CFT
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# Introduction to [two-dimensional] conformal field theory

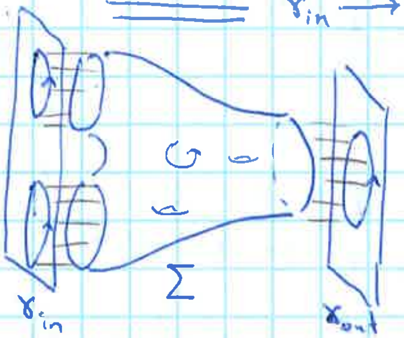
## Segal's picture of a (D-dimensional) quantum field theory

[G. Segal "The definition of CFT", 1988
long version: 2002]

a QFT is an assignment: * oriented closed (D−1)-manifolds $\gamma$ $\longrightarrow$ ~~Hilbert~~ vector space over $\mathbb{C}$ $\mathcal{H}_\gamma$, – "space of states" $\Leftrightarrow$

* D- ~~cobordism~~ (oriented) manifold $\Sigma$ with boundary $\partial\Sigma = \gamma_{out} \sqcup \bar{\gamma}_{in}$ (reversed orientation) $\longrightarrow$ linear map $Z_\Sigma : \mathcal{H}_{\gamma_{in}} \longrightarrow \mathcal{H}_{\gamma_{out}}$ – "partition function"

= cobordism $\gamma_{in} \xrightarrow{\Sigma} \gamma_{out}$

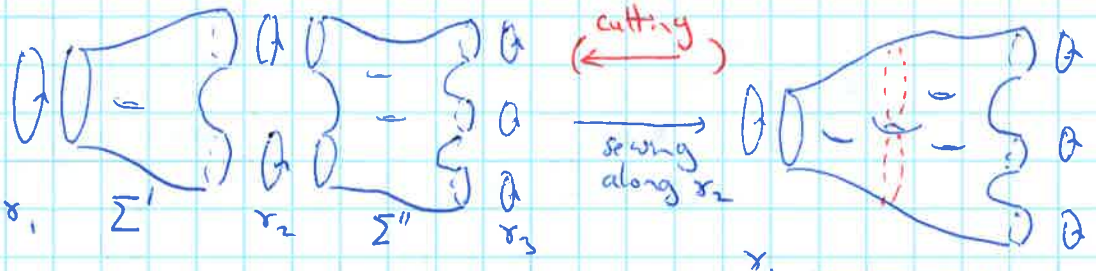


axioms:
- multiplicativity: $\mathcal{H}_{\gamma_1 \sqcup \gamma_2} = \mathcal{H}_{\gamma_1} \otimes \mathcal{H}_{\gamma_2}$

  $Z_{\Sigma_1 \sqcup \Sigma_2} = Z_{\Sigma_1} \otimes Z_{\Sigma_2} : \mathcal{H}_{\gamma_1^{in}} \otimes \mathcal{H}_{\gamma_2^{in}} \to \mathcal{H}_{\gamma_1^{out}} \otimes \mathcal{H}_{\gamma_2^{out}}$

- sewing axiom:

  given two cobordisms



$\gamma_1 \xrightarrow{\Sigma = \Sigma' \cup_{\gamma_2} \Sigma''} \gamma_3$

axiom: $Z_\Sigma = Z_{\Sigma''} \circ Z_{\Sigma'} : \mathcal{H}_{\gamma_1} \longrightarrow \mathcal{H}_{\gamma_3}$

- normalization:
  - $\mathcal{H}_\emptyset = \mathbb{C}$
  - $\lim_{\varepsilon \to 0} Z_{\gamma \times [0,\varepsilon]} = \mathrm{id} \circlearrowleft \mathcal{H}_\gamma$ – very short cylinder

- symmetry:

  $\mathcal{H}_{\pi(\gamma_1 \sqcup \cdots \sqcup \gamma_n)} = \pi(\mathcal{H}_{\gamma_1} \otimes \cdots \otimes \mathcal{H}_{\gamma_n})$
  permutation

  $Z_{\pi(\Sigma_1 \sqcup \cdots \sqcup \Sigma_n)} = \pi(Z_{\Sigma_1} \otimes \cdots \otimes Z_{\Sigma_n})$

additional data:

* for ~~every~~ $\varphi : \gamma \to \tilde{\gamma}$ diffeomorphism, we have a map $\rho(\varphi) : \mathcal{H}_\gamma \to \mathcal{H}_{\tilde{\gamma}}$ $\begin{cases} \text{- linear if } \varphi \text{ preserves orientation} \\ \text{- anti-linear otherwise} \end{cases}$

* cobordisms are equipped with local geometric data $\xi_\Sigma \in \mathrm{Geom}_\Sigma$ E.g.
  - metric
  - conformal structure*
  - nothing (topological theories – Atiyah)

  boundaries also, $\xi_\gamma \in \mathrm{Geom}_\gamma$ E.g.
  - Riemannian collar
  - parametrization of a circle*

> in the sewing axiom, we sew the geom. data

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## Axioms cont'd:

Naturality (Diff-equivariance)

- for $\varphi : \Sigma \to \widetilde{\Sigma}$ a diffeomorphism,

we have

$$\begin{tikzcd} \mathcal{H}_{\gamma_{in}} \arrow[r, "Z_{\Sigma,\xi}"] \arrow[d, "\rho(\varphi|_{in})"'] \arrow[dr, phantom, "\square"] & \mathcal{H}_{\gamma_{out}} \arrow[d, "\rho(\varphi|_{out})"] \\ \mathcal{H}_{\tilde{\gamma}_{in}} \arrow[r, "Z_{\widetilde{\Sigma},\tilde{\xi}=\varphi_* \xi}"] & \mathcal{H}_{\tilde{\gamma}_{out}} \end{tikzcd}$$

- commutes ← in particular, diffeo preserving $\xi$ are symmetries of the theory

- for $r : \gamma \xrightarrow{id} -\gamma$ ↖ or. reversal, $\rho(r) : \mathcal{H}_\gamma \xrightarrow{c.c.} \mathcal{H}_{-\gamma}$ - complex conjugation

$\psi \longmapsto \overline{\psi}$

"crossing axiom"

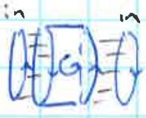
$Z_\Sigma : \mathcal{H}_{\gamma_1} \otimes \cdots \otimes \mathcal{H}_{\gamma_n} \to \mathcal{H}_{\gamma_{n+1}} \otimes \cdots \otimes \mathcal{H}_{\gamma_{n+m}}$

$\Big\}$

$Z_{\widetilde{\Sigma}} : \mathcal{H}_{\gamma_1} \otimes \cdots \otimes \mathcal{H}_{\gamma_n} \otimes \mathcal{H}_{-\gamma_{n+1}} \to \mathcal{H}_{\gamma_{n+2}} \otimes \cdots \otimes \mathcal{H}_{\gamma_{n+m}}$

re-assigning $\gamma_{n+1}$ as incoming

- $\mathcal{H}_{-\gamma} = \mathcal{H}_\gamma^*$ ← linear dual $\longleftarrow\!\!\sim$



↑ short cylinder

$\to Z(\cdots) : \mathcal{H}_\gamma \otimes \mathcal{H}_{-\gamma} \to \mathbb{C}$

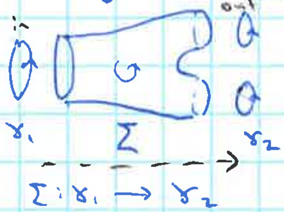
= canonical pairing $(\ ,\ )$

- Hermitian structure:

$\langle\ ,\ \rangle : \mathcal{H}_\gamma \otimes \mathcal{H}_\gamma \to \mathbb{C}$

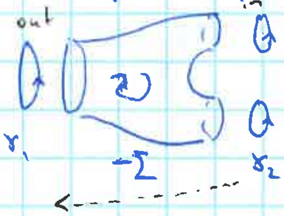
$= (c.c.(-), -)$ $\quad \psi, \chi \longmapsto \langle \psi, \chi \rangle = (\overline{\psi}, \chi)$

- Unitarity (or "reflection-positivity") - optional!



$\Sigma : \gamma_1 \to \gamma_2$

or. reversal $\rightsquigarrow$



$-\Sigma : \gamma_2 \to \gamma_1$

axiom: $Z_{-\Sigma} = \overline{Z}_\Sigma^*$

- $Z(\boxed{\ }) : \mathcal{H}_\gamma \xrightarrow{\text{unitary}} \mathcal{H}_\gamma$

$\Sigma = \gamma \times [0,t]$

- $\rho : \mathrm{Diff}(\gamma) \to \mathrm{End}(\mathcal{H}_\gamma)$ - unitary representation

**Example** D - any $\geq 1$ $\quad \mathcal{H}_\gamma = \mathbb{C}$ $\forall \gamma$ $\quad , \quad Z_\Sigma : \mathbb{C} \xrightarrow{\ \cdot e^{\chi(\Sigma, \gamma_{in})} = \cdot e^{\chi(\Sigma) - \chi(\gamma_{in})}\ } \mathbb{C}$

(relative) Euler characteristic

(trivial TQFT) $\quad$ Geom $= \{\emptyset\}$ no geom data

multiplicativity & sewing follow from additivity of $\chi$

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**Example:** D = 1 - quantum mechanics

1-cobordisms equipped with Riemannian metric (and orientation). Note: $\mathrm{Met}(\bullet\!\!\to)/\mathrm{Diff}(\bullet\!\!\to) = \mathbb{R}_+$ ↑ length

$\overset{+}{\bullet} \to \mathcal{H}$

$\overset{\bullet}{-} \to \mathcal{H}^*$

$\longmapsto Z_t : \mathcal{H} \to \mathcal{H}$

sewing axiom →

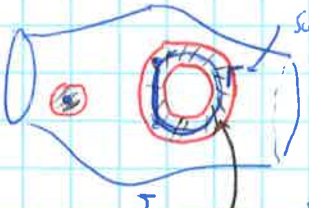
$\longmapsto Z_{t_1+t_2} = Z_{t_2} \circ Z_{t_1}$ - semi-group law

(strengthened) normalization: $Z_\varepsilon = id + \frac{i}{\hbar}\hat{H}\varepsilon + O(\varepsilon^2)$

↑ some lin. operator

$\Rightarrow$ $\boxed{Z_t = e^{-\frac{i}{\hbar}\hat{H}t}}$ - evolution operator (evol. in time $t$) in quantum mechanics, with quantum Hamiltonian $\hat{H}$

## Observables (idea):



submanifold, or sub-CW complex

$U_\varepsilon(\Gamma)$ - $\varepsilon$-thickening / tubular nbhd in $\Sigma$

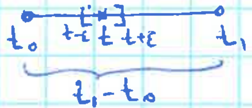
family, $\varepsilon\to0$

$\hat{O}_{\Gamma,\varepsilon}\in\mathcal{H}_{\partial U_\varepsilon(\Gamma)}$ - (quantum) observable "at $\Gamma$"

$$\langle\hat{O}_\Gamma\rangle_\Sigma=\lim_{\varepsilon\to0}\{Z_{\Sigma\setminus U_\varepsilon(\Gamma)}\circ\hat{O}_{\Gamma,\varepsilon}$$

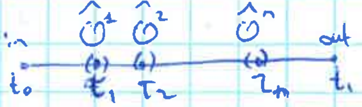
expectation value / correlator

**Example** - QM



$\hat{O}\in\mathcal{H}\otimes\mathcal{H}^*=\mathrm{End}(\mathcal{H})$

expectation value: ~~$\langle\psi^{out}_{t_1}|\hat{O}_t|\psi^{in}_{t_0}\rangle$~~

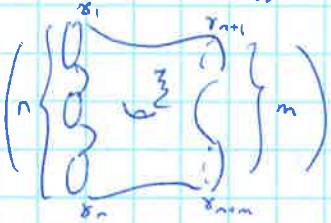


~~$\langle\psi^{out}_{t_1}|\hat{O}$~~ $\underbrace{\langle\psi^{out}_{t_1}|}_{\in\mathcal{H}^*}Z\underbrace{|\psi^{in}_{t_0}\rangle}_{\in\mathcal{H}}=\langle\psi^{out}_{t_1}|\,e^{-\frac{i}{\hbar}\hat{H}(t_1-\tau_n)}\hat{O}^n\cdots e^{-\frac{i}{\hbar}\hat{H}(\tau_2-\tau_1)}\hat{O}^1e^{-\frac{i}{\hbar}\hat{H}(\tau_1-t_0)}|\psi^{in}_{t_0}\rangle$

* * *

**CFT**: 2D cobordisms - equipped with conformal structure $=$ metric $g/_{g\sim\underbrace{\Omega}_{\text{positive function}}\cdot g}$

boundaries $=$ unions of $S^1$'s

$Z\Big($


$\Big):\ \mathcal{H}_{S^1}^{\otimes n}\to\mathcal{H}_{S^1}^{\otimes m}$

boundary circles are equipped with parametrization.

- $\mathcal{H}_{S^1}$ carries a projective representation of $\mathrm{Diff}(S^1)$.

- self-sewing: if $\tilde{\Sigma}=\Sigma$ with $\overset{\text{in}}{\gamma_i}$ identified with $\overset{\text{out}}{\gamma_j}$, then $\boxed{Z(\tilde{\Sigma})=\mathrm{tr}_{\mathcal{H}}Z(\Sigma)}$

viewed as ~~$\mathcal{H}\to\mathcal{H}$~~ $\mathrm{End}(\mathcal{H})\otimes\mathrm{Hom}(\dots,\dots)$
$\otimes\dots\ \otimes\dots$ $\mathcal{H}_i\to\mathcal{H}_j$

* assumption: $Z(\Sigma)$ is trace-class - "compactness" of CFT

genus 1 partition functions

- for $\mathbb{T}_\tau=\mathbb{C}/_{\mathbb{Z}\oplus\tau\mathbb{Z}}$ - torus, $\mathrm{Im}\,\tau>0$, $Z(\mathbb{T}_\tau)=Z(\mathbb{T}_{-1/\tau})$ conformally equivalent torus - modular invariance

- conformal anomaly: $Z(\Sigma,\overset{e^{2\sigma}}{\Omega}\cdot g)=e^{ic\,S_{\text{Liouville}}(\sigma,g)}\cdot Z(\Sigma,g)$

$S_{\text{Liouville}}(\sigma,g)=\int\frac{1}{2}(d\sigma\wedge*d\sigma+4\sigma R_g\,d\mathrm{vol}_g)$

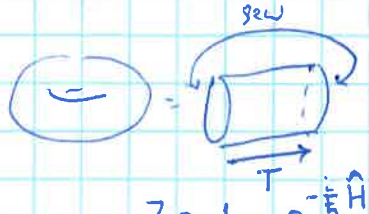
Ricci scalar ~~Ricci scalar curvature 2-form~~ of $g$

$c$: number - "central charge"

So, $Z$ depends (mildly) on the particular metric representing a conformal class.

$Z(\Sigma,g/_\sim)$ - "line of trace-class operators"

a related point: action $\mathrm{Diff}(S^1)\circlearrowright\mathcal{H}_{S^1}$ is projective.

or:


$Z_T=\mathrm{tr}\,e^{-\frac{i}{\hbar}\hat{H}T}=Z_{1/T}$ ← because it is a conf. equivalent torus

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## Why care about CFT?

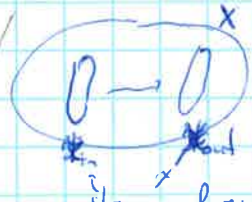
- a "tame" QFT with explicit nontrivial answers
- relation to critical phenomena in statistical physics (Polyakov)

2D Ising model  in the limit of infinite lattice has a 2nd order phase transition.

at $T_{crit}$, correlation radius for a 2-point correlator $\langle \sigma(x)\,\sigma(y)\rangle$ $\to \infty$ and $\langle \sigma(x)\sigma(y)\rangle \sim \frac{1}{|x-y|^{\boxed{1/4}}}$ critical exponent

→ one can "zoom out" of the lattice and see a scaling-invariant field theory on a plane
⟶ improved to conformal invariance

- relation to string theory



$\rightsquigarrow K(x_{out}, x_{in})$ - string propagation amplitude

$$\text{“}\int_{x: S^1\times[0,t]\to X} \text{”}\, e^{\frac{i}{\hbar}S(x)}\, Dx\text{”}$$

$x|_{S^1\times 0} = x_{in}$

$x|_{S^1\times t} = x_{out}$

$\mathcal{H}_{S^1} = \mathrm{Fun}_{\mathbb{C}}(LX) \ni \Psi(x \in LX)$ - wave-function

- integral kernel of

$Z($  $): |x_{in}\rangle \longmapsto \int |x_{out}\rangle\, K(x_{out}, x_{in})$

$$\Psi_{in} \longmapsto \left(\Psi_{out}: x_{out} \mapsto \int_{x_{in}\in LX} K(x_{out}, x_{in})\,\Psi_{in}(x_{in})\, Dx_{in}\right)$$

string theory = quantization of a Lagrangian field theory on $\mathrm{Maps}(\overset{\text{worldsheet}}{\Sigma}, X = \mathbb{R}^N)$ $\rightsquigarrow$ CFT of $N$ scalar fields on $\Sigma$.

- relation to 3D TQFT, especially Wess-Zumino-Witten on $\Sigma = \partial M$ $\overset{2d\,CFT}{\longleftarrow}$ Chern-Simons 3d TQFT on $M$

- CFT ↔ rep. theory of $\mathrm{Diff}(S^1)$, Virasoro algebra, affine Lie algebras $\hat{\mathfrak{g}}$.
  ↕ interesting structures on $\mathcal{M}_{g,n}$
  representations of mapping class group of $\Sigma$.

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Segal's axioms formalize the heuristic construction: $\langle \varphi_{out}| Z($  $)|\varphi_{in}\rangle = $ "$\int_{\substack{\varphi \in \Gamma(\Sigma, \mathbb{F}) = \mathrm{Fields}_\Sigma \\ \varphi|_{\gamma_{in}} = \varphi_{in} \\ \varphi|_{\gamma_{out}} = \varphi_{out}}} e^{\frac{i}{\hbar}S(\varphi)} D\varphi$" $=: K_\Sigma(\varphi_{out}, \varphi_{in})$

"matrix element" of Z

action functional

sheaf of fields

$$Z: |\Psi\rangle = \int_{\varphi_{in} \in \Gamma(\gamma_{in}, \mathbb{F})} \underbrace{\Psi_{in}(\varphi_{in})}_{\text{wave-function}} |\varphi_{in}\rangle \longmapsto \int\!\!\int_{\substack{\varphi_{in} \in \Gamma(\gamma_{in}, \mathbb{F}),\\ \varphi_{out} \in \Gamma(\gamma_{out}, \mathbb{F})}} \left(K_\Sigma(\varphi_{out}, \varphi_{in})\,\Psi_{in}(\varphi_{in})\right) |\varphi_{out}\rangle$$

sewing axiom ~ Fubini theorem for path integrals

* in quantum mechanics, $\mathrm{Fields}_{[0,t]} = \mathrm{Map}([0,t], \underset{\text{configuration space}}{X})$ $\quad \varphi_{in}, \varphi_{out} \in X$

* in string theory, $\mathrm{Fields}_\Sigma = \mathrm{Map}(\Sigma, X)$ $\quad \varphi_{in}, \varphi_{out} \in \underset{\text{loops}}{LX}$.

