{H}$) $\qquad$ previous notation $(\hat{J}^a_n = T^a\otimes t^n)$

with $[\hat{J}^a_n, \hat{J}^b_m] = f^{abc}\hat{J}^c_{n+m} + k\, n\, \delta_{n,-m}$ - comm. rel. of $\hat{\mathfrak{g}}_k$

$\updownarrow$

basic OPE: $J^a(w)\, J^b(z) \sim \frac{k\,\delta^{ab}\,\mathbb{1}}{(w-z)^2} + \frac{f^{abc}}{w-z} J^c(z) + \text{reg.}$

$\hat{\mathfrak{g}}$-primary field (or, rather, multiplet): $\varphi_{(\lambda)}$ with values in $M^{\mathfrak{g}}_\lambda$, $\qquad (*)\ J^a(w)\,\varphi_{(\lambda)}(z,\bar z) \sim \frac{T^a_{(\lambda)}}{w-z}\varphi_{(\lambda)}(z,\bar z) + \text{reg}$ ($T^a_{(\lambda)}$: matrix of rep. $M^{\mathfrak{g}}_\lambda$)

stress-energy tensor: $T(z) = \frac{1/2}{k+c^\vee}\sum_{a=1}^{\dim G} :J^a(z)\,J^a(z):$

- satisfies TT OPE with $c = \frac{k\dim G}{k+c^\vee}$

5/1/19 3 ~~4/29/19 2~~

Ward identity for $\hat{\mathfrak{g}}$-symmetry:

$$(**) \quad \langle J^a(z)\, \underset{\substack{\uparrow \\ \hat{\mathfrak{g}}\text{-primary field}}}{\varphi_{(\lambda_1)}(z_1,\bar z_1)} \cdots \varphi_{(\lambda_n)}(z_n,\bar z_n)\rangle = \sum_{j=1}^n \frac{T^a_{(\lambda_j)}}{z-z_j}\langle \varphi_{(\lambda_1)}(z_1,\bar z_1)\cdots \varphi_{(\lambda_n)}(z_n,\bar z_n)\rangle$$

- follows from $(*)$

Sugawara $\rightarrow$ $L_{-1} = \frac{1/2}{k+C^\vee}\sum_{j\in\mathbb{Z}} :J^a_{-j} J^a_{-1+j}:$ $\rightsquigarrow$ for $\varphi_{(\lambda)}$ $\hat{\mathfrak{g}}$-primary, $\underset{\substack{\uparrow\\ \text{local Vir generator}}}{L_{-1}}\varphi_{(\lambda)} = \frac{1}{k+C^\vee}\underset{\substack{\uparrow\\ \text{local } \hat{\mathfrak{g}}\text{-generator}}}{J^a_{-1}} T^a_{(\lambda)}\varphi_{(\lambda)}$

$\Rightarrow$

$$0 = \langle \varphi_{(\lambda_1)}(z_1,\bar z_1)\cdots\Big(\underbrace{L^{(z_j)}_{-1}}_{\frac{\partial}{\partial z_j}} - \frac{1}{k+C^\vee}\underbrace{J^{(z_j)a}_{-1}}_{=\oint_{C_j}\frac{dw}{2\pi i}\frac{1}{w-z_j}J^a(w)} T^a_{(\lambda_j)}\Big)\varphi_{(\lambda_j)}(z_j,\bar z_j)\cdots\varphi_{(\lambda_n)}(z_n,\bar z_n)\rangle$$

$$\underset{\text{using }(**)}{=} \boxed{\Big(\frac{\partial}{\partial z_j} - \frac{1}{k+C^\vee}\sum_{i\neq j}\frac{T^a_{(\lambda_j)}T^a_{(\lambda_i)}}{z_j-z_i}\Big)\langle\varphi_{(\lambda_1)}(z_1,\bar z_1)\cdots\varphi_{(\lambda_n)}(z_n,\bar z_n)\rangle}$$

- Knizhnik-Zamolodchikov equation

$\forall j=1,\ldots,n$

Space of conformal blocks

let $\mathfrak{g}(z_1,\ldots,z_n) = \mathfrak{g}\otimes\left\{\begin{matrix}\text{merom. functions on } \mathbb{CP}^1\\ \text{with poles at } z_1\ldots z_n \text{ allowed}\end{matrix}\right\}$

