Topics in Topology II, Spring 2019
MATH 80440
"Introduction to Conformal Field Theory"
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## Tentative program of the course

- Introduction: Segal's picture of CFT – surfaces with boundaries and sewing [Segal]. CFT as a set of correlators. Special fields: identity, stress-energy tensor. Action of conformal vector fields.
- Bits of conformal geometry.
  - Weyl transformations, conformal maps. Conformal diffeomorphisms of $\mathbb{R}^{p,q}$ with $p+q>2$: translations+rotation+scaling+special canonical transformations. [Schottenloher, Ginsparg]
  - Conformal diffeomorphisms of $\mathbb{R}^2$ (and $\mathbb{R}^{1,1}$) – global (Möbius group) and infinitesimal (holomorphic vector fields). Witt algebra.
  - Conformal structures = complex structures. Moduli space $\mathcal{M}_{g,n}$. Tangent space to $\mathcal{M}_{g,n}$: Beltrami differentials, Kodaira-Spencer equation (Newlander-Nirenberg theorem).
- Some classical field theory.
  - Examples of Weyl-invariant action functionals. Free boson, sigma model, free fermion, classical WZW. [Kohno?]
  - Noether theorem. Symmetries $\mapsto$ conserved quantities; Noether currents.
  - Stress-energy tensor. Examples. [DMS]
- Generalities of CFT on a plane. [BPZ]
  - Correlators. Constraints from global conformal symmetry. Conformal Ward identities.
  - OPEs. OPE with $T$ vs transformation properties. Primary fields.
  - Mode operators for $T$. Virasoro algebra. $TT$ OPE. Central charge.
  - The meaning of central charge – conformal anomaly. (Possibly: $T$ as a connection on the moduli space.)
  - Correlators as built out of conformal blocks.
- Quantum free boson [Ginsparg].
  - Canonical quantization on a cylinder. Expansion in harmonic oscillators (creation/annihilation operators). Fock space.
  - Path integral quantization on the plane. Correlators, OPEs.
  - Vertex operators.
  - Free boson with target a circle (or $S^1\times\cdots\times S^1$). Case $r^2\in\mathbb{Q}$ (first example of a *rational* CFT): Electric/magnetic numbers, instantons, vertex operators. Partition function on a torus.
- Free fermion.
  - Aside: Ising model, critical phenomena, Polyakov's view of CFT as coming from a lattice statistical model.
  - (Bosonization formulae?)
- $bc$ system (reparametrization ghosts). Where does dimension 26 in bosonic string theory come from?
- Minimal models of CFT.
  - Representation theory of Virasoro algebra, Verma modules, singular vectors, Kac determinant.

  - (Coset construction ?)
- Wess-Zumino-Witten model. [DMS, Kohno, GawedzkiWZW]
  - Affine Lie (Kac-Moody) algebras. Embedding of the Virasoro algebra – Sugawara construction.
  - The space of conformal blocks. Verlinde formula. Index theorem (Riemann-Roch) approach.
  - Knizhnik-Zamolodchikov connection on the configuration space of points [KZ]. Its holonomy – Drinfeld associator.
  - Modular functor [Segal?], representations of the mapping class group.
  - Relation to Chern-Simons theory [WittenCS]. (KZ connection vs. Hitchin connection on the bundle of conformal blocks over $\mathcal{M}_{g,n}$ [ADPW].)
- The formalism of vertex algebras. [Kac, Frenkel-Ben-Zvi, Borcherds?]
- Topological and supersymmetric CFTs.
  - Topological conformal field theories. Operadic structure of OPEs. Framed $E_2$ and BV operads appearing in TCFT. [Getzler]
  - Supersymmetric CFTs. Supersymmetric sigma model and Witten's topological A and B models ("topological twists"). [WittenMirror]

Some references:


[ADPW ] S. Axelrod, S. Della Pietra, E. Witten, "Geometric quantization of Chern Simons gauge theory," representations 34 (1991): 39.

[BPZ ] A. A. Belavin, A. M. Polyakov, A. B. Zamolodchikov, "Infinite conformal symmetry in two-dimensional quantum field theory," Nuclear Physics B 241.2 (1984): 333–380.

[DMS ] P. Di Francesco, P. Mathieu, D. Sénéchal, "Conformal Field Theory," Springer 1997.

[Frenkel-Ben-Zvi ] E. Frenkel, D. Ben-Zvi, "Vertex algebras and algebraic curves," No. 88. AMS, 2004

[GawedzkiSurvey ] K. Gawedzki, "Conformal field theory," Séminaire Bourbaki 31.1988–1989.

[GawedzkiWZW ] K. Gawedzki, "Conformal field theory: a case study," arXiv:hep-th/9904145.

[Getzler ] E. Getzler, "Batalin-Vilkovisky algebras and two-dimensional topological field theories," Commun. Math. Phys. 159.2 (1994): 265–285.

[Ginsparg ] P. Ginsparg,"Applied conformal field theory," hep-th/9108028.

[Kac ] V. G. Kac, "Vertex algebras for beginners," No. 10. AMS, 1998.

[Kohno ] T. Kohno, "Conformal field theory and topology," Vol. 210 AMS 2002.

[KZ ] V. G. Knizhnik, A. B. Zamolodchikov, "Current algebra and Wess-Zumino model in two dimensions," Nuclear Physics B 247.1 (1984): 83-103.

[Schottenloher ] M. Schottenloher, "A mathematical introduction to conformal field theory," Vol. 759, Springer 2008.

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[Segal ] G. Segal, "The definition of conformal field theory," Differential geometrical methods in theoretical physics. Springer, Dordrecht, 1988. 165–171.

[WittenCS ] E. Witten, "Quantum field theory and the Jones polynomial," Commun. Math. Phys. 121.3 (1989) 351–399.

[WittenMirror ] E. Witten, "Mirror manifolds and topological field theory," hep-th/9112056.