- enrichment by observables: $Z_{\Sigma, \hat{O}_\gamma} \Leftarrow$ "$\int e^{\frac{i}{\hbar}S(\varphi)}\, \underbrace{O(\varphi|_\gamma)}_{\text{classical observable}}\, D\varphi$"
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## Example: 1D Segal's QFT - Quantum Mechanics

$\overset{+}{\bullet} \longrightarrow \mathcal{H}$

$\overset{-}{\bullet} \longrightarrow \mathcal{H}^*$

Z 1-cobordisms equipped with Riemannian metric

$$Z\left(\underset{+}{\bullet} \xrightarrow[g\text{-metric}]{} \underset{+}{\bullet}\right) \in \mathrm{End}(\mathcal{H}) \otimes \underbrace{\mathrm{Fun}(\mathrm{Met})^{\mathrm{Diff}}}_{\mathrm{Fun}(\underbrace{\mathbb{R}_{>0}}_{t\text{-length of the interval}})}$$

Denote $Z_t := Z\left(\bullet \xrightarrow[t>0]{} \bullet\right)$

Sewing axiom: $Z\left(\bullet \xrightarrow[t_1]{} \bullet \xrightarrow[t_2]{} \bullet\right) = Z\left(\bullet \xrightarrow[t_2]{} \bullet\right) \circ Z\left(\bullet \xrightarrow[t_1]{} \bullet\right)$ or $\boxed{Z_{t_1+t_2} = Z_{t_2} \circ Z_{t_1}}$ Semi-group law

(improved) normalization: $Z\left(\underset{t\text{-small}}{\bullet\!\to\!\bullet}\right) = \mathrm{id} - \frac{i}{\hbar} \underbrace{\hat{H}}_{\text{"quantum Hamiltonian"}} t + O(t^2)$

$\Rightarrow Z_t = \left(Z_{\frac{t}{N}}\right)^N \underset{(N\to\infty)}{=} \boxed{e^{-\frac{i}{\hbar}\hat{H} t}}$ - evolution operator of QM, $\underset{\sim}{U(t)}$ - another notation.

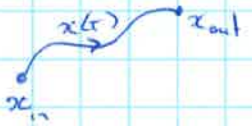
- From Feynman: classical mechanics $x(\tau) \in \mathrm{Maps}([0,t], \underset{\text{target Riemannian manifold}}{X})$ - parameterized path in $X$

$$S[x(\tau)] = \int_0^t \underbrace{\left(\frac{\|\dot{x}(\tau)\|^2}{2m} - V(x(\tau))\right)}_{L(x,\dot{x})} d\tau$$ - action functional

Classical motion: $x(\tau)$ s.t. $\delta S[x(\tau)] = 0$

$x(0) = x_{in}$ ← initial position

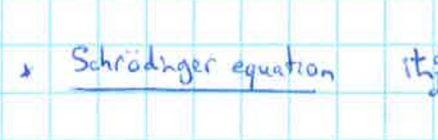
$x(t) = x_{out}$ ← final position



Quantization: states $\mathcal{H} = L^2_{\mathbb{C}}(X) \ni \Psi(x)$ - wavefunction

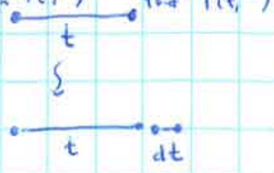
evolution $U(t): \psi_{in}(x) \longmapsto \psi_{out}(x) = \int_{X \ni y} \underset{\sim}{K(t;x,y)}\, \psi_{in}(y)$ where $K(t,x,y) =$ "$\displaystyle\int_{\substack{x(0)=y\\ x(t)=x}} e^{\frac{i}{\hbar}S[x(\tau)]} \mathcal{D}[x(\tau)]$" $= \langle x | U(t) | y \rangle$ - "matrix element" of $U(t)$

convolution property $U(t_1+t_2) = U(t_2) \circ U(t_1) \rightsquigarrow K(t_1+t_2; x,y) = \int_{X \ni z} K(t_2;x,z) K(t_1;z,y)$



$$\int_{\substack{\text{paths } y\to x\\ p}} e^{\frac{i}{\hbar}S(p)} = \int_X dz \int_{\substack{\text{paths } y\to z\\ p_1}} e^{\frac{i}{\hbar}S(p_1)} \int_{\substack{\text{paths } z\to x\\ p_2}} e^{\frac{i}{\hbar}S(p_2)}$$ ← "Fubini theorem"

- Schrödinger equation $i\hbar\frac{d}{dt}\psi(t;x) = \underbrace{\left(-\frac{\hbar^2}{2m}\Delta + V(x)\right)}_{\hat{H}} \psi(t,x)$ expresses infinitesimal sewing



motivation for Segal's axioms

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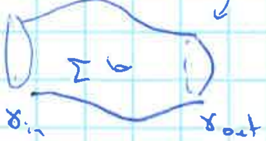
Field theory on



classically: $\text{Fields}_\Sigma = \Gamma(\Sigma, \mathcal{F}) \ni \varphi$ — bundle/sheaf over $\Sigma$; field on $\Sigma$

$S_\Sigma(\varphi) = \int_\Sigma L(\varphi, d\varphi, ..)$ - action.

Classical equation of motion: $\delta S_\Sigma = 0$

$\varphi|_{in} = \varphi_{in}$ ← fixed

$\varphi|_{out} = \varphi_{out}$ ↙ - b.c.

P.I. quantization

$$\langle \varphi_{out} | Z_\Sigma | \varphi_{in} \rangle = \text{“}\int\limits_{\substack{\varphi|_{in}=\varphi_{in}\\ \varphi|_{out}=\varphi_{out}}} e^{\frac{i}{\hbar}S_\Sigma(\varphi)} \mathcal{D}\varphi\text{”}$$

, $\mathcal{H}_\gamma = \text{“}L^2\text{”}(\text{Fields}_\gamma)$.

$\ni \int \mathcal{D}\varphi_\gamma \underbrace{\Psi(\varphi_\gamma)}_{\text{wavefunction}} |\varphi_\gamma\rangle$

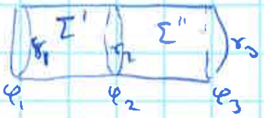
**Sewing:**

Fubini

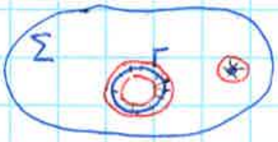
$$\langle \varphi_3 | Z_\Sigma | \varphi_1 \rangle = \int \mathcal{D}\varphi_2 \int\limits_{\substack{\varphi'|_1=\varphi_1\\ \varphi'|_2=\varphi_2}} \mathcal{D}\varphi'_{\text{on }\Sigma'} e^{\frac{i}{\hbar}S_{\Sigma'}} \int\limits_{\substack{\varphi''|_2=\varphi_2\\ \varphi''|_3=\varphi_3}} \mathcal{D}\varphi''_{\text{on }\Sigma''} e^{\frac{i}{\hbar}S_{\Sigma''}}$$

$$= \int \mathcal{D}\varphi_2 \langle \varphi_3 | Z_{\Sigma''} | \varphi_2 \rangle \langle \varphi_2 | Z_{\Sigma'} | \varphi_1 \rangle$$

$$\sim \quad Z_\Sigma = Z_{\Sigma''} \circ Z_{\Sigma'}$$



**Observables**



$\Gamma \subset \Sigma$ submanifold / sub-CW complex

$U_\varepsilon(\Gamma)$ - $\varepsilon$-thickening

quantum observable supported on $\Gamma$: $\hat{O}_\Gamma \in \mathcal{H}_{\partial U_\varepsilon(\Gamma)}$

(surface of a thin tube around $\Gamma$)

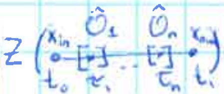
expectation value (correlator) $Z_{\Sigma, \hat{O}_\Gamma} = \langle Z_{\Sigma \setminus U_\varepsilon(\Gamma)}, \hat{O}_\Gamma \rangle \in \mathbb{C}$

$= \text{“}\langle \hat{O}_\Gamma \rangle_\Sigma\text{”}$ if $\Sigma$ was closed

PI expression: $\langle \hat{O}_\Gamma \rangle_\Sigma = \int e^{\frac{i}{\hbar}S(\varphi)} O_\Gamma(\varphi|_\Gamma) \mathcal{D}\varphi$

observables in QM

$\hat{O} \in \mathcal{H} \otimes \mathcal{H}^* = \text{End}(\mathcal{H})$ - operator

$Z\Big($$\Big)$

$$= \langle x_{out} | e^{-\frac{i}{\hbar}\hat{H}(t_1-\tau_n)} \hat{O}_n \cdots \hat{O}_2 e^{-\frac{i}{\hbar}\hat{H}(\tau_2-\tau_1)} \hat{O}_1 \cdot e^{-\frac{i}{\hbar}(\tau_1-t_0)} | x_{in} \rangle$$

$$= \int\limits_{\substack{x(t_0)=x_{in}\\ x(t_1)=x_{out}}} e^{\frac{i}{\hbar}S[x(\tau)]} O_1(\tau_1) \cdots O_n(\tau_n) \, D[x(\tau)]$$

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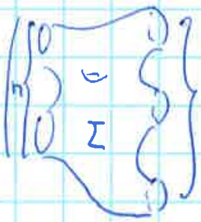
CFT: $D=2$, cobordisms equipped with conformal structure = metric $g / g \sim \Omega g$ ($\Omega$: positive function)

boundary circles equipped with parametrization

$Z\Big($$\Big) : \mathcal{H}_{S^1}^{\otimes n} \longrightarrow \mathcal{H}_{S^1}^{\otimes m}$

- $\mathcal{H} = \mathcal{H}_{S^1}$ is equipped with an action of $\text{Diff}(S^1)$

self-sewing: $\Sigma \longrightarrow \tilde{\Sigma} = \Sigma / \gamma_i^{in} = \gamma_j^{out}$ — identify two bdry circles

then $Z_{\tilde{\Sigma}} = \text{tr}_\mathcal{H} Z_\Sigma$ ↗ as $\text{End}(\mathcal{H}) \otimes \text{Hom}(\mathcal{H}^{\otimes n-1}, \mathcal{H}^{\otimes m-1})$

- WANT $Z_\Sigma$ to be trace-class ("compactness")

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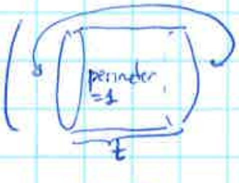
Corrections to the picture - "conformal anomaly"

- action $\mathrm{Diff}(S^1) \circlearrowright \mathcal{H}$ is projective
- $$Z(\Sigma, \underbrace{\Omega}_{e^{2\sigma}} \cdot g) = e^{i\boxed{c} S_{\mathrm{Liouville}}(\sigma)} Z(\Sigma, g) \quad , \quad S_{\mathrm{Liouville}}(\sigma) = \frac{1}{2}\int d\sigma \wedge * d\sigma + 4\sigma R_g \, dvol_g$$

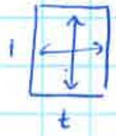
  $c \in \mathbb{R}$ - <u>central charge</u>.

- genus 1 partition function $Z\Big($ 

$\Big) = \mathrm{tr}_{\mathcal{H}}\, e^{-\frac{i}{\hbar}\hat{H} t} \in \mathbb{C}$

  $=: Z(\mathbb{T}_t)$ Then $Z(\mathbb{T}_t) = Z(\mathbb{T}_{1/t})$ ← modular invariance

  conformally equivalent tori



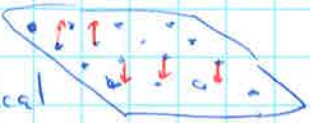
or: $\mathbb{T}_\tau = \mathbb{C}/\mathbb{Z} \oplus \textcircled{\tau}\mathbb{Z}$ , $Z(\mathbb{T}_\tau) = Z(\mathbb{T}_{-1/\tau})$ ~ $Z(\mathbb{T}_\tau)$ - modular function of $\tau$.

modular parameter

since tori are equivalent

* * *

Why CFT?

- Ising model  at temperature $T_{crit}$ $\quad$ 2-point correlation radius → ∞,

  - lattice statistical system
  - 2nd order phase transition

  2-point correlator becomes $\langle \sigma(x)\,\sigma(y)\rangle \sim \frac{1}{|x-y|^{\textcircled{1/4}}}$ ← crit exponent

  Zooming out of the lattice, the model attains $\underset{\text{invariance}}{\text{scaling}} \to \underset{\text{invariance}}{\text{conformal}}$ ,

  can be described by a CFT on the plane.

- string theory

  classically $\mathrm{Maps}\big(\underbrace{\text{}}_{\text{worldsheet}}, \underbrace{X = \mathbb{R}^N}_{\text{target}}\big)$ → CFT of n bosons on $\Sigma$

  (+ reparametrization ghost system)

- relation to 3D TQFT

  (and to representation theory of $MCG_\Sigma$)
- <u>interesting</u> structures on $\mathcal{M}_{g,n}$ ; reps of $\widehat{\mathrm{Diff}(S^1)}$.

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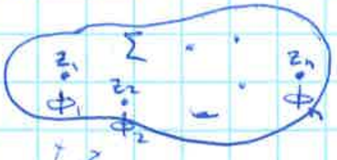
CFT as a set of correlators

"$\int \Phi_1(z_1) \cdots \Phi_n(z_n) e^{-S(\varphi)} D\varphi$"



punctures ~ infinitesimal circles

$\underbrace{\langle \Phi_1(z_1) \cdots \Phi_n(z_n) \rangle}_{\text{correlator}} \in \mathbb{C}$ $\sim$ $\langle \Phi_1 \cdots \Phi_n \rangle \in \Gamma(\mathcal{M}_{g,n}, \underbrace{\mathcal{L}}_{\text{certain line bundle}})$

$\Phi_i \in \underset{\mathcal{H}_{S^1}}{\underset{\shortparallel}{V}}$ bi-graded by $\underbrace{(h, \bar{h})}_{\text{"conformal dimension"}}$

space of (composite) field - admissible decorations of a puncture

special field: · $\mathbb{1}$ - trivial (identity) field

· $T$ - stress-energy tensor

· primary fields - highest weight vectors of $\mathrm{Diff}(S^1) \circlearrowright V$

$$\Phi(z) \in \cancel{\mathcal{B}}^{0} K_z^{\otimes h} \otimes \bar{K}_z^{\otimes \bar{h}}$$

$$K = (T^{1,0})^* \Sigma \ , \ \bar{K} = (T^{0,1})^* \Sigma$$

· instead of sewing, one studies OPEs (operator product expansions) governing singularities of correlators as $z_i \to z_j$

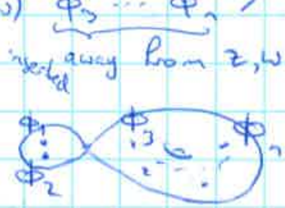
OPE: $$\Phi_1(z)\,\Phi_2(w) \sim \sum_{\substack{\tilde{\Phi}_\gamma \text{ runs} \\ \text{over a basis in } V}} \underbrace{f^{\tilde{\Phi}}_{\Phi_1 \Phi_2}(z,w)}_{\substack{\text{real-analytic functions} \\ \text{on a nbhd of } \mathrm{Diag} \subset \Sigma \times \Sigma, \\ \text{singular on Diag}}} \tilde{\Phi}_\gamma(w) + \underbrace{\text{reg}}_{\text{regular part}} \qquad (*)$$

its meaning: substitution (*)

can be made inside any correlator $\langle \Phi_1(z)\,\Phi_2(w) \underbrace{\Phi_3 \cdots \Phi_n}_{\text{inserted away from } z,w} \rangle$

yielding asymptotics, as $z \to w$.

(~ nbhd of a nodal point on $\mathcal{M}_{g,n}$,



)

Idea: we can recover ~~(inductively in n)~~

n-point correlators from (n-1)-point correlators, using singular parts of OPEs

< similar to recovering a ~~[illegible]~~ rational function from singular parts of its Laurent expansion at poles >

Bits of
## Conformal geometry

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Let $(M,g)$ - (pseudo-) Riemannian manifold

def a Weyl transformation is $\underset{\overset{\cup}{x}}{(M,g)} \to \underset{\overset{\cup}{x}}{(M,g')}$ with $x \longmapsto x$
$g'(x) = \underbrace{\Omega(x)}_{\text{everywhere positive function.}} \cdot g(x)$

def two (pseudo-) Riemannian mflds $(M,g)$, $(M',g')$ are conformally equivalent if there exists a diffeomorphism $\varphi: M \to M'$ s.t. $\boxed{(\varphi^* g')(x) = \Omega(x)\cdot g(x)}$

then $\varphi$ is called a conformal map
$\Omega$ - the associated conformal factor

* a composition of conf. maps
$$\begin{tikzcd} (M,g) \arrow[r, "\varphi_1"] \arrow[rr, bend right, "\varphi_2\circ\varphi_1"'] & (M',g') \arrow[r, "\varphi_2"] & (M'',g'') \end{tikzcd}$$
is a conf. map

* inverse $\varphi^{-1}$ of a conf map
$$\begin{tikzcd} (M,g) \arrow[r, bend left, "\varphi"] & (M',g') \arrow[l, bend left, "\varphi^{-1}"] \end{tikzcd}$$
is a conf. map

def conformal automorphisms ~~of~~ $\varphi:(M,g) \to$ ~~form~~ $(M,g)$ comprise the conformal group $\mathrm{Conf}(M,g)$.

def a conformal structure on $M$ = a choice of metric modulo Weyl transformations.