$\mathfrak{g}(z_1,\ldots,z_n)$ acts on $H_{\lambda_1}\otimes\cdots\otimes H_{\lambda_n}$ — $\alpha\in\mathfrak{g}(z_1\ldots z_n)$ expand in a Laurent series near z_j, use the embedding $\alpha = \sum_{n=-N}^{\infty}\alpha^a_n (T^a\otimes t_j^n)$, $t_j = z-z_j$; $\mathfrak{g}\otimes\mathbb{C}[t_j^{-1}, t_j]] \hookrightarrow \hat{\mathfrak{g}}$, act with $\hat{\mathfrak{g}}$ on H_{λ_j}

WZW

Space of conformal blocks:

$$\mathcal{H}(z_1,\ldots,z_n;\lambda_1,\ldots,\lambda_n) := \boxed{\mathrm{Hom}_{\mathfrak{g}(z_1,\ldots,z_n)}(H_{\lambda_1}\otimes\cdots\otimes H_{\lambda_n},\mathbb{C})}$$

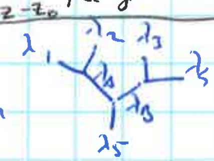
idea: $\Rightarrow$ $\langle \underset{\substack{\cap\\ H_{\lambda_1}}}{\varphi_1(z_1)}\cdots \underset{\substack{\cap\\ H_{\lambda_n}}}{\varphi_n(z_n)}\rangle$ - correlator of the chiral theory

Ward identity:

$\alpha\circ\langle\varphi_1(z_1)\cdots\varphi_n(z_n)\rangle = 0$.

Rem: applying to $\langle \mathbb{1}_{z_0}\varphi_1(z_1)\cdots\varphi_n(z_n)\rangle$ (primary) with $\alpha = \frac{T^a}{z-z_0}$, we get $(**)$

* $\mathcal{H}$ is finite dimensional!
* for $n=3$, $\dim\mathcal{H} = \begin{cases} 1 & \text{if } \begin{cases}\lambda_1+\lambda_2+\lambda_3\in 2\mathbb{Z}\\ |\lambda_1-\lambda_2|\le\lambda_3\le\lambda_1+\lambda_2\\ \lambda_1+\lambda_2+\lambda_3\le 2k\end{cases}\\ 0 & \text{otherwise}\end{cases}$ ("fusion rules" = "quantum Klebsch-Gordan condition")



* for general n, can associate a basis in $\mathcal{H}$ to a 3-valent tree with leaves decorated with $\lambda_1\ldots\lambda_n$
  basis vectors $\sim$ decorations of edges e by labels $\lambda_e\in\{0,\ldots,k\}$ s.t. fusion rules hold at each vertex.

~~Weeh~~ • $\mathbb{CP}^1 \to$ higher genus Σ of : replace trees with 3-valent graphs with g loops.

• Verlinde formula: $\dim\mathcal{H}(z_1\ldots z_n;\lambda_1\ldots\lambda_n) = \sum_{0\le\lambda\le k}\frac{S_{\lambda_1\lambda}\cdots S_{\lambda_n\lambda}}{(S_{0\lambda})^{n-2}}$ where $S_{\lambda\mu} = \sqrt{\frac{2}{k+2}}\sin\frac{(\lambda+1)(\mu+1)}{k+2}$

• $\mathcal{H}(z_1\ldots z_n;\lambda_1\ldots\lambda_n) \to \mathcal{E}_{\lambda_1\ldots\lambda_n}$ - $\mathbb{C}$-v.bun. of conformal blocks
$\qquad\downarrow$
$\quad\mathrm{Conf}_n(\mathbb{CP}^1)$

$\nabla_{\frac{\partial}{\partial z_j}} = \frac{\partial}{\partial z_j} - \underset{\substack{\uparrow\\ \text{Sugawara operator on } H_{\lambda_j}}}{L^{(j)}_{-1}}$ - a(holom.) flat connection on $\mathcal{E}_{\lambda_1\ldots\lambda_n}$

if $\Psi\in\Gamma(\mathcal{E}_{\lambda_1\ldots\lambda_n})$ is a horizontal section, then Ψ_0^a = restriction of Ψ to $M^{\mathfrak{g}}_{\lambda_1}\otimes\cdots\otimes M^{\mathfrak{g}}_{\lambda_n}$ satisfies KZ equation: $\frac{\partial\Psi_0^a}{\partial z_i} = \frac{1}{k+2}\sum_{j\neq i}\frac{T^{a(i)}T^{a(j)}}{z_i-z_j}\Psi_0$