- for $g \sim g'$ two Weyl-equivalent metrics on $M$, $\mathrm{Conf}(M,g) = \mathrm{Conf}(M,g')$.
  - conformal maps are the same (but conf. factors may differ)

thus, $\mathrm{Conf}(M, g/_{g\sim\Omega g})$ is well-defined
(depends only on conf. structure on $M$)

~~def~~ · $\{\text{Isometries}\} \hookrightarrow \{\text{conf. maps}\}$, singled out by the property $\Omega = 1$

Examples of conformal maps

- translations and $O(n)$-rotations of Euclidean $\mathbb{R}^n$

( or translations and $O(p,q)$-rotations of $\mathbb{R}^{p,q}$ with $g = (dx^1)^2 + \dots + (dx^p)^2 - (dx^{p+1})^2 - \dots - (dx^{p+q})^2$ )

thus $\underbrace{O(n)\ltimes\mathbb{R}^n}_{\text{Poincaré group}} \subset \mathrm{Conf}(\mathbb{R}^n)$, with $\Omega = 1$.

- dilatations $\mathbb{R}^n \to \mathbb{R}^n$ for $\lambda > 0$ (also $\mathbb{R}^{p,q} \to \mathbb{R}^{p,q}$)
  $\vec{x} \longmapsto \lambda\vec{x}$  $\vec{x} \longmapsto \lambda\vec{x}$
  - conf. factor $\boxed{\Omega = \lambda^2}$

~~· stereographic projection~~

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- <u>Stereographic projection</u>

$$\begin{tikzcd} \mathbb{R}^{n+1} \supset S^n \setminus \{\text{North pole}\} \arrow[d, "\varphi"] & (x^0, x^1, \ldots, x^n) \text{ s.t. } \sum_{i=0}^{n} (x^i)^2 = 1 \arrow[d, mapsto] \\ \mathbb{R}^n & \frac{1}{1-x^0}(x^1, \ldots, x^n) \end{tikzcd}$$

<u>Exercise</u>: show that $\varphi$ is conformal with $\boxed{\Omega = \frac{1}{(1-x^0)^2}}$

- Any diffeo $\varphi: \mathbb{R}^1 \to \mathbb{R}^1$ with $g = (dx)^2$, $\Omega = \left(\frac{d\varphi}{dx}\right)^2$

- <u>inversion</u> $\varphi: \mathbb{R}^n - \{0\} \to \mathbb{R}^n - \{0\}$, $\vec{x} \longmapsto \frac{\vec{x}}{\|\vec{x}\|^2}$ is an orientation-reversing conformal map, $\boxed{\Omega = \frac{1}{\|\vec{x}\|^4}}$ ← <u>Exercise</u>

$z = x+iy \longmapsto u+iv = w$

- Any (bi)<u>holomorphic</u> map $\varphi: D \xrightarrow{\sim} D'$, $D \subset \mathbb{C}$, $D' \subset \mathbb{C}$, with $g = (dx)^2 + (dy)^2 = dz \cdot d\bar{z}$ $\left(= \frac{1}{2}(dz \otimes d\bar{z} + d\bar{z} \otimes dz)\right)$

$z = x+iy$ - complex coordinate on $\mathbb{C}$

[ ~~then $\varphi^* \underbrace{g}_{(du)^2+(dv)^2} = \left(\frac{\partial u}{\partial x}\right)^2 dx^2 + \left(\frac{\partial v}{\partial y}\right)^2 dy^2 + 2\frac{\partial u}{\partial y}$~~

$$\varphi^* \underbrace{g}_{du^2+dv^2} = \underbrace{(u_x^2 + v_x^2)}_{} dx^2 + \underbrace{(u_y^2 + v_y^2)}_{\text{by CR}} dy^2 + \underbrace{(u_x v_y + u_y v_x)}_{0 \text{ by CR}} dx\,dy$$

Cauchy-Riemann eq.: $u_x = v_y$, $u_y = -v_x$ ]

$$\varphi^* \underbrace{g}_{dw\,d\bar{w}} = \left(\frac{\partial w}{\partial z} dz + \underbrace{\frac{\partial w}{\partial \bar{z}}}_{0} d\bar{z}\right) \cdot \left(\underbrace{\frac{\partial \bar{w}}{\partial z}}_{0} dz + \frac{\partial \bar{w}}{\partial \bar{z}} d\bar{z}\right) = \left|\frac{dw}{dz}\right|^2 dz\,d\bar{z} \quad \to \varphi \text{ is conformal}, \quad \boxed{\Omega = \left|\frac{dw}{dz}\right|^2}$$

since $\varphi$ holomorphic

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- Möbius transformations $PSL_2(\mathbb{C}) \circlearrowleft \overline{\mathbb{C}} = \mathbb{CP}^1$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \longmapsto \frac{az+b}{cz+d} = z'$ with $ad - bc = 1$ $\quad \Omega = \left|\frac{dz'}{dz}\right|^2 = \boxed{\frac{1}{|cz+d|^4}}$

$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ translation by $b \in \mathbb{C}$

$\begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix}$ rotation by angle $\varphi$

$\begin{pmatrix} \lambda^{1/2} & 0 \\ 0 & \lambda^{-1/2} \end{pmatrix}$ - dilatation by factor $\lambda$, $\lambda > 0$

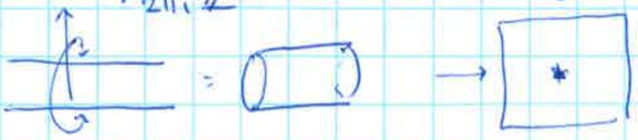
$\begin{pmatrix} 1 & 0 \\ \mathbb{C} & 1 \end{pmatrix}$ - "special ~~conformal~~ conf. transf."

$z \mapsto \frac{z}{cz+1} = \frac{1}{c + 1/z}$

maps $-1/c \to \infty$

$\infty \to 1/c$

- $\mathbb{C}/_{2\pi i \mathbb{Z}} \xrightarrow{\exp} \mathbb{C} \setminus \{0\}$ - find $\Omega$



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## "Infinitesimal conf. maps"

"a conformal (Killing) vector field on $(M,g)$ (pseudo-)Riem. mfd is a v.f. $v \in \mathfrak{X}(M)$ satisfying

(*) $\boxed{\mathcal{L}_v g = \omega \cdot g}$ $\quad \omega \in C^\infty(M)$ - infinitesimal conf. factor.

Lie derivative along $v$ of the metric

Properties:
- if $u, v$ conf. v.f. w/ conf. factors $\omega_u, \omega_v$, then
  - $u+v$ is a c.v.f. with $\omega = \omega_u + \omega_v$
  - $[u,v]$ is a c.v.f. with $\omega = \mathcal{L}_u\omega_v - \mathcal{L}_v\omega_u$
- c.v.f.s form a Lie subalgebra $\mathrm{conf}(M,g) \subset \mathfrak{X}(M)$
- if $M$ compact, then $\mathrm{conf}(M,g) = \mathrm{Lie}\ \mathrm{Conf}(M,g)$

  with $\exp: \mathrm{conf} \longrightarrow \mathrm{Conf}$

  $v \longmapsto \mathrm{Flow}_1(v)$ - flow along $v$ in unit time

## Conformal vector fields on $\mathbb{R}^{p,q}$

$g = g_{ij}(x)\, dx^i dx^j$ $\qquad g_{ij}(x) = \eta_{ij} = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & \ddots \\ & & & & & -1 \end{pmatrix}$ $\quad \underbrace{\phantom{xxx}}_{p}\ \underbrace{\phantom{xxx}}_{q}$

looking for $\varepsilon = \varepsilon^k(x)\partial_k$ a c.v.f.

$p+q=n$

(*) $\rightsquigarrow$ ~~$\partial_k \eta$~~ ① $\boxed{\partial_i \varepsilon_j + \partial_j \varepsilon_i = \omega\, \eta_{ij}}$, $\quad \varepsilon_i = \eta_{ij}\varepsilon^j$

Solving ①:
- ① $\xrightarrow[\text{with } \eta^{ij}]{\text{contract}}$ $\boxed{\partial_i \varepsilon^i = \frac{n}{2}\omega}$ ② $\quad \underbrace{\phantom{\partial_i\varepsilon^i}}_{\operatorname{div}\varepsilon = \partial\cdot\varepsilon}$
- ① $\xrightarrow[\partial^j]{}$ $\partial_i(\partial\cdot\varepsilon) + \Delta\varepsilon_i = \partial_i\omega$ $\xrightarrow[(2)]{}$ $\boxed{\Delta\varepsilon_i = \left(1 - \frac{n}{2}\right)\partial_i\omega}$ ③
- ③ $\xrightarrow[\text{and symmetrize } i\leftrightarrow j \text{, use ①}]{\text{apply } \partial_j}$ $\boxed{\frac{1}{2}\eta_{ij}\Delta\omega = \left(1-\frac{n}{2}\right)\partial_i\partial_j\omega}$ ④
- ③ $\xrightarrow[\partial^i]{}$ $\Delta(\underbrace{\partial\cdot\varepsilon}_{\frac{n}{2}\omega}) = \left(1-\frac{n}{2}\right)\Delta\omega \rightsquigarrow \boxed{(n-1)\Delta\omega = 0}$ ⑤
- ④,⑤ $\rightsquigarrow$ for $n \neq 1,2$ $\boxed{\partial_i\partial_j\omega = 0}$ ⑥ $\to$ $\omega$ at most linear in coord.
- deriving ① $\rightsquigarrow$

$$\begin{aligned}
+\ & \partial_i\partial_j\varepsilon_k + \partial_i\partial_k\varepsilon_j = \partial_i\omega\,\eta_{jk} \\
& \partial_j\partial_i\varepsilon_k + \partial_j\partial_k\varepsilon_i = \partial_j\omega\,\eta_{ik} \\
-\ & \partial_k\partial_i\varepsilon_j + \partial_k\partial_j\varepsilon_i = \partial_k\omega\,\eta_{ij}
\end{aligned} \quad \begin{matrix} \} i\leftrightarrow j \\ \} j\leftrightarrow k \end{matrix}$$

$$\boxed{2\,\partial_i\partial_j\varepsilon_k = \partial_i\omega\cdot\eta_{jk} + \partial_j\omega\,\eta_{ik} - \partial_k\omega\cdot\eta_{ij}} \quad ⑦$$

- ⑥⑦ $\rightsquigarrow$ for $n\neq 1,2$ $\boxed{\partial_i\partial_j\partial_k\varepsilon_l = 0}$ $\Longrightarrow$ $\varepsilon$ at most quadratic in coord.

Ansatz: $\left.\begin{aligned} \varepsilon_i(x) &= a_i + b_{ij}x^j + c_{ijk}x^jx^k \\ \omega(x) &= 2\mu + 4\nu_i x^i \end{aligned}\right\} \xrightarrow[①]{}$

$a_i$ - no restriction

$b_{ij} + b_{ji} = 2\mu\eta_{ij} \Rightarrow b_{ij} = \overset{\text{anti-sym in } i\leftrightarrow j}{\beta_{ij}} + \mu\eta_{ij}$

$c_{ijk} + c_{jik} = 2\nu_k\eta_{ij} \xRightarrow[\text{like ⑦}]{} c_{ijk} = \nu_j\eta_{ik} + \nu_k\eta_{ij} - \nu_i\eta_{jk}$

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**Thm** (Liouville)

$$\mathrm{conf}(\mathbb{R}^{p,q}) = \underset{\simeq\mathbb{R}^n}{\{\text{translations}\}} \oplus \underset{\simeq SO(p,q)}{\{\text{rotations}\}} \oplus \underset{\simeq\mathbb{R}}{\{\text{dilatations}\}} \oplus \underset{\simeq\mathbb{R}^n}{\left\{\begin{matrix}\text{special}\\ \text{conf.}\\ \text{transformations}\end{matrix}\right\}}$$

| | conf. v.f. | $\omega$ | Conf. map | $\Omega$ |
|---|---|---|---|---|
| translation | $\varepsilon^i(x)=a^i$ | $0$ | $x^i\mapsto x^i+a^i$ , $\vec{a}\in\mathbb{R}^{p,q}$ | $1$ |
| rotation | $\varepsilon^i(x)=\beta^i{}_j x^j$ , with $\beta_{ij}=-\beta_{ji}$ | $0$ | $\vec{x}^i\mapsto O^i{}_j x^j$ , $O^i{}_j\in SO(p,q)$ | $1$ |
| dilatation | $\varepsilon^i(x)=\mu x^i$ | $2\mu$ | $x^i\mapsto\lambda x^i$ , $\lambda\in\mathbb{R}_+$ | $\lambda^2$ |
| SCT | $\varepsilon^i(x)=2(\vec{x},\vec{\nu})x^i-\nu^i\|\vec{x}\|^2$ | $4(\vec{\nu},\vec{x})$ | $x^i\mapsto\dfrac{x^i-\|\vec{x}\|^2b^i}{1-2(\vec{b},\vec{x})+\|\vec{b}\|^2\cdot\|\vec{x}\|^2}$ , $\vec{b}\in\mathbb{R}^{p,q}$ | $(1-2(\vec{b},\vec{x})+\|\vec{b}\|^2\|\vec{x}\|^2)^{-2}$ |

**Rem**: · finite SCT $=(\text{inversion})\circ\left(\begin{matrix}\text{translation}\\ \text{by } -\vec{b}\end{matrix}\right)\circ(\text{inversion})$

$$\boxed{\vec{x}\longmapsto\vec{x}' \qquad \frac{\vec{x}'}{\|\vec{x}'\|^2}=\frac{\vec{x}}{\|\vec{x}\|^2}-\vec{b}}$$

· finite SCT is not everywhere defined as a map $\mathbb{R}^{p,q}\to\mathbb{R}^{p,q}$
can find a "conformal compactification" $\underbrace{N^{p,q}}_{\text{compact}}\supset\mathbb{R}^{p,q}$ s.t. SCT are everywhere well-defined on $N^{p,q}$.

**Thm** for $p+q>2$, · $\mathrm{conf}(\mathbb{R}^{p,q})\cong so(p+1,q+1)$ as a Lie algebra

· $\underset{\text{conn. component of } 1}{\mathrm{Conf}_0(\mathbb{R}^{p,q})}\cong SO_0(p+1,q+1)$ [or $SO_0(p+1,q+1)/\mathbb{Z}_2$ if $-1$ is in connected component of $1$]

proof: see M. Schottenloher "A mathematical introduction to CFT"

* dimension counting

$$\mathrm{conf}(\mathbb{R}^{p,q}) = \{\text{translations}\}\oplus\overset{so(p,q)-}{\{\text{rotations}\}}\oplus\{\text{dilatations}\}\oplus\{\text{SCTs}\}$$

$$\text{dim}: \underset{\text{dim } so(p+1,q+1)}{\overset{}{\frac{(n+1)(n+2)}{2}}} = n + \frac{n(n-1)}{2} + 1 + n$$

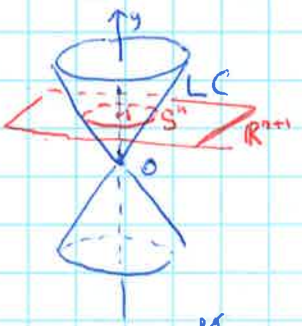
---

Action of $SO(p+1,q+1)$ on $\mathbb{R}^{p,q}$

· Case of $\mathbb{R}^n$   $SO_*(n+1,1)$ acts on $\mathbb{R}^{n+1,1}$ by linear isometries and preserves the light cone $LC=\{(x^0,\dots,x^n,y) \text{ s.t. } (x^0)^2+\dots+(x^n)^2-y^2=0\}$

$SO(n+1,1)\circlearrowright LC \circlearrowleft \underset{\text{dilatations}}{\mathbb{R}^*=\mathbb{R}-\{0\}}$ - commuting actions $\rightarrow$ $SO(n+1,1)$ acts on $LC\text{-}\{0\}/\mathbb{R}^*$

$LC$ inherits a degenerate metric from $\mathbb{R}^{n+1,1}$; its kernel is killed by quotienting over $\mathbb{R}^*$



$\rightarrow LC-\{0\}/\mathbb{R}^*$ inherits a conformal structure
$\simeq S^n$   and $SO(p+1,1)$ acts by conf. maps

and $S^n\setminus\text{north pole}\xrightarrow[\text{stereographic proj}]{\sim}\mathbb{R}^n$ . So: $S^n$ - conf. compactification of $\mathbb{R}^n$:
c.v.f.s on $\mathbb{R}^n$ extend to $S^n$,
finite conf. maps are everywhere defined on $S^n$.

General $p,q$: $\mathbb{R}^{p+1,q+1}\supset LC=\{(x^0,\dots,x^p,y^0,\dots,y^q) \mid \sum_0^p(x^i)^2-\sum_0^q(y^j)^2=0\}\circlearrowleft SO(p+1,q+1)$

$$\begin{tikzcd} SO(p+1,q+1)\circlearrowright LC-\{0\}\circlearrowleft\mathbb{R}^* \arrow[d, "\pi"] \\ \mathbb{RP}^{n+1} \end{tikzcd}$$

denote $N^{p,q}=\mathrm{im}\,\pi$. - it has conf. str. inherited from $\mathbb{R}^{p+1,q+1}$, $SO(p+1,q+1)$ acts by conf. maps

- $\iota: \mathbb{R}^{p,q} \longrightarrow N^{p,q}$ $\quad (x^1,\dots x^p, y^1,\dots,y^q) \mapsto \left(\frac{1-\sum_1^p (x^i)^2 + \sum_1^q (y^j)^2}{2} : x^1 : \dots : x^p : \frac{1+\sum_1^p (x^i)^2 - \sum_1^q (y^j)^2}{2} : y^1 : \dots : y^q\right)$ $\boxed{\frac{1/23/2019}{3}}$

injective, with image open-dense

- $N^{p,q}$ = conf. compactification
- $S^p \times S^q \xrightarrow{2-1} N^{p,q}$ - double cover

$\boxed{N^{p,q} = S^p \times S^q / \{\pm 1\}}$

## Conformal ~~vector~~ symmetry of $\mathbb{R}^2$

eq. for c.v.f. $\varepsilon = \varepsilon_i(x,y)\partial_i$ : $\quad \partial_i \varepsilon_j + \partial_j \varepsilon_i = \omega \delta_{ij} \iff \begin{cases} \partial_x \varepsilon_x = \partial_y \varepsilon_y = \frac{1}{2}\omega \\ \partial_x \varepsilon_y = -\partial_y \varepsilon_x \end{cases} \iff$ $\varepsilon_x + i\varepsilon_y$ satisfies Cauchy-Riemann eq.

$x = x^1, y = x^2$

$\iff \varepsilon_i \partial_i$ is of the form $\underbrace{\varepsilon(z)\frac{\partial}{\partial z}}_{\text{holom. v.f.}} + \underbrace{\bar{\varepsilon}(\bar{z})\frac{\partial}{\partial \bar{z}}}_{\text{conjugate anti-hol. v.f.}}$ , $\omega = \partial\varepsilon + \bar{\partial}\bar{\varepsilon}$

notations: $z = x+iy$, $\bar{z} = x - iy$

$\partial = \partial_z = \frac{1}{2}(\partial_x - i\partial_y)$, $\bar{\partial} = \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$

So: $\underset{\simeq \mathbb{C}}{\mathrm{conf}(\mathbb{R}^2)} \simeq \left\{\begin{matrix}\text{holom. v.f.} \\ \text{on } \mathbb{C}\end{matrix}\right\}$

$\varepsilon_x \partial_x + \varepsilon_y \partial_y \longmapsto (\varepsilon_x + i\varepsilon_y)\partial_z$

$(\mathrm{Re}\,\varepsilon)\partial_x + (\mathrm{Im}\,\varepsilon)\partial_y \longleftarrow\!\shortmid\ \varepsilon(z)\partial_z$

**Finite version**: a diffeo $\varphi: \underset{\underset{\mathbb{C}}{\cap}}{\mathcal{D}} \longrightarrow \underset{\underset{\mathbb{C}}{\cap}}{\mathcal{D}'}$ is conformal iff it is either holomorphic or anti-holomorphic

Indeed: $\underset{dz\,d\bar{z}}{\varphi^* g} = \underbrace{\frac{\partial \varphi}{\partial z}\frac{\partial \bar{\varphi}}{\partial z}}_{\overset{\text{WANT}}{=} 0}(dz)^2 + \underbrace{\left(\frac{\partial \varphi}{\partial z}\frac{\partial \bar{\varphi}}{\partial \bar{z}} + \frac{\partial \varphi}{\partial \bar{z}}\frac{\partial \bar{\varphi}}{\partial z}\right)}_{\Omega} dz\,d\bar{z} + \underbrace{\frac{\partial \varphi}{\partial \bar{z}}\frac{\partial \bar{\varphi}}{\partial \bar{z}}}_{\overset{\text{WANT}}{=} 0}(d\bar{z})^2 \overset{\text{WANT}}{=} \Omega\, dz\, d\bar{z}$

~~$\partial\varphi = 0$ or $\partial\bar{\varphi} = 0$~~

and ~~$\Omega \neq 0$ $\quad \Omega \neq 0$~~

~~$\bar{\partial}\varphi = 0$ or $\bar{\partial}\bar{\varphi} = 0$~~

$\rightsquigarrow$ ~~$\partial\bar{\varphi} \neq 0, \bar{\partial}\varphi \neq 0$~~ either $\bar{\partial}\varphi = 0$ , then $\Omega = |\partial\varphi|^2$

or $\partial\varphi = 0$ , then $\Omega = |\bar{\partial}\varphi|^2$ $\quad \square$

- $\mathrm{conf}(\mathbb{C} \setminus \{0\}) = \left\{\begin{matrix}\text{real parts of merom. v.fields} \\ \text{on } \mathbb{C} \text{ with pole at 0 allowed}\end{matrix}\right\}$

Introduce $\mathcal{A} = \left\{ \sum_{n=-\infty}^{\infty} c_n \ell_n \,\middle|\, \begin{matrix} c_n \in \mathbb{C} \\ \boxed{\ell_n = -z^{n+1}\frac{\partial}{\partial z}} \end{matrix} \right\} = \left\{\begin{matrix}\text{merom. v.f. on } \mathbb{C} \\ \text{with pole at 0 allowed}\end{matrix}\right\}$ - Witt algebra

Generators $\ell_n$ obey $\boxed{[\ell_m, \ell_n] = (m-n)\ell_{m+n}}$

- $\mathrm{conf}(\mathbb{C}\setminus\{0\}) \hookrightarrow \underset{\mathrm{Span}_{\mathbb{C}}\{\ell_n\}}{\mathcal{A}} \oplus \underset{\mathrm{Span}_{\mathbb{C}}\{\bar{\ell}_n\}}{\bar{\mathcal{A}}}$ $\quad \leftarrow \ell_n = -z^{n+1}\frac{\partial}{\partial z}$, $\bar{\ell}_n = -\bar{z}^{n+1}\frac{\partial}{\partial \bar{z}}$

$\quad //$

$\mathrm{Span}_{\mathbb{R}}\{\ell_n + \bar{\ell}_n, i(\ell_n - \bar{\ell}_n)\}_{n=-\infty}^{\infty}$

- $\mathrm{conf}(\mathbb{C}) = \mathrm{Span}_{\mathbb{R}}\{\ell_n + \bar{\ell}_n, i(\ell_n - \bar{\ell}_n)\}_{n \geq -1}$
- $\left\{\begin{matrix}\text{c.v.f. on } \mathbb{C} \\ \text{vanishing at 0}\end{matrix}\right\} = \mathrm{Span}_{\mathbb{R}}\{\cdots\}_{n \geq 0}$

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- $\mathrm{conf}(\mathbb{CP}^1) = \mathrm{Span}_{\mathbb{R}}\{\ell_n+\overline{\ell_n}, i(\ell_n-\overline{\ell_n})\}_{n\in\{-1,0,1\}} \simeq sl_2(\mathbb{C}) \simeq so(3,1)$

$\overset{\shortparallel}{\overline{\mathbb{C}}}$

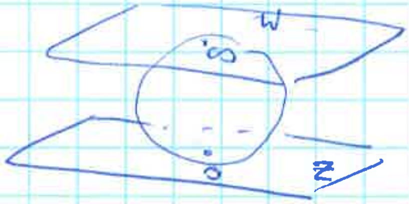
$\mathrm{conf}(\overline{\mathbb{C}}\setminus\{0\}) = \mathrm{Span}_{\mathbb{R}}\{\cdots\}_{n\leq 1}$

$\simeq \mathbb{C}$

- conf. v. fields extending to $\{\infty\}$

$$\boxed{-z^{n+1}\frac{\partial}{\partial z} \longmapsto = w^{-n+1}\frac{\partial}{\partial w}}$$

$w=z^{-1}$ - coord. at $\{\infty\}$



So e.g. $-z^3\partial_z = w^{-1}\partial_w$ - singular at $\infty$

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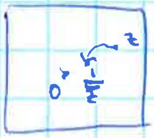
- Inversion II: $z \mapsto \frac{1}{\bar{z}}$ maps $\boxed{\ell_n \longmapsto -\overline{\ell_n}}$



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$-(\ell_{-1}+\overline{\ell_{-1}}) = \partial_x$
$-i(\ell_{-1}-\overline{\ell_{-1}}) = \partial_y$ } translations

$-(\ell_0+\overline{\ell_0}) = x\partial_x + y\partial_y$ dilatation

$-i(\ell_0-\overline{\ell_0}) = -y\partial_x + x\partial_y$ rotation

$-(\ell_1+\overline{\ell_1}) = (x^2-y^2)\partial_x + 2xy\partial_y$
$-i(\ell_1-\overline{\ell_1}) = -2xy\partial_x + (x^2-y^2)\partial_y$ } special coronical transformations

---

$\mathrm{Conf}_+(\mathbb{CP}^1) = PSL_2(\mathbb{C}) \simeq SO_+(3,1)$

or.-preserving part Möbius transformations

- ~~Conf~~ $\mathrm{conf}(\mathbb{C})$ is $\infty$-dimensional, but global conf. automorphisms of $\overline{\mathbb{C}}$ comprise only a finite-dim. group
- $\mathbb{C}$ does not have a conf. compactification (a compact mfd $M \underset{\text{open dense}}{\hookleftarrow} \mathbb{C}$ to which any c.v.f. on $\mathbb{C}$ can be extended)

- naively, $\underset{\text{punctured plane}}{\mathbb{C}\setminus\{0\}}$, $\underset{\text{punctured disk}}{D\setminus\{0\}} = \{z\in\mathbb{C} \mid 0<|z|<1\}$ and annulus $\mathrm{Ann}_r^R = \{z\in\mathbb{C} \mid r<|z|<R\}$

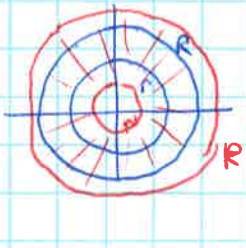
have same $\mathrm{conf}(-) \simeq \mathcal{A} = \mathrm{Span}_{\mathbb{C}}\{\ell_n\}_{n=-\infty}^{\infty}$.

Subtlety: different convergence restrictions on coeff. $\{c_n\}$ in $\varepsilon(z)\partial_z = \sum_{n=-\infty}^{\infty} c_n\ell_n$

$$\left\{\begin{array}{l} c_n\rho^n = O(n^{-\infty}) \text{ for } \rho<R \text{, at } n\to+\infty \\ c_n\rho^n = O(|n|^{-\infty}) \text{ for } \rho>r \text{ at } n\to-\infty \end{array}\right.$$

In fact, $\mathrm{conf}(\mathrm{Ann}_{r'}^{R'}) \subsetneq \mathrm{conf}(\mathrm{Ann}_r^R)$

if $\mathrm{Ann}_{r'}^{R'} \supsetneq \mathrm{Ann}_r^R$



- <u>vector fields on $S^1$ vs. Witt algebra</u>

$v = f(\theta)\partial_\theta = \sum a_n e^{in\theta}\partial_\theta$, $a_{-n} = \overline{a_n}$

- real vector field tangent to $S^1$

$e^{in\theta}\partial_\theta = -i(\ell_n - \overline{\ell_{-n}})$ - tangent v.f. to $S^1$

$e^{in\theta}\partial_r = -(\ell_n + \overline{\ell_{-n}})$ - normal v.f. to $S^1$

$$\sum_{n=-\infty}^{\infty} c_n\ell_n \longmapsto \mathrm{Re}\sum c_n \underbrace{(\ell_n - \overline{\ell_{-n}})}_{ie^{in\theta}\partial_\theta}$$

$$\begin{array}{ccl} \mathcal{A} & \xrightarrow{\sim} & \Gamma(S^1, T\mathbb{C}|_{S^1}) \text{ - real vector fields on } S^1 \text{ - tangent + normal!} \\ \cup & & \\ \{\sum c_n\ell_n \mid c_{-n} = \overline{c_n}\} & \longrightarrow & \text{tangent v. fields} \end{array}$$

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Thus, $\mathcal{A}$ = complexification of $\mathfrak{X}(S^1) = \mathrm{Lie\,Diff}(S^1)$

- $\mathrm{Conf}(\mathbb{H}) = PSL_2(\mathbb{R})$

$$z \mapsto \frac{az+b}{cz+d} \qquad a,b,c,d \in \mathbb{R}$$

$z \mapsto \frac{z-i}{z+i}$ conjugate in $PSL_2(\mathbb{C})$

- $\mathrm{Conf}(D) = PSU(1,1)$ ← holds for closed and open disks

$$z \longmapsto e^{i\varphi}\frac{z-a}{\bar{a}z-1}, \quad \varphi \in S^1, |a| < 1$$

- which vector fields on $S^1$ extend to c.v.f. on $D$?

- only v.f. $\left\{\sum_{n=-1}^{1} c_n \ell_n\right\}_{c_n = -\bar{c}_n} \simeq sl_2(\mathbb{R})$ if we want v.f. to be tangent to $S^1$
- $\infty$-dim. algebra $\left\{\sum_{n\geq -1} c_n \ell_n\right\}$ if we allow to move the boundary in normal directions

**Recall:** Riemann mapping theorem:
any two simply-connected domains $D, D' \subset \mathbb{C}$ are conformally equivalent

* Conformal symmetry of $\mathbb{R}^1$ (trivial case)

$$\frac{\mathbb{R}^1}{g=(dx)^2} \qquad \left\{\begin{matrix}\text{conf diffeos} \\ \varphi: \mathbb{R}^1 \to \mathbb{R}^1\end{matrix}\right\} = \left\{\begin{matrix}\text{all diffeos} \\ \varphi: \mathbb{R}^1 \to \mathbb{R}^1\end{matrix}\right\} \qquad \text{conf. factor } \Omega = \left(\frac{d\varphi}{dx}\right)^2$$

$\mathrm{conf}(\mathbb{R}^1) = \mathfrak{X}(\mathbb{R}^1)$ v.f. $\varepsilon(x)\partial_x$ has conf. factor $\omega = 2\partial_x \varepsilon(x)$

$$\underset{\overset{\shortparallel}{\overline{\mathbb{R}^1}}}{\mathrm{Conf}(S^1)} = \mathrm{Diff}(S^1) \supset \underbrace{PSL_2(\mathbb{R})}_{\text{Möbius transf. of } S^1} \simeq SO_+(2,1)$$ - "restricted conformal group".

$S^1$ is a conf. compactification for a subalgebra of $\mathrm{conf}(\mathbb{R}^1)$ - v.f. well-behaved at $\infty$.

* Conf. symmetry of Minkowski plane $\mathbb{R}^{1,1}$

$$g = (dx)^2 - (dy)^2 = \eta_{ij}\,dx^i dx^j \qquad \eta_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

switch to light-cone coordinates $\boxed{\begin{matrix} x_+ = x+y \\ x_- = x-y \end{matrix}}$

$$\left(\text{then: } \begin{matrix} x = \frac{x_+ + x_-}{2} & \partial_+ = \frac{\partial}{\partial x_+} = \frac{\partial_x + \partial_y}{2} & \partial_x = \partial_+ + \partial_- \\ y = \frac{x_+ - x_-}{2} & \partial_- = \frac{\partial_x - \partial_y}{2} & \partial_y = \partial_+ - \partial_- \end{matrix}\right)$$

$$g = dx_+ dx_- \qquad \eta_{ij} = \begin{array}{c|cc} & + & - \\ \hline + & & 1/2 \\ - & 1/2 & \end{array}$$

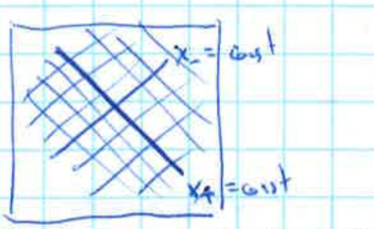
equation $\partial_i \varepsilon_j + \partial_j \varepsilon_i = \omega \eta_{ij}$ for c.v.f. ~~becomes~~ $\varepsilon^i \partial_i = \varepsilon_+(x_+, x_-)\partial_+ + \varepsilon_-(x_+,x_-)\partial_-$ becomes:

$$\left.\begin{matrix} \partial_- \varepsilon_+ = 0 \\ \partial_+ \varepsilon_- = 0 \\ \partial_+ \varepsilon_+ + \partial_- \varepsilon_- = \omega \end{matrix}\right\} \Rightarrow$$ generic c.v.f. on $\mathbb{R}^{1,1}$ is of form $\varepsilon = \varepsilon_+(x_+)\partial_+ + \varepsilon_-(x_-)\partial_-$,
$\omega = \partial_+ \varepsilon_+ + \partial_- \varepsilon_-$ - conf. factor

So: $$\mathrm{conf}(\mathbb{R}^{1,1}) = \underbrace{\mathfrak{X}(\mathbb{R})}_{\varepsilon_+\partial_+} \oplus \underbrace{\mathfrak{X}(\mathbb{R})}_{\varepsilon_-\partial_-}$$

functions of $x_-$ - "right-movers"
———— $x_+$ - "left-movers"

which convention?



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Conformal maps $\varphi: \mathbb{R}^{1,1} \to \mathbb{R}^{1,1}$; ~~$\varphi(x_+, x$~~

$(x_+, x_-) \mapsto (\varphi_+, \varphi_-)$

① $\varphi_+ = \varphi_+(x_+)$, $\varphi_- = \varphi_-(x_-)$ i.e. $\varphi \in \mathrm{Diff}(\mathbb{R}) \times \mathrm{Diff}(\mathbb{R})$ - reparametrization of $x_+$ and $x_-$ independently $\leftarrow \Omega(\partial_+\varphi_+)(\partial_-\varphi_-)$

② $\varphi_+ = \varphi_+(x_-)$, $\varphi_- = \varphi_-(x_+)$ i.e. $\varphi = \begin{pmatrix} \text{reparam of} \\ x_+, x_- \end{pmatrix} \circ \begin{pmatrix} \text{reflection} \\ (x,y) \to (x,-y) \end{pmatrix}$ $\leftarrow \Omega = (\partial_+\varphi_-)(\partial_-\varphi_+)$

So, $\mathrm{Conf}_0(\mathbb{R}^{1,1}) = \mathrm{Diff}_+(\mathbb{R}) \times \mathrm{Diff}_+(\mathbb{R})$ (the whole $\mathrm{Conf}(\mathbb{R}^{1,1})$ has 8 connected components)

$\overline{\mathbb{R}}^{1,1} = \underset{\overset{\uparrow}{x^+}}{S^1} \times \underset{\overset{\uparrow}{x^-}}{S^1}$ - compactification for a large subgroup of $\mathrm{Conf}(\mathbb{R}^{1,1})$

- $\mathrm{Conf}_0(\overline{\mathbb{R}^{1,1}}) = \mathrm{Diff}_+(S^1) \times \mathrm{Diff}_+(S^1) \supset \underset{\text{Möbius}_+}{PSL_2(\mathbb{R})} \times \underset{\text{Möbius}_-}{PSL_2(\mathbb{R})} \simeq SO(2,2)$ "restricted conformal group"

* * *

**def** A (pseudo-) Riemannian mfd $(M,g)$ is conformally flat if one can choose coord. charts on M s.t. in each chart $g = \Omega(x)\,\eta_{ij}\,dx^i dx^j$ with $\eta_{ij} = \begin{pmatrix} \underbrace{\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}}_{p} & \\ & \underbrace{\begin{matrix} -1 & & \\ & \ddots & \\ & & -1 \end{matrix}}_{q} \end{pmatrix}$ and $\Omega(x) > 0$

<being conformally flat is a local property>.

- for dim $M = 1, 2$ all mfds are conformally flat (for dim $M=1$ in fact globally flat)
- for dim $M \geq 3$, $(M,g)$ is conf. flat iff certain tensor vanishes $\left(\begin{matrix} \text{Weyl curvature } (0,4)\text{-tensor for } D \geq 4 \\ \text{Cotton } (0,3)\text{-tensor for } D = 3 \end{matrix}\right)$

**def** Moduli space of conformal structures on M: $\mathcal{M}_M = \{\text{conf. structures on } M\} / \mathrm{Diff}(M)$

Action of $\mathrm{Diff}(M)$ can be non-free; ~~its~~ stabilizer of $(M, \underset{\text{conf. str}}{\xi})$ is $\mathrm{Conf}(M,\xi)$

~~1-~~ ~~conf~~

* Case $M = \Sigma$ a surface. $\Sigma_{\underset{\text{genus}}{g}, \underset{\# \text{punctures}}{n}, m}$ - 2d smooth oriented mfd; $m$ ← # bdry circles; $\Sigma_{g,n} := \Sigma_{g,n,0}$

$\left\{ \begin{matrix} (2,0)\text{-} \\ \text{conformal} \\ \text{structures} \\ \text{on } \Sigma_{g,n,m} \end{matrix} \right\} = \left\{ \begin{matrix} \text{complex} \\ \text{structures} \\ \text{on } \Sigma \end{matrix} \right\}$ <compatible with orientation>

**Recall**: almost complex structure on M is $J \in \Gamma(M, \mathrm{End}(TM))$ s.t. $J_x^2 = -1$ for all $x \in M$.

then $T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$ — $\pm i$ eigenspaces of $J$

induces a splitting $\Omega = \bigoplus_{p,q} \Omega^{p,q}$

complex structure = additional integrability condition, automatic for dim $M = 2$. $(\bar\partial^2 = 0)$

$\bar\partial : \Omega^{p,q} \to \Omega^{p,q+1}$ - component of $d$

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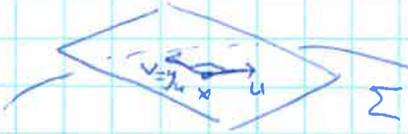
on $\Sigma$:

Conf. structure $\xi = g/_\sim \xrightarrow{(*)}$ cx str. $J: \underset{u}{T_x\Sigma} \to \underset{v}{T_x\Sigma}$

$\vec{v}$ s.t. • $v$ is orthog to $u$
- $\|v\| = \|u\|$ w.r.t. $g$
- $(u,v)$ - positively oriented pair



given a cx str. $J \longrightarrow$ choose $\sigma \in \Omega^2(\Sigma)$ vol. form, set $g_x(\vec{u},\vec{v}) = \sigma_x(\vec{u}, J\vec{v})$ then:
- conf. class $g/_\sim$ does not depend on $\sigma$
- $g$ is symmetric and non-deg
- this construction inverts $(*)$

Also, equivalences are the same: $\left(\begin{array}{c} \text{diffeo } (\Sigma,\xi) \xrightarrow{\varphi} (\Sigma',\xi') \\ \text{is a conf. equiv.} \end{array}\right) \Leftrightarrow \left(\begin{array}{c} (\Sigma, J) \xrightarrow{\varphi} (\Sigma', J') \\ \text{is a biholom map} \end{array}\right)$.

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~~Cross-ratio~~

- ~~Deformations~~ Infinitesimal Deformations of an (almost) complex structure $J$ on $M$

$$\boxed{\bar\partial \longmapsto \bar\partial + \mu\partial}$$ with $\mu \in \Omega^{0,1}(M, T^{1,0}M)$ - "Beltrami differential"

$d = \partial + \bar\partial$

$\Omega^{p+1,q} \overset{\partial}{\leftarrow} \Omega^{p,q} \underset{\bar\partial}{\rightrightarrows} \Omega^{p,q+1}$

(and $\partial \longmapsto \partial + \bar\mu\bar\partial$) acts as 1st order diff op. (corresponds to deforming $J_x \longmapsto J_x + (\mu_x + \bar\mu_x)$ ) signs/factors?

so, $\bar\partial_\mu = d\bar z^i \left(\frac{\partial}{\partial \bar z^i} - \mu^j_i \frac{\partial}{\partial z^j}\right)$ locally

$\mu = \mu^j_i(z,\bar z)\, d\bar z^i \frac{\partial}{\partial z^j}$ locally

- for $\mu$ to define a deformation of (strict) cx structure, we need $\bar\partial_\mu^2 = 0 \Leftrightarrow \boxed{\bar\partial\mu - \frac{1}{2}[\mu,\mu] = 0}$ $(**)$ - "Kodaira-Spencer equation"

thus: deformations of a cx structure are governed by Maurer-Cartan elements of the dgLa $\underline{\Omega^{0,*}(M, T^{1,0}M), \bar\partial, [,]}$

wedge product of forms ⊗ Lie bracket of vector fields

- In case ~~the~~ $M = \Sigma$ surface, KS equation $(**)$ is satisfied trivially (no $(0,2)$-forms)

- $\underset{[\text{space of cx str.}]}{T_J \mathcal{M}_\Sigma} \simeq \Omega^{0,1}(\Sigma, T^{1,0}\Sigma)$ - tangent to the moduli space $= \{$Beltrami differentials$\}$, $T_J \mathcal{M}_\Sigma \simeq H^{0,1}_{\bar\partial}(\Sigma, T^{1,0}\Sigma)$

$\underset{[\text{space of cx str.}]}{T^*_J \mathcal{M}_\Sigma} \simeq \Gamma(\Sigma, ((T^{1,0})^*)^{\otimes 2})$ • cotangent space $= \{$~~holom~~. quadratic differentials$\}$

$T^*\mathcal{M}_\Sigma = \{$holom. quadratic differentials $f(z)dz^2\}$

////////////////////

## Cross-ratio

In $\mathbb{R}^D$ (or $\mathbb{R}^{p,q}$): consider functions on $C_n(\mathbb{R}^D) = \{(\vec{x}_1, \ldots, \vec{x}_n) \in \mathbb{R}^D \mid \vec{x}_i \neq \vec{x}_j\}$ invariant under $\mathrm{Conf}(\mathbb{R}^D)$

open conf space

Invariants under translations $= \{$ ~~$f(\{\vec{x}_i - \vec{x}_j\})$~~ functions of differences $\vec{x}_i - \vec{x}_j$

Invariants under translations + rotations $=$ functions of distances $\|\vec{x}_i - \vec{x}_j\|$

—— translations + rotations + dilatations $=$ functions of ratios of distances $\frac{\|\vec{x}_i - \vec{x}_j\|}{\|\vec{x}_i - \vec{x}_k\|}$

—— translations + rotations + dilatations + SCTs $=$ functions of cross-ratios $[\vec{x}_1, \vec{x}_2 : \vec{x}_3, \vec{x}_4] = \frac{\|\vec{x}_1 - \vec{x}_3\| \cdot \|\vec{x}_2 - \vec{x}_4\|}{\|\vec{x}_1 - \vec{x}_4\| \cdot \|\vec{x}_2 - \vec{x}_3\|}$

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Given 4 points $z_1,z_2,z_3,z_4$ on $\mathbb{C}P^1$, one can form the

cross-ratio: $\boxed{[z_1,z_2:z_3,z_4] := \frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)}} = \frac{z_1-z_3}{z_1-z_4} : \frac{z_2-z_3}{z_2-z_4}$

it is invariant under $PSL_2(\mathbb{C})$, thus defines a ~~function~~ cx coordinate on $C_4(\mathbb{C}P^1)/PSL_2(\mathbb{C})$

Exercise!

where $C_n(M) = \{\{x_1,\dots,x_n\} \in M^n \mid x_i \neq x_j\}$ - open configuration space of $n$ points

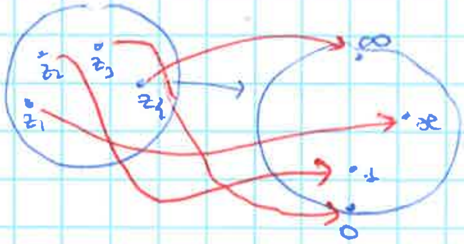
- $PSL_2(\mathbb{C})$ acts on $\mathbb{C}P^1$ 3-transitively* (translations = 1-transitive, transl + rotation + dil = 2-transitive)

thus one can map $\{z_1,z_2,z_3,z_4\} \longrightarrow \{0,1,\infty,\lambda\}$

Möbius transf.

Then $[z_1,z_2:z_3,z_4] = \frac{\lambda-1}{\lambda}$

or $\rightarrow \{x,1,0,\infty\}$ $\rightarrow [\cdots] = x$



- $S_4$ acts on the cross-ratio by permuting the 4 points:

$$\lambda \sim \underbrace{\frac{1}{\lambda} \sim 1-\lambda \sim \frac{\lambda}{\lambda-1}}_{\text{order 2}} \sim \underbrace{\frac{1}{1-\lambda} \sim \frac{\lambda-1}{\lambda}}_{\text{order 3}}$$

$$1 \rightarrow \underbrace{\mathbb{Z}_2 \times \mathbb{Z}_2}_{\text{symmetries of the cross-ratio}} \longrightarrow S_4 \longrightarrow S_3 \rightarrow 1$$

e.g. $[z_2,z_1:z_4,z_3] = [z_1,z_2:z_3,z_4]$
$\quad \parallel$
$[z_3,z_4:z_1,z_2]$
$\quad \parallel$
$[z_4,z_3:z_2,z_1]$

Or: every simply-connected (without boundary) Riem. surface $\Sigma$ is conf equiv to either
- open unit disk
- complex plane $\mathbb{C}$
- Riem. sphere $\mathbb{C}P^1$

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Uniformization Theorem (Klein-Koebe-Poincaré): * ~~closed~~ for $(\Sigma_{g,m,n}, \overset{\text{conf structure}}{\xi})$ :

ⓐ if ~~$\chi>0$ (i.e.~~ $\Sigma = S^2$ ), there is a unique metric representative of $\xi$ with $R=+1$
~~$=2-2g-m-n$~~

ⓑ if $\chi=0,1$, there is unique up to scaling flat $(R=0)$ metric

ⓒ if $\chi<0$ unique complete hyperbolic metric

~~ⓐ $\Sigma \simeq S^2$~~ ** a Riem. surface $(\Sigma_{g,m,n},\xi)$ is conf. equivalent to one of the following:

ⓐ' $\mathbb{C}P^1$

ⓑ' $\mathbb{C}$, $\mathbb{C}\setminus\{0\}$ $\underset{\text{infinite cylinder}}{\simeq}$ $\mathbb{C}/\mathbb{Z}$, annulus $\simeq$ finite cylinder $\simeq$ strip $\{0 \leq \operatorname{Im} z < 1\}/\mathbb{Z}$, punctured disk $\simeq$ semi-infinite cylinder, torus $\mathbb{C}/\mathbb{Z}_2$

$\mathbb{H}_+/\mathbb{Z}$

ⓒ' $\boxed{\mathbb{H}_+/\Gamma, \text{ for some } \Gamma \subset PSL_2(\mathbb{R}) \text{ "Fuchsian group" a discrete subgroup, } \Gamma \simeq \pi_1\Sigma}$

with standard cx/conf structure

ⓒ' $\mathbb{H}_+$ with $g = \frac{1}{y^2}(dx^2+dy^2)$, $R=-1$

or $\mathbb{D}$ with $g = \frac{4\,dz\,d\bar{z}}{(1-z\bar{z})^2}$

ⓐ' $\mathbb{C}P^1$ with $g = \frac{4\,dz\,d\bar{z}}{(1+z\bar{z})^2}$ - Fubini-Study metric $R=+1$

ⓑ' admits flat metric s.t.
- a geodesic cannot run into puncture in a finite time
- boundaries are geodesics

ref: Penner "Decorated Teichmüller Theory"

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$\mathcal{M}_{g,n} = \{\text{cx. str. on } \Sigma\} / \mathrm{Diff}_+(\Sigma_{g,n})$ ← punctures are not allowed to move

$\widetilde{\mathcal{M}}_{g,n}$ - with ordered punctures

Idea: do this quotient in steps: ~~$MCG(\Sigma_{g,n}) \hookrightarrow \mathrm{Diff}_+(\Sigma_{g,n})$~~ mapping class group

$$\mathrm{Diff}_0(\Sigma_{g,n}) \hookrightarrow \mathrm{Diff}_+(\Sigma_{g,n}) \twoheadrightarrow \boxed{MCG(\Sigma_{g,n}) \simeq \pi_0 \mathrm{Diff}_+}$$

$\mathrm{Diff}_0$ ↑ conn. comp. of 1

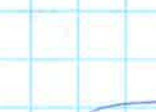
- mapping class gp.

$$\begin{tikzcd}
\mathrm{Diff}_0 \circlearrowright \{\text{cx. str. on } \Sigma\} \arrow[d] \\
pMCG \circlearrowright \{\text{cx. str. on } \Sigma\}/\mathrm{Diff}_0 \arrow[d] \\
S_n \circlearrowright \widetilde{\mathcal{M}}_{g,n} \arrow[d] \\
\mathcal{M}_{g,n}
\end{tikzcd}$$

pure = not permuting the punctures

$\{\text{cx. str. on } \Sigma\}/\mathrm{Diff}_0 = T_{g,n}$ "Teichmüller space" $\simeq \mathbb{R}^{6g-6+2n}$ = { equiv classes of conf. structures on $\Sigma_{g,n}$ endowed with a "marking" (- a diffeo $\Sigma^{\text{stand}}_{g,n} \xrightarrow{\varphi} \Sigma_{g,n}$ up to homotopy }

Ex: ~~$\mathcal{M}_{0,3} = pt$~~ $\widetilde{\mathcal{M}}_{0,n}$ : $S^2 \setminus n$ points with conf. structure $\xrightarrow[\text{uniformization of } S^2]{}$ $(\mathbb{CP}^1 \setminus \{z_1, \ldots, z_n\}) / PSL_2(\mathbb{C})$ ← a point in $\widetilde{\mathcal{M}}_{0,n}$



for $n \le 3$, $\widetilde{\mathcal{M}}_{0,3} = pt$ since we can ~~put~~ move the points into $0, 1, \infty$

(~~similarly~~ in fact, for $n \le 3$, $\widetilde{\mathcal{M}}_{0,3} = pt$)

- $\widetilde{\mathcal{M}}_{0,4} \simeq \mathbb{CP}^1 \setminus \{0, 1, \infty\}$

coordinate = cross-ratio $\{z_1, z_2 ; z_3, z_4\}$

DM compactification: * gluing in points $x = 0, 1, \infty$ ~ "nodal curves"

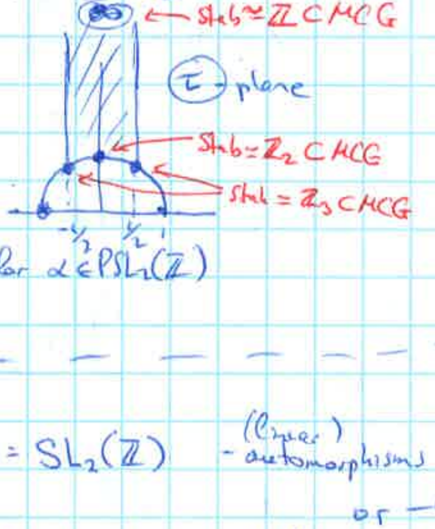
- $\widetilde{\mathcal{M}}_{0,5} = \{(x_1, x_2) \in \mathbb{CP}^1 \setminus \{0,1,\infty\} \mid x_1 \ne x_2\}$

two cross-ratios

- $\widetilde{\mathcal{M}}_{1,0} \simeq \underset{\tau \in}{\mathbb{H}_+} / \underbrace{PSL_2(\mathbb{Z})}_{MCG = SL_2(\mathbb{Z})}$

$\ni \left[\text{torus } \mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}\right]$ $= \mathbb{T}_\tau$

$\mathbb{T}_\tau \simeq \mathbb{T}_{\alpha\tau}$ for $\alpha \in PSL_2(\mathbb{Z})$



Also:

$\mathcal{M}_{1,1} \simeq \mathcal{M}_{1,0}$

has no conf. auto (preserving the puncture) ↗ ↖ has conf. automorphisms

notation: $MCG_{g,n} = \Gamma_{g,n}$ modular group

ref: ⟨Farb, Margalit "Primer on MCGs"⟩

* Mapping class group examples:

- $MCG_{\Sigma_{1,0}} = SL_2(\mathbb{Z})$ - (linear) automorphisms of $\mathbb{R}^2$ preserving the lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$ or — " — $\mathbb{C}$ —— " —— $\mathbb{Z} \oplus \tau\mathbb{Z} \subset \mathbb{C}$

- $MCG_{0,n}$ = "spherical braid group on n strands" $= \pi_1 \, C_n(S^2)$ ← non-ordered; open config. space of n points

$pMCG_{0,n}$ = "pure — " — " $= \pi_1 \, C_n^{\text{ordered}}(S^2)$

- $MCG$(  ) $= \mathbb{Z}$ generator: Dehn twist

generally, $MCG_{g,n}$ is generated by Dehn twists around cycles and Dehn 1/2-twists for pairs of punctures

# Symmetries in classical field theory

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- **Class. mechanics**

"configuration space"

$$S: \mathrm{Maps}([t_0,t_1],X) \to \mathbb{R}$$

$$[x(t)]_{t_0}^{t_1} \longmapsto \int_{t_0}^{t_1} dt\, L(x(t),\dot{x}(t)) \quad \text{- action}$$

$L \in C^\infty(TX)$ - Lagrangian

variation of $S$:

$$\delta S = \int_{t_0}^{t_1} dt \underbrace{\left(\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i}\right)}\delta x^i + \left.\underbrace{\frac{\partial L}{\partial \dot{x}^i}\delta x^i}\right|_{t_1}^{t_0}$$

$$\shortparallel$$

$$\left.\frac{d}{d\xi}\right|_{\xi=0} S[x(t)+\xi\,\delta x(t)] \quad \text{- Fréchet derivative}$$

no Umlaut in Noether, apparently!

Nöther 1-form

$$\alpha = \frac{\partial L}{\partial v^i}dx^i \in \Omega^1(TX)$$

classical trajectories $[x(t)]_{t_0}^{t_1}$ = extremals of $S$ with fixed b.c. $\delta x^i|_{t_0,t_1}=0$ = solutions of Euler-Lagrange eq. $\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} = 0$

**Ex:** ~~$X=\mathbb{R}^n$~~ a particle of mass $m$ on $\mathbb{R}^n = X$ in a potential $U \in C^\infty(\mathbb{R}^n)$

$$L = \frac{m|v|^2}{2} - U(x) \qquad \text{E-L eq: } m\ddot{x}^i + \partial_i U(x) = 0$$

- Newton's eq. of motion in a force field $F^i(x) = -\partial_i U(x)$

$\alpha = mv^i dx^i$

**Ex:** free particle on $(X,g)$ - Riem. mfd: $L = \frac{m}{2}\langle v,v\rangle_g = \frac{m}{2}g_{ij}v^iv^j$

then, EL eq: $\ddot{x}^i + \underbrace{\Gamma^i_{jk}}(x)\dot{x}^j\dot{x}^k = 0$ - eq. of geodesic motion ← Exercise!

↑ Christoffel symbol for $g$

$\alpha = \langle \underbrace{m\vec{v}}_{p\text{ - momentum}}, d\vec{x}\rangle_g$ - canonical 1-form on $T^*X$ ~~prated~~ pulled back to $TX$ by $g$. $TX \xrightarrow[g]{\sim} T^*X$ $\langle p, dx\rangle$

Symmetries & conserved quantities (integrals of motion) - in class. mech.

target symmetry (continuous) - group action $F: \mathbb{R} \circlearrowright X$

~~F is a symmetry~~

$\xi \mapsto F_\xi \in \mathrm{Diff}(X) \rightsquigarrow (F_\xi)_*: \mathrm{Maps}([t_0,t_1],X) \circlearrowleft$

$[x(t)] \longmapsto [F_\xi\, x(t)]$

$F$ is a symmetry if $S$ is invariant under $F_*$.

infinitesimally: $f = f^i(x)\frac{\partial}{\partial x^i} = \left.\frac{d}{d\xi}\right|_{\xi=0} F_\xi \in \mathfrak{X}(X)$ - symmetry if $\delta_f S = 0$

$$\shortparallel$$

$$\left.\frac{d}{d\xi}\right|_{\xi=0} S[F_\xi(x(t))]_{t_0}^{t_1}$$

Notation: change $\xi, x \to \lambda$ parameter of the 1-parametric group of symmetries

$$\delta_f S = \underbrace{\int_{t_0}^{t_1} dt\left(\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i}\right) f^i(x(t))}_{=0 \text{ mod EL}} + \left.\frac{\partial L}{\partial \dot{x}^i}f^i\right|_{t_0}^{t_1}$$

tangent lift of $f$, $\tilde{f} = f^i(x)\frac{\partial}{\partial x^i} + \partial_j f^i(x)\, v^j \frac{\partial}{\partial v^i}$

Nöther 1-form

Thus: $f \in \mathfrak{X}(X)$ a symmetry $\Rightarrow$ $\boxed{I_f := \frac{\partial L}{\partial \dot{x}^i}f^i = \iota_{\tilde{f}}\,\alpha} \in C^\infty(TX)$ is an integral of motion, i.e. $\frac{d}{dt}I_f(x(t),\dot{x}(t)) = 0$ if $[x(t)]$ is a class. trajectory

**Ex:** free particle in $\mathbb{R}^D = X$,

$L = \frac{m|v|^2}{2}$, $F_{\xi,\vec{u}}: \vec{x} \mapsto \vec{x} + \xi\vec{u}$ - translation

$\vec{u} \in \mathbb{R}^D$ fixed vector

→ $I_{\vec{u}} = m\langle \vec{v}, \vec{u}\rangle$ - integral of motion $\forall \vec{u}$

→ $m\vec{v}$ - vector-valued integral of motion - momentum

**Ex** $n$ particles in $\mathbb{R}^D$

$X = \underbrace{\mathbb{R}^D \times \cdots \times \mathbb{R}^D}_{n}$, $L = \sum_{i=1}^n \frac{m|v_i|^2}{2} - \sum_{i<j} U(\|\vec{x}_i - \vec{x}_j\|)$

↙ pairwise interaction

$F_{\xi,\vec{u}}: (\vec{x}_1,\ldots,\vec{x}_n) \mapsto (\vec{x}_1 + \xi\vec{u},\ldots,\vec{x}_n + \xi\vec{u})$ - simultaneous translation

$\vec{P}$ - total momentum

$\to I_{\vec{u}} = \langle \overbrace{\sum m\vec{v}_i}, \vec{u}\rangle$

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## Source symmetry

$R_\xi : \mathbb{R} \longrightarrow \mathbb{R}$ - family of diffeo

$[t_0,t_1] \mapsto [R_\xi(t_0), R_\xi(t_1)]$

$(R_\xi^{-1})^*: \mathrm{Maps}([t_0,t_1],X) \longrightarrow \mathrm{Maps}([R_\xi(t_0),R_\xi(t_1)],X)$ - right action of $\overset{\xi}{R}$ on trajectories

$[x(t)]_{t_0}^{t_1} \longmapsto [x'(t')=x(t)]_{t_0'=R_\xi(t_0)}^{t_1'=R_\xi(t_1)}$

$R$ is a symmetry if $S$ is $(R_\xi^{-1})^*$-invariant

Infinitesimally: $r=r(t)\frac{\partial}{\partial t} = \frac{d}{d\xi}\Big|_{\xi=0} R_\xi \in \mathfrak{X}(\mathbb{R})$ , $t \longmapsto t' = t+r(t)$ infin. transf

$-r^*: x(t) \mapsto x'(t) = x(t-r(t)) = x(t) - r(t)\dot{x}(t)$  $(*)$

$$0 = \delta_r S = \underbrace{-\int_{t_0}^{t_1} dt\; r\,\dot{x}^i\left(\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i}\right)}_{\text{change due to variation of fields } (*)} - \Big| r\left(\dot{x}^i\frac{\partial L}{\partial \dot{x}^i} - L\right)\Big|_{t_0}^{t_1} \;\circled{=}$$

$:= \frac{d}{d\xi}\Big|_{\xi=0} S[x(R_\xi^{-1}(t))]_{R_\xi(t_0)}^{R_\xi(t_1)}$

due to variation of $t_0, t_1$

$\circled{=} -\Big| r\, H(x,\dot{x})\Big|_{t_0}^{t_1}$ with $\boxed{H := \dot{x}^i\frac{\partial L}{\partial \dot{x}^i} - L} \in C^\infty(TX)$ - Hamiltonian

* since constant $r$ (indep. of $t$) is a symmetry, (H is an int of motion)

* $H=0 \iff$ any $r$ is a symmetry $\iff$ theory is diff-invariant. <topological mechanics>

Ex: $X=\mathbb{R}^D$

$L = \frac{m|v|^2}{2} - U(x) \rightsquigarrow H = \frac{m|v|^2}{2} + U(x)$

Ex ("relativistic particle") $X, g$ - (pseudo) Riemannian.

$L = m\sqrt{(\vec{v},\vec{v})_g} \rightarrow S = \int m\sqrt{\left(\frac{dx}{dt},\frac{dx}{dt}\right)_g}\,dt$ - diff-invariant $\rightsquigarrow H=0$

* One can consider mixed source-target symmetries $(F_\xi)_*(R_\xi^{-1})^*: \mathrm{Maps}([t_0,t_1],X) \to \mathrm{Maps}([R_\xi(t_0),R_\xi(t_1)],X)$

infinitesimally: $x'^i(t) = x^i(t) - r(t)\dot{x}^i(t) + f^i(x(t))$.  $\boxed{I_{r,f} = \frac{\partial L}{\partial \dot{x}^i} f^i - H r}$ - ~~conserved~~ int. of motion

Ex: $X=\mathbb{R}^D$, $L = \frac{m|v|^2}{2}$  $R_\alpha: t \mapsto e^\alpha t$  $F_\alpha: \vec{x} \mapsto e^{\alpha/2}\vec{x}$

Exercise: - find $I_{r,f}$

* In Hamiltonian mechanics:

For $\{x^i, p_i\}$ Darboux chart on $\Phi$, $\omega = \sum_i dp_i \wedge dx^i$, $\{f,g\} = \sum \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial x^i} - \frac{\partial f}{\partial x^i}\frac{\partial g}{\partial p_i}$

$(\Phi,\omega)$ - symplectic mfd (phase space), $H \in C^\infty(\Phi)$ - Hamiltonian $\longrightarrow \check{H} = \{H,\cdot\} \in \mathfrak{X}(\Phi)$ - Ham. v.f. (s.t. $L_{\check{H}}\omega = -dH$)

time-evolution in time $t$ = flow of $\check{H}$ in time $t$

for $\{Y^a\}$ coord. functions on $\Phi$, $\boxed{\dot{Y}^a = \{H, Y^a\}}$ Hamilton's eq. of motion

phase-space symmetry: a 1-param. family of symplectomorphisms $\Psi_\xi: \Phi \to \Phi$ commuting with $\mathrm{Flow}_t(\check{H})$

assume: $\frac{d}{d\xi}\Big|_{\xi=0}\Psi_\xi = \{\psi,\cdot\}$. Then symmetry condition: $[\check{\psi},\check{H}]=0 \iff \{\psi,H\} = \mathrm{Const}$

assume $\mathrm{Const}=0$. Then $\psi$ is an integral of motion: $\frac{d}{dt}\psi(Y(t)) = \{H,\psi\} = 0$

if $\{H,\psi\} = C$, then $\psi - Ct$ is an int. of motion

Ex: particle in a ~~magnetic field~~ linear potential (= in const. ~~[illegible]~~ field) $H = \frac{p^2}{2m} + \mu x \rightsquigarrow \check{\psi} = -\frac{\partial}{\partial x}$, $\psi = p$  $\circled{p+\mu t}$ = int. of motion

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## Classical (Lagrangian) field theory

(possibly also higher derivatives)

$$\underset{\text{"field" on }(M,g)\leftarrow\text{metric}}{S_{g}[\varphi]} = \int_M \underbrace{d\,vol_g}_{\sqrt{g}\,d^n x}\, \mathcal{L}(\varphi, d\varphi) \qquad \varphi \in \text{Fields}_M = \Gamma(M, \underset{\text{sheaf of fields}}{\text{Fields}})$$

$\underbrace{\sqrt{g}\,d^n x \cdots \mathcal{L}(\varphi,d\varphi)}_{\mathcal{L}^{dens}}$

For simplicity: $\text{Fields} = \text{Maps}(M, \underset{=\mathbb{R}^N}{X})$

\+ "covariance": for a diffeo $m: M \to M'$, we have a map $(m^{-1})^*: \text{Fields}_{M} \to \text{Fields}_{M'}$ and $S_{M,g}(\varphi) = S_{M',(m^{-1})^*g}((m^{-1})^*\varphi)$

Variation: $$\delta S = \int_M \sqrt{g}\,d^n x \left(\frac{\partial\mathcal{L}}{\partial\varphi^a} - \underbrace{\frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^a)}\right)}_{\nabla_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^a)} = \text{div}_{\sqrt{g}d^nx}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^a)}\partial_\mu\right)}\right)\delta\varphi^a + \underbrace{\int_{\partial M}\underbrace{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^a)}\iota_{\partial_\mu}(\sqrt{g}\,d^n x)\,\delta\varphi^a(x)}_{\underline{\alpha}\ \in\ \Omega^1_{loc}(\text{Fields}, \Omega^{n-1}(M))}}_{\text{flux of the v.field } \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^a)}\delta\varphi^a\partial_\mu \text{ through } \partial M}$$

- covariant divergence

structure: $\delta S = \int_M \underbrace{(\cdots)}_{EL}\delta\varphi + \int_{\partial M}\underline{\alpha}$

EL equations: $$\left(\frac{\delta S}{\delta\varphi^a(x)} = \right)\ \boxed{\frac{\partial\mathcal{L}}{\partial\varphi^a} - \nabla_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^a)} = 0}$$ - PDE determining class. dynamics

$\frac{d}{d\xi}\Big|_{\xi=0} S[\varphi + \xi\underbrace{\delta\varphi}_{\text{compactly-supported fluctuation}}] = 0$

notation: $\vec{P}_a := \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^a)}\partial_\mu$

$\frac{\partial\mathcal{L}}{\partial\varphi^a} - \text{div}\,\vec{P}_a = 0$

**Ex**:* free (massive) scalar field

$\text{Fields}_M = C^\infty(M)$ $$S_{M,g}[\varphi] = \int_M \sqrt{g}\,d^n x\left(\frac{1}{2}\langle d\varphi, d\varphi\rangle_{g^{-1}} + \frac{m^2}{2}\varphi^2\right) = \int_M \frac{1}{2}d\varphi\wedge * d\varphi + \frac{m^2}{2}\varphi\wedge *\varphi$$

EL eq: $\Big(\underbrace{\frac{1}{\sqrt{g}}\partial_\mu\sqrt{g}\partial^\mu}_{\Delta = *d*d} - m^2\Big)\varphi = 0$ - "Klein-Gordon eq." (for $M = \mathbb{R}^{3,1}$)

$$\boxed{\begin{aligned}\delta S &= \int_M \frac{(-1)^{n+1}}{\cancel{2}}(d\delta\varphi\wedge *d\varphi - m^2\delta\varphi\,\varphi) \\ &= \int_M (-1)^{n+1}\delta\varphi(d*d\varphi - m^2\varphi) + \\ &+ \int_{\partial M}\underbrace{(-1)^{n+1}\delta\varphi *d\varphi}_{\underline{\alpha}}\end{aligned}}$$

* non-free scalar field: $S = \int_M \frac{1}{2}d\varphi\wedge *d\varphi + \cancel{\frac{m^2}{2}\varphi\wedge *\varphi}\ \underbrace{U(\varphi)}_{}\,d\,vol$ - polynomial of deg ≥ 3

EL eq: $\Delta\varphi - U'(\varphi) = 0$ - non-linear PDE !

## Symmetries:

Assume we have a vector field $V \in \mathfrak{X}(\text{Fields}_M)$ - infinitesimal symmetry

$\left.\begin{array}{l}\varphi^a(x) \mapsto \varphi^a(x) + \epsilon V^a(\varphi(x), \partial_\mu\varphi(x)) \\ \text{- inf. transformation}\end{array}\right\}$ such that $\overset{L_V S}{\overset{\shortparallel}{\delta_V S}} = \int_{\partial M}\Lambda$ - boundary term

or: $L_V\mathcal{L}^{dens} = d\Lambda$ ← field-dependent (n-1)-form on M

then $$\boxed{J = L_V\underline{\alpha} + (-1)^n\Lambda} \in \Omega^0_{loc}(\text{Fields}, \Omega^{n-1}(M))$$

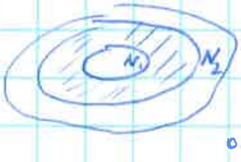
- Noether current associated to the symmetry

Noether theorem: $\boxed{dJ \underset{EL}{\sim} 0 \text{ mod. EL equation.}}$

**Proof**: $L_V S = \delta_V S = L_V \int_M EL\,\delta\varphi + d\underline{\alpha} = \int_M \underbrace{\pm EL}_{\underset{EL}{\sim}0}\cdot L_V\delta\varphi + (-1)^n L_V d\,\underline{\alpha}$
$= \int_M d\Lambda$

- true for any $\underset{\text{submanifold}}{N \subset M}$ $\Rightarrow d\Lambda \underset{EL}{\sim} -(-1)^n d\,L_V\underline{\alpha}$
$\Rightarrow d(L_V\underline{\alpha} + (-1)^n\Lambda) \sim 0$

Thus, for $N_1, N_2$ two cobordant codim = 1 submflds in $M$,



we have $\oint_{N_1} J = \oint_{N_2} J$ on a classical field configuration allowed

$\oint_N J$ - "Noether charge"

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**Ex:** free massless boson $S = \int \frac{1}{2} d\varphi \wedge * d\varphi$  symmetry: $\varphi \longrightarrow \varphi + x$

Nöther current: $J = \iota_v \alpha = \iota_{\frac{\partial}{\partial\varphi}} (-1)^{n+1} \delta\varphi * d\varphi = (-1)^{n+1} * d\varphi$ - closed form, modulo EL

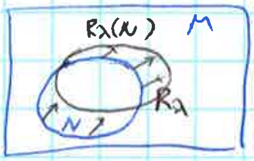
note that for $m \neq 0$, EL is: $\Delta\varphi = m^2\varphi$ and is not conserved!

- For Mixed symmetry:
  $x' = R_\lambda(x)$
  $\varphi'(x') = F_\lambda(\varphi(x))$ or $\varphi(x) \longmapsto F_\lambda(\varphi(R_\lambda^{-1}(x))$ infinitesimally: $\varphi^a(x) \longmapsto \varphi^a(x) + \lambda\underbrace{(f^a(\varphi(x)) - r^\mu(x)\partial_\mu\varphi^a(x))}_{V \in \mathfrak{X}(\text{Fields})}$

~~then~~ symmetry condition:

for any $\underset{\text{submersed}}{N} \subset M$, $\boxed{S_N[\varphi] = S_{R_\lambda(N)}[(F_\lambda)_*(R_\lambda^{-1})^*\varphi]}$ (*)

(**)



$\rightarrow \nabla_\mu J^\mu \underset{\text{mod EL}}{\sim} 0$ with $\boxed{J^\mu_{f,r} = r^\nu\left(\mathcal{L}\delta^\mu_\nu - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^a)}\partial_\nu\varphi^a\right) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^a)} f^a}$

or: $\vec{J} = \mathcal{L}\vec{r} - (L_{\vec{r}}\varphi^a)\vec{p}_a + f^a\vec{p}_a$

from $\frac{d}{d\lambda}\Big|_{\lambda=0}(*) = \underbrace{\int_M EL \cdots}_{\sim 0} + \int_{\partial M} \iota_v\alpha + \int_{\partial M} \mathcal{L}\, \iota_{r^\mu\partial_\mu}(\sqrt{g}\, d^n x)$ ← from shift of integration domain

Noether current: vector field $J^\mu\partial_\mu \rightleftarrows$ $(n-1)$-form $J = \iota_{J^\mu\partial_\mu}(\sqrt{g}\, d^n x)$

conservation: $\nabla_\mu J^\mu = 0$ mod EL or $\text{div}_{\sqrt{g}d^n x} \vec{J} = 0$ $\rightleftarrows$ $dJ = 0$ mod EL

Noether charge: flux of ~~$J^\mu\partial_\mu$~~ $\vec{J}$ through $N \subset M$ $\rightleftarrows$ $\oint_N J$

**Ex:** for a field theory on (flat) $\mathbb{R}^n$, translation symmetry (on the source!) $x^\mu \rightarrow x^\mu + a^\mu$ - symmetry due to covariance, since translations are isometries

$J^\mu_{\vec{x}\rightarrow\vec{x}+\vec{a}} = -T^\mu{}_\nu a^\nu$ with $\boxed{T^\mu{}_\nu := \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^a)}\partial_\nu\varphi^a - \mathcal{L}\delta^\mu_\nu}$ - the "canonical" stress-energy tensor (or energy-momentum)

$\partial_\mu T^\mu{}_\nu \underset{\text{mod EL}}{=} 0$ by Noether thm ← $T^\mu{}_\nu\partial_\mu \otimes dx^\nu = \vec{p}_a \otimes d\varphi^a - \mathcal{L}\cdot \text{id}$ $\in \Gamma(M, \text{End}\,TM)$

$P_\mu(x^0) = \int_{\substack{x^0 \text{ fixed slice} \\ \text{of } \mathbb{R}^n}} T^0{}_\mu(x)$ - conserved energy-momentum (co-)vector, i.e. $\frac{d}{dx^0} P_\mu(x^0) \underset{EL}{\sim} 0$

**Ex:** for a scalar field, $\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi + \frac{m^2}{2}\varphi^2$ $\rightsquigarrow$ $T^\mu{}_\nu = \partial^\mu\varphi\,\partial_\nu\varphi - \delta^\mu_\nu\left(\frac{1}{2}\partial_\lambda\varphi\,\partial^\lambda\varphi + \frac{m^2}{2}\varphi^2\right)$

or: $T = (d\varphi)^\# \otimes d\varphi - \left(\frac{1}{2}\langle d\varphi, d\varphi\rangle_g + \frac{m^2}{2}\varphi^2\right)\text{id}$

check the conservation: $\partial_\mu T^\mu{}_\nu = \Delta\varphi\,\partial_\nu\varphi + \cancel{\partial^\mu\varphi\,\partial_\mu\partial_\nu\varphi} -$
$- \cancel{\partial_\nu\partial_\lambda\varphi\,\partial^\lambda\varphi} - m^2\varphi\,\partial_\nu\varphi =$
$= (\Delta\varphi - m^2\varphi)\,\partial_\nu\varphi \underset{EL}{\sim} 0$

## Hilbert stress-energy tensor

2/6/2019

$$\boxed{T^{\mu\nu} := -\frac{2}{\sqrt{g}}\frac{\delta S_g}{\delta g_{\mu\nu}}}$$ or: $$\overset{\text{variation w.r.t. variation of metric}}{\delta_g S} = -\frac{1}{2}\int_M \sqrt{g}\, d^n x\, T^{\mu\nu}\delta g_{\mu\nu}$$ $$\mathbb{T} = T^{\mu\nu}\partial_\mu\cdot\partial_\nu \in \Omega^0_{\text{loc}}(\text{Fields}, \Gamma(M, \text{Sym}^2 TM))$$

it is
- symmetric, $T^{\mu\nu} = T^{\nu\mu}$
- conserved: $\nabla_\mu T^{\mu\nu} \underset{EL}{\approx} 0$, ← Idea: let $R_\lambda$ - family of diffeo of $M$ (rel. $\partial M$)

$$S_{M,g}(\varphi) = S_{M,(R_\lambda^{-1})^* g}((R_\lambda^{-1})^*\varphi) \quad \text{by covariance}$$

→ apply $\frac{d}{d\lambda}\Big|_{\lambda=0}$ → $$0 = -\frac{1}{2}\int_M \sqrt{g}\, d^n x\, T^{\mu\nu}(\nabla_\mu r_\nu + \nabla_\nu r_\mu) + \underbrace{\left(\overset{\text{coming from variation of fields}}{\delta_r^{(g\text{ fixed})}} S\right)}_{\sim 0 \text{ mod } EL} \quad (*)$$

$\xrightarrow{\text{Stokes'}}$ $\nabla_\mu T^{\mu\nu} \underset{EL}{\sim} 0$

- $T^{\mu\nu}$ does not necessarily coincide with $T^{\mu\nu}_{\text{can.}}$ on flat spacetimes

- by (*), $\delta_r^{(g\text{ fixed})} S \sim \int \sqrt{g}\, d^n x\, T^{\mu\nu}\nabla_\mu r_\nu$ ⇒ ~~then~~ if $r$ is a source symmetry, then $\boxed{J^\mu = T^\mu{}_\nu r^\nu}$ is conserved.

Thus, $T^\mu{}_\nu\, d x_\mu \otimes dx^\nu \in \Gamma(M, \text{End}\, TM)$ - tensor transforming source-symmetries $r$ into conserved currents $J = ~~T~~ \iota_r T$

**Ex:** for free massive scalar field: $$T^{\mu\nu}_{\text{Hilb}} = \partial^\mu\varphi\,\partial^\nu\varphi - (g^{-1})^{\mu\nu}\left(\frac{1}{2}\partial_\lambda\varphi\,\partial^\lambda\varphi + \frac{m^2}{2}\varphi^2\right) = T^{\mu\nu}_{\text{canonical}} \text{ for } \mathbb{R}^n.$$

## Classically conformally invariant field theories

Consider $S_{M,g}$ invariant under Weyl transformations, $S_{M,g}[\varphi] = S_{M,\Omega\cdot g}[\varphi]$ $\forall \Omega(x) > 0$

$$\Rightarrow\ 0 = \delta_{g\mapsto(1+\omega)g} S = -\frac{1}{2}\int\sqrt{g}\, d^n x\, T^{\mu\nu}(x)\, g_{\mu\nu}(x)\, \omega(x) \quad \forall\omega(x) \quad \Rightarrow \boxed{\text{tr}\, T = T^\mu{}_\mu = T^{\mu\nu}(x)\, g_{\mu\nu}(x) = 0}$$

i.e. theory is conformally invariant $\Leftrightarrow$ $\boxed{\text{tr}\, T = 0}$

- Weyl-invariance $\underset{\text{by covariance}}{\Longleftrightarrow}$ $\delta_r^{(g\text{ fixed})} S_N = 0$ for any $r \in \text{conf}(N, g)$ ⇒ for any conf v.f. $r = r^\mu\partial_\mu$, $J^\mu_r = T^\mu{}_\nu r^\nu$ is a conserved current

- $T^{\mu\nu}$ depends on the metric; Weyl transf. act by $g_{\mu\nu} \to \Omega\, g_{\mu\nu}$
$$T^{\mu\nu} \to \Omega^{-1-\frac{n}{2}}T^{\mu\nu}$$

In particular, for $\underset{\text{CFT}}{n = 2}$ on, $T_{\mu\nu}$ is Weyl-invariant or $T_{\mu\nu} \to \Omega^{1-\frac{n}{2}}T_{\mu\nu}$

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**Ex** ① scalar field $S = \int \frac{1}{2} d\varphi\wedge * d\varphi + \frac{m^2}{2}\varphi^2\, d\text{vol}$, $$T^{\mu\nu} = \partial^\mu\varphi\,\partial^\nu\varphi - (g^{-1})^{\mu\nu}\left(\frac{1}{2}\partial_\lambda\varphi\,\partial^\lambda\varphi + \frac{m^2}{2}\varphi^2\right)$$
$$\text{tr}\, T = \frac{2-n}{2}\partial_\mu\varphi\,\partial^\mu\varphi - n\frac{m^2}{2}\varphi^2$$

So: $\text{tr}\, T = 0$ iff $n = 2, m = 0$ → only massless 2D scalar field is conformal

Explicitly: $$\boxed{S_{\Omega\cdot g} = \int \Omega^{\frac{n}{2}-1}\frac{1}{2} d\varphi\wedge * d\varphi + \Omega^{\frac{n}{2}}\frac{m^2}{2}\varphi^2\, d\text{vol}} = S_g \text{ iff } \boxed{n=2}, m = 0$$

② Electromagnetic field
$$\underset{\in\Omega^1(M)}{S[A]} = \frac{1}{2}\int \underset{F = dA}{F\wedge * F}, \quad EL: d^* F = 0; \quad T^{\mu\nu} = g_{\alpha\beta}F^{\mu\alpha}F^{\nu\beta} - \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}(g^{-1})^{\mu\nu}$$
$$\text{tr}\, T = \frac{4-n}{4}F_{\alpha\beta}F^{\alpha\beta} \to \text{tr}\, T = 0 \Leftrightarrow \boxed{n=4}$$

Explicitly:
$$S_{\Omega g} = \frac{1}{2}\int \Omega^{\frac{n}{2}-2} F\wedge * F$$

# Classical (Lagrangian) field theory

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action: $S_{M,g}[\varphi] = \int_M \overbrace{d\,\mathrm{vol}_g\, \mathcal{L}(\varphi, \partial\varphi | g)}^{\mathcal{L}_{dens}}$

$\underset{\sim}{M,g}$ Riemannian mfd (possibly with boundary)

"field" on $M$, $\varphi \in \Gamma(M, \mathbb{F}\text{ields}) = \mathcal{F}_M$

we assume "covariance": for $m: M \xrightarrow{\sim} M'$ diffeo,

$S_{M,g}[\varphi] = S_{M', (m_*^{-1})^* g}[(m^{-1})^* \varphi]$

transform both $g$ and $\varphi$ by $m$

Euler-Lagrange equation: $\langle \psi, \delta S\rangle$ $\quad \frac{d}{d\lambda}\Big|_{\lambda=0} S[\varphi + \lambda \underline{\psi}] = 0 \quad (*)$ → PDE on $\varphi$. "equation of motion"

variation of the field supported away from $\partial M$

Interpretation: $\iota_\psi \overset{dR}{\delta} S$, $\psi \in T_\varphi \mathcal{F}$

**Ex:**

- free (massive) scalar field $\varphi \in C^\infty(M)$ $\quad S[\varphi] = \int_M \frac{1}{2} d\varphi \wedge *_g d\varphi + \frac{m^2}{2}\varphi^2\, d\mathrm{vol}_g$

$$\delta S = \int_M (-1)^{n+1} \underbrace{d\delta\varphi \wedge * d\varphi}_{d(\delta\varphi \wedge * d\varphi) + \delta\varphi \wedge d * d\varphi} + (-1)^n m^2 \delta\varphi\, \overset{d\mathrm{vol}_g}{d\varphi} = \int_M (-1)^{n+1} \delta\varphi (\underbrace{\Delta}_{*d*d} - m^2)\varphi + d\underbrace{\left((-1)^{n+1} \delta\varphi * d\varphi\right)}_{\underline{\alpha} \in \Omega^{n-1}(M, \Omega^1(\mathcal{F}))_{loc}}$$

$$= \int_M \delta\varphi \left((-1)^{n+1}(\Delta - m^2)\varphi\right) + \int_{\partial M} (-1)^{n+1} \delta\varphi * d\varphi$$

- density of the Noether 1-form

$\Omega^{p,q}_{loc}(M \times \mathcal{F}) \ni \omega$

$\omega|_x$ depends on the jet of fields $(\varphi, \partial\varphi, \partial^2\varphi, ...)$ at $x$.

EL eq: $\overset{(*)}{} \boxed{(\Delta - m^2)\varphi = 0}$ - linear PDE

- non-free scalar field $\quad S[\varphi] = \int_M \frac{1}{2} d\varphi \wedge * d\varphi + \underbrace{U(\varphi)}_{\text{polynomial in } \varphi,\ \text{of degree} \geq 3} d\,\mathrm{vol}_g$

EL eq: $\boxed{\Delta\varphi - U'(\varphi) = 0}$ non-linear PDE

- Yang-Mills theory field $A$ - connection in a $G$-bundle $P$ over $M$.

$S[A] = \int_M \frac{1}{2} \langle F_A \overset{\wedge}{,} * F_A \rangle_{\text{Killing}}$, $F_A$ = curvature of $A$ $\in \Omega^2(M, \mathrm{ad}(P))$

E-L eq: $\boxed{\underset{\Omega^p(M, \mathrm{ad}(P)) \to \Omega^{p+1}(M, \mathrm{ad}(P))}{d_A} * F_A = 0}$ ← Yang-Mills equation

# Symmetries & Noether currents

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assume we have an infinitesimal (local) symmetry $\varphi(x) \longmapsto \varphi(x) + \overset{\text{infinitesimal parameter}}{\lambda} V(\varphi(x), \partial\varphi(x))$ (*)

$V \in \mathfrak{X}(\mathcal{F})_{loc}$

such that $L_V S_M[\varphi] = \int_{\partial M} \underset{\in\, \Omega^{n-1}(M, \mathrm{Fun}(\mathcal{F}))_{loc}}{\Lambda}$ or: $\boxed{L_V \mathcal{L}^{dens} = d\Lambda}$ (**)

Note (**) implies that symmetry ~~transformation~~ (*) transforms a sol. of EL into a solution of EL.

Noether theorem: Let $\boxed{J := (-1)^{n+1} L_V \underline{\alpha} - \Lambda}$ ← Noether current associated to a inf. symmetry

$\in \Omega^{n-1}(M, \mathrm{Fun}(\mathcal{F}))_{loc}$

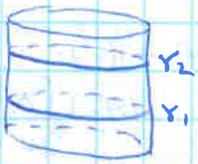
then $\boxed{dJ \sim 0 \mod EL}$

Proof: $L_V S_N = L_V \delta S_N = L_V \int_N (\delta\varphi\, EL + d\underline{\alpha}) = \int_N \pm (L_V \delta\varphi)\cdot EL + (-1)^n L_V d\underline{\alpha} \underset{EL}{\sim} \int_N (-1)^{n+1} d L_V \underline{\alpha}$

any $N \subset M$ submersed submfd

$\overset{=}{(**)} \int_N d\Lambda$ thus $\int_N d(\Lambda + (-1)^n L_V \underline{\alpha}) \underset{EL}{\sim} 0$ ▨

- If we have a symmetry $V$ and the associated Noether current $J$, then for $\gamma_1, \gamma_2$ two cobordant codim = 1 submanifolds in $M$, we have $\underbrace{\oint_{\gamma_1} J}_{} \underset{EL}{\sim} \underbrace{\oint_{\gamma_2} J}_{}$



conserved Noether charge

Example: free massless ~~boson~~ scalar field $S[\varphi] = \int_M \frac{1}{2} d\varphi \wedge * d\varphi$ symmetry: $\varphi \to \varphi + \lambda$ (constant shift of the value of the field) - preserves $S$ (and EL) (thus $\Lambda = 0$)

Noether current: $J = (-1)^{n+1} \iota_{\underbrace{\frac{\partial}{\partial\varphi}}_{V}} \underbrace{\left((-1)^{n+1} \delta\varphi * d\varphi\right)}_{\underline{\alpha}} = \boxed{*d\varphi}$

check: $d(*d\varphi) \underset{EL}{\sim} 0$ since EL reads $\Delta\varphi = 0$.

note: $*d\varphi$ is not conserved (closed mod EL) for a massive scalar field.

Example: massive ~~free boson~~ scalar field on $\mathbb{R}^n$, $S[\varphi] = \int_{\mathbb{R}^n} \frac{1}{2} d\varphi \wedge * d\varphi + \frac{m^2}{2}\varphi^2\, d\,vol_g$

symmetry: source-translation $R_\lambda: \vec{x} \mapsto \vec{x} + \underbrace{\lambda\vec{a}}_{\text{fixed vector}} = \vec{x}'$ i.e. $\varphi(\vec{x}) \longmapsto \varphi'(\vec{x}') = \varphi(\vec{x})$

or $\varphi'(\vec{x}) = \varphi(\vec{x} - \lambda\vec{a})$

$S_N[\varphi] = S_{R_\lambda N}[(R_\lambda^{-1})^* \varphi]$ ← action is conserved but on a shifted domain

infinitesimally: $\varphi(\vec{x}) \mapsto \varphi(\vec{x}) - \lambda\langle \vec{a}, d\varphi(x)\rangle$

$\Rightarrow$ we get a nontrivial $\Lambda$ in (**)! → conserved current:

Exercise: obtain this, write $T_{can}$, check conservation mod EL

$J_{\vec{a}} = *d\varphi \langle \vec{a}, d\varphi\rangle - \iota_{\vec{a}}\left(\frac{1}{2} d\varphi \wedge * d\varphi + \frac{m^2}{2}\varphi^2\, d\,vol_g\right)$

- a contraction of $\vec{a}$ with $T_{can} \in \Omega^{n-1}_M \otimes \Omega^1_M$ ← "canonical" stress-energy tensor

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Rem

| Noether current | as a vector field | as an (n-1)-form |
|---|---|---|
| | $\vec{J} = J^\mu \frac{\partial}{\partial x^\mu} \in \mathfrak{X}(M)$ | $\overset{\text{Hodge dual}}{\rightleftarrows}$ $J \in \Omega^{n-1}(M)$ |
| conservation: | $\mathrm{div}\,\vec{J} = \nabla_\mu J^\mu \underset{EL}{\sim} 0$ | $dJ \underset{EL}{\sim} 0$ |
| Noether charge: | flux of $\vec{J}$ through $\gamma \subset M$, codim = 1 | $\oint_\gamma J$ |

(canonical) stress-energy tensor - can be understood as

- $T_{can} \in \Omega^{n-1}\otimes\Omega^1$ (Hodge dual)
- $T_{can}{}^{\bullet}{}_{\bullet} \in \Gamma(M, \mathrm{End}\,TM)$
- $T_{can}^{\bullet\bullet}$, $T_{can\,\bullet\bullet}$ ← raising/lowering indices using the metric

(Hilbert) stress-energy tensor:

variation w.r.t. variation of the metric

$$T = T^{\mu\nu}(x)\,\partial_\mu\cdot\partial_\nu \quad \text{with} \quad T^{\mu\nu}(x) := -\frac{2}{\sqrt{\det g}}\frac{\delta S}{\delta g_{\mu\nu}(x)} \iff \delta_g S = -\frac{1}{2}\int d\mathrm{vol}_g \underbrace{\langle T, \delta g\rangle}_{T^{\mu\nu}\delta g_{\mu\nu}}$$

$\in \Gamma(M, \mathrm{Sym}^2 TM)$

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* $T$ is conserved $\boxed{\nabla_\mu T^{\mu\nu} \underset{EL}{\sim} 0}$ (or $(\mathrm{div}\otimes\mathrm{id})T \sim 0$)

IDEA: $R_\lambda$ - family of diffeo of $M$ (rel. $\partial M$) "= $\mathrm{Flow}_\lambda(r)$" ← vector field

$$\frac{d}{d\lambda}\Big|_{\lambda=0}\Big|\; S_{M,g}[\varphi] \underset{\substack{\uparrow\\ \text{covariance}}}{=} S_{R_\lambda(M),(R_\lambda^{-1})^* g}[(R_\lambda^{-1})^*\varphi]$$

$$\to 0 = \underbrace{-\frac{1}{2}\int_M d\mathrm{vol}_g\, T^{\mu\nu}(\nabla_\mu r_\nu + \nabla_\nu r_\mu)}_{\text{from } \frac{d}{d\lambda} \text{ change of the metric}} + \Big(\underbrace{\delta^{g \text{ fixed}} S}\Big) \xrightarrow[\text{Stokes'}]{\text{using } R_\lambda \text{ rel. } \partial M} \nabla_\mu T^{\mu\nu} \underset{EL}{\sim} 0$$

(*)

$$\Big(\Big(\frac{d}{d\lambda}\Big|_{\lambda=0} S_{R_\lambda(M),g}[(R_\lambda^{-1})^*\varphi] \underset{EL}{\sim} \text{surface term}$$

* If $\vec{r} \in \mathfrak{X}(M)$ is a source-symmetry, (i.e. ~~$\frac{d}{d\lambda}\Big|$~~ $S_{\underset{\mathrm{Flow}_\lambda(\vec{r})}{R_\lambda(N)},g}[(R_\lambda^{-1})^*\varphi] = S_{N,g}[\varphi]$) then $\vec{j}_r := T^{\mu\nu}(x)\, r_\nu \frac{\partial}{\partial x^\mu}$ is conserved: $\mathrm{div}\,\vec{j}_r \underset{EL}{\sim} 0$

IDEA: by (*), $0 = \int_N d\mathrm{vol}_g \underbrace{T^{\mu\nu}\nabla_\mu r_\nu}_{\nabla_\mu(T^{\mu\nu}r_\nu) - \underbrace{(\nabla_\mu T^{\mu\nu})}_{\underset{EL}{\sim} 0} r_\nu}$ for $r$ source-symmetry

Thus $T^\mu{}_\nu\, \partial_\mu \otimes dx^\nu \in \Gamma(M, \mathrm{End}\,TM)$ is a tensor transforming source-symmetries $r$ into conserved currents $\vec{j}_r = \langle T^{\bullet}{}_{\bullet}, r\rangle$

Ex: $T$ for a scalar field,

$$T_{Hilb} = (d\varphi)^\# \otimes (d\varphi)^\# - \underbrace{g^{-1}}_{\in\Gamma(M,\mathrm{Sym}^2 TM)}\Big(\frac{1}{2}\langle d\varphi, d\varphi\rangle_{g^{-1}} + \frac{m^2}{2}\varphi^2\Big) = T_{can}^{\bullet\bullet}$$

Ex: Electromagnetic field (Yang-Mills with $G=\mathbb{R}$): $A \in \Omega^1(M)$

$$S = \frac{1}{4}\int d\mathrm{vol}_g \underbrace{F_{\mu\nu}}_{(dA)_{\mu\nu}} F^{\mu\nu} \rightsquigarrow T^{\mu\nu} = g_{\alpha\beta}F^{\mu\alpha}F^{\nu\beta} - \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}(g^{-1})^{\mu\nu}$$

Ex: Chern-Simons: $\dim M = 3$

A - connection in a trivial G-bundle over M $\quad S[A] = \int_M \mathrm{tr}\,\frac{1}{2}A\wedge dA + \frac{1}{6}A\wedge[A,A] \rightsquigarrow \boxed{T \equiv 0}$

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## Classically conformally invariant field theories

Consider $S_{M,g}$ invariant under Weyl transformations: $S_{M,g}[\varphi] = S_{M,\Omega g}[\varphi]$ , $\forall \Omega(x) > 0$

Then $0 = \frac{d}{d\lambda}\Big|_{\lambda=0} S_{M, ~~g~~ (1+\lambda\omega)g}[\varphi] = -\frac{1}{2}\int_M d\mathrm{vol}_g \underbrace{T^{\mu\nu}(x)\, g_{\mu\nu}(x)}_{\mathrm{tr}\,T^{\cdot}_{\cdot} = T^{\mu}{}_{\mu}(x)}\, \omega(x) \quad \forall \omega(x) \Leftrightarrow \boxed{\mathrm{tr}\,T = 0}$

So: theory is conformally invariant iff $\mathrm{tr}\,T = 0$

- Weyl invariance $\underset{\text{by covariance}}{\Leftrightarrow}$ $\delta_r^{(g \text{ fixed})} S_N = 0$ $\forall r \in \mathrm{conf}(N,g)$ $\Rightarrow$ for any conf. v.f., $\vec{J}_r = \langle T, r\rangle$ is a conserved current

- $T^{\mu\nu}$ depends on ~~the~~ the metric $g \to \Omega g$
  $T^{\mu\nu} \to \Omega^{-1-\frac{n}{2}} T^{\mu\nu}$
  $T_{\mu\nu} \to \Omega^{1-\frac{n}{2}} T_{\mu\nu}$ $\nearrow$ so, for $\underset{\text{CFT}}{n=2}$, $T_{\mu\nu}$ is Weyl-invariant

Ex: scalar field $S = \int \frac{1}{2} d\varphi \wedge * d\varphi + \frac{m^2}{2}\varphi^2 d\mathrm{vol}_g$ $\quad T^{\mu\nu} = \partial^\mu\varphi\,\partial^\nu\varphi - (g^{-1})^{\mu\nu}\left(\frac{1}{2}\partial_\lambda\varphi\,\partial^\lambda\varphi + \frac{m^2}{2}\varphi^2\right)$

$\to \mathrm{tr}\,T = \frac{2-n}{2}\partial^\mu\varphi\,\partial_\mu\varphi - n\frac{m^2}{2}\varphi^2$

So: $\mathrm{tr}\,T = 0$ iff $\boxed{n=2 \text{ and } m=0}$ $\to$ only massless 2D scalar is conformal

Another way to see this: $S_{M,\Omega g}[\varphi] = \int \frac{\zeta}{2}\,\Omega^{\frac{n}{2}-1}\frac{1}{2}d\varphi\wedge * d\varphi + \Omega^{\frac{n}{2}}\frac{m^2}{2}\varphi^2 d\mathrm{vol}_g$ $= S_{M,g}[\varphi]$ iff $\begin{matrix} n=2 \\ \text{and } m=0\end{matrix}$

Ex electromagnetic field

$S[A] = \frac{1}{2}\int \underbrace{F}_{dA} \wedge * F$ $\quad \mathrm{tr}\,T = \frac{4-n}{4}F_{\alpha\beta}F^{\alpha\beta}$ $\quad \to \mathrm{tr}\,T = 0$ iff $\boxed{n=4}$

Alternatively: $S_{\Omega\cdot g}[A] = \frac{1}{2}\int \Omega^{\frac{n}{2}-2} F \wedge * F$ $= S_g[A]$ iff $\frac{n}{2} - 2 = 0$

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## Class. CFT on $\mathbb{R}^2 \simeq \mathbb{C}$

Symmetry + $\mathrm{tr} = 0$ $\Rightarrow T_{\mu\nu} = \begin{pmatrix} T_{11} & T_{12} \\ T_{12} & -T_{11}\end{pmatrix}$ $\quad$ Conservation: $\partial^\mu T_{\mu\nu} = 0 \Leftrightarrow$ ⊛ $\partial_1 T_{11} + \partial_2 T_{12} \sim 0$, $\partial_1 T_{12} - \partial_2 T_{11} \sim 0$

$\Rightarrow \underset{\substack{\uparrow \\ \text{Weyl-invariant}}}{T_{\cdot\cdot}} = T_{\mu\nu}dx^\mu dx^\nu = T_{11}\underbrace{(dx^2 - dy^2)}_{\frac{1}{2}(dz^2 + d\bar{z}^2)} + T_{12}\underbrace{2dx\,dy}_{\frac{1}{2i}(dz^2 - d\bar{z}^2)} = \underbrace{\frac{T_{11} - iT_{12}}{2}}_{T_{zz}}(dz)^2 + \underbrace{\frac{T_{11} + iT_{12}}{2}}_{T_{\bar{z}\bar{z}}}(d\bar{z})^2 = \boxed{T_{zz}(dz)^2 + T_{\bar{z}\bar{z}}(d\bar{z})^2}$

- no mixed $dz\,d\bar{z}$ term!
<due to $\mathrm{tr}\,T = 0$>

conservation (*): $\partial_{\bar{z}} T_{zz} \underset{EL}{\simeq} 0$ $\quad$ (so, $T_{zz}(dz)^2$ - holom. quadratic differential modulo EL)
$\partial_z T_{\bar{z}\bar{z}} \underset{EL}{\sim} 0$

Standard notation: $T_{zz} =: T$ , $\partial_z =: \partial$
$T_{\bar{z}\bar{z}} =: \bar{T}$ , $\partial_{\bar{z}} =: \bar{\partial}$

So: $\bar{\partial} T \sim 0$
$\partial \bar{T} \sim 0$

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- for $\vec{\varepsilon} = \varepsilon\partial + \bar{\varepsilon}\bar{\partial}$ a conf. v.f.,
the assoc. Noether ~~cha~~ current: $J_{\vec{\varepsilon}} = \iota_{\vec{\varepsilon}} T_{\cdot\cdot} = \boxed{\varepsilon T\, dz + \bar{\varepsilon}\,\bar{T}\, d\bar{z}}$ - closed 1-form (for $\varepsilon$ holom.)

Noether charge:
assoc. to conf.
sym. on $\mathbb{C}\setminus\{0\}$

$C_{\vec{\varepsilon}} = \oint_\gamma J_{\vec{\varepsilon}}$ $\quad$ on a sol. of EL, $C_{\vec{\varepsilon}}$ does not depend on $\gamma$ by Cauchy thm.

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massless scalar field on $\mathbb{C}$

$$S[\varphi] = \int\!\!\int dx\, dy\ \tfrac{1}{2}(g^{-1})^{\mu\nu}\partial_\mu\varphi\,\partial_\nu\varphi \ \overset{\nearrow}{=} \int \tfrac{i}{2} dz\, d\bar z\ 2\,\partial\varphi\,\bar\partial\varphi$$

switch to $z, \bar z$ - coordinates

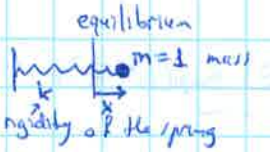
EL eq: $\Delta\varphi = 0 \iff \partial\bar\partial\varphi = 0$ i.e. $\varphi$ is harmonic

- for a general metric $g$ on $\mathbb{R}^2$, EL eq. $0 = \Delta\varphi = \frac{1}{\sqrt g}\partial_\mu \underbrace{\sqrt g (g^{-1})^{\mu\nu}}_{\sim \Omega^{\frac{n}{2}-1}} \partial_\nu \varphi$ is Weyl-invariant
- stress-energy tensor: $T = \partial\varphi\cdot\partial\varphi$

  $\bar T = \bar\partial\varphi\cdot\bar\partial\varphi$

[Quantum free scalar field]

Harmonic oscillator



$\sim \omega^2 = k$

I. Classical Mechanics (Hamiltonian formalism):

phase space $\Phi = T^*\mathbb{R}$ $\qquad H = \frac{p^2}{2} + \omega^2\frac{x^2}{2}$ $\qquad$ eq. of motion $\dot x = p$, $\dot p = -\omega^2 x$

$\omega_{symp} = dp\wedge dx$ $\qquad \omega$ = "frequency"

~~$\{p, x\} = 1$~~

~~$\{x, p\}$~~ $= \{p, x\} = 1$

Lagrangian formalism: $S[x(t)] = \int_{t_0}^{t_1} dt \left(\frac{\dot x^2}{2} - \frac{\omega^2}{2}x^2\right)$

EL eq: $\boxed{\ddot x + \omega x = 0}$

Canonical quantization:

$\left.\begin{array}{l} x \longmapsto \hat x \\ p \longmapsto \hat p \end{array}\right\}$ operators on $\mathcal{H}$ satisfying $[\hat p, \hat x] = -i\hbar$

$\hat H = \frac{\hat p^2}{2} + \frac{\omega^2}{2}\hat x^2$ - quantum Hamiltonian

Schrödinger representation $\mathcal{H} = L^2(\mathbb{R})$,

$\hat x : \psi(x) \longmapsto x\,\psi(x)$

$\hat p : \psi(x) \longmapsto -i\hbar\frac{\partial}{\partial x}\psi(x)$

Spectral problem $\underbrace{\hat H \psi}_{\left(-\frac{\hbar^2}{2}\frac{\partial^2}{\partial x^2} + \frac{\omega^2}{2}x^2\right)\psi} = E\psi$ can be solved explicitly, with

$$\boxed{\begin{array}{l} E_n = \hbar\omega\left(n + \frac{1}{2}\right) \\ \psi_n = C_n e^{-\frac{\omega x^2}{2\hbar}} H_n\left(\sqrt{\frac{\omega}{\hbar}}\,x\right) \end{array}}, n \geq 0$$

normalization constant s.t. $(\psi_n, \psi_n) = 1$ | Hermite polynomials $H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}$

Creation/annihilation operators

$$\left.\begin{array}{l} \hat a = \sqrt{\frac{\omega}{2\hbar}}\left(\hat x + \frac{i}{\omega}\hat p\right) \\ \hat a^+ = \sqrt{\frac{\omega}{2\hbar}}\left(\hat x - \frac{i}{\omega}\hat p\right) \end{array}\right\} \iff \begin{array}{l} \hat x = \sqrt{\frac{\hbar}{2\omega}}\left(\hat a^+ + \hat a\right) \\ \hat p = i\sqrt{\frac{\hbar\omega}{2}}\left(\hat a^+ - \hat a\right) \end{array}, \quad \hat H = \hbar\omega\left(\hat a^+\hat a + \frac{1}{2}\right)$$

then: $[\hat a, \hat a^+] = 1$, $[\hat H, \hat a] = -\hbar\omega\hat a$

$[H, \hat a^+] = \hbar\omega\hat a^+$

$\implies$ if $\hat H\psi = E\psi$, then $\hat H(\hat a\psi) = (E - \hbar\omega)\,\hat a\psi$

$\hat H(\hat a^+\psi) = (E + \hbar\omega)\,\hat a^+\psi$

So: $\hat a^+$ raises energy by $\hbar\omega$

$\hat a$ lowers —"—

$\mathrm{Heis} = \mathrm{Span}_{\mathbb{C}}\{\hat a, \hat a^+, \mathbb{1}\}$ - Heisenberg Lie algebra

central element

Heisenberg Lie algebra

$(V, \omega_{\text{symp}})$ - symplectic v.space $\longrightarrow$ $\text{Heis}(V, \omega_{\text{symp}}) = V \oplus \mathbb{R}\cdot \overset{\text{central element}}{K}$,

$[\hat{u}, \hat{v}] = K\,\omega_{\text{symp}}(u,v)$. ~~$(-i\hbar)$~~

u,v viewed as elements of Heis

Note: $\underset{\text{stand symp plane}}{\text{Heis}(T^*\mathbb{R})} \simeq \left\{ \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} \right\}$

Lie algebra of 3×3 upper-triangular matrices

$0 \to \mathbb{R} \to \text{Heis} \to \underset{\text{as abelian Lie alg}}{V} \to 0$ - SES of Lie alg.

$$\boxed{\begin{aligned} \text{Weyl}(\mathbb{R}^n) &:= \mathbb{C}[\hat{x}^1 .. \hat{x}^n, \hat{p}_1, \dots, \hat{p}_n] \Big/ {\scriptstyle [\hat{p}_i, \hat{x}^j] = \delta^j_i (-i\hbar)} \\ &\simeq U\,\text{Heis}(T^*\mathbb{R}^n) \Big/ \mathbb{K} = -i\hbar 1 \end{aligned}}$$

- alg. of polynomial diff op

Hermite polynomials

$H_0(x) = 1$

$H_1(x) = 2x$

$H_2(x) = 4x^2 - 2$

$H_3(x) = 8x^3 - 12x$

$H_4(x) = 16x^4 - 48x^2 + 12$

$\dots$

$$\int_{-\infty}^{\infty} dx\, e^{-x^2} H_n(x) H_m(x) = \sqrt{\pi}\, 2^n n!\, \delta_{nm}$$

$$\{\text{central extensions of } \mathfrak{g}\}/\sim \;\; \simeq H^2_{CE}(\mathfrak{g})$$

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$$S = \int \sqrt{\det g}\, d^n x \left( \tfrac{1}{2} (g^{-1})^{\alpha\beta} \partial_\alpha \varphi\, \partial_\beta \varphi + \frac{m^2}{2} \varphi^2 \right)$$

$$\delta_g S = -\tfrac{1}{2} \int \sqrt{\det g}\, d^n x\, \delta g_{\mu\nu} \underbrace{\left( (g^{-1})^{\mu\alpha} \partial_\alpha \varphi\, \partial_\beta \varphi\, (g^{-1})^{\beta\nu} - (g^{-1})^{\mu\nu} \left( \tfrac{1}{2} (g^{-1})^{\alpha\beta} \partial_\alpha \varphi\, \partial_\beta \varphi + \frac{m^2}{2} \varphi^2 \right) \right)}_{T^{\mu\nu}}$$

$\sqrt{\det g} = e^{\frac{1}{2} \operatorname{tr} \log g}$

$\delta_g \sqrt{\det g} = \sqrt{\det g}\, \tfrac{1}{2} \operatorname{tr}(g^{-1} \delta g)$

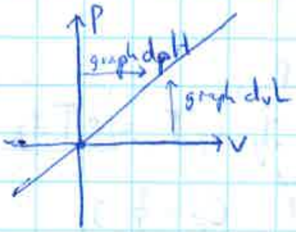
$\delta_g (g^{-1})^{\mu\nu} = -(g^{-1} \delta g\, g^{-1})^{\mu\nu}$

extremal of $S[x(t)] = \int dt\, L \rightleftarrows$ Flow of $\check{H}$ on $T^*X$

$$L \in C^\infty(\underset{v\ \ x}{TX}) \underset{}{\overset{\text{Legendre transform}}{\rightleftarrows}} H \in C^\infty(\underset{p\ \ x}{T^*X})$$

graph ~~$(d_{\text{vert}} TX\, L)$~~

graph $d_v L$ = graph $d_p H$ $\subset (T \oplus T^*)X$



Legendre transform is a fiberwise ~~diffeo~~ map ~~(diffeo in good cases)~~

$TX \xrightarrow{\mu} T^*X$ - diffeo in good cases.

defined by $L$ such that

$\mu(x,v) = (x, d_v L|_{(x,v)})$

or: $(x^i, v^i) \mapsto (x^i, p_i = \frac{\partial L}{\partial v^i})$

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"vacuum state" $|0\rangle \in \mathcal{H}$ s.t. $\boxed{\hat{a}|0\rangle = 0}$ $\Rightarrow \hat{H}|0\rangle = \frac{\hbar\omega}{2}|0\rangle$

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excited states $|n\rangle \in \mathcal{H}$ ~~st~~ , $|n\rangle = \underbrace{\alpha_n}_{\alpha_n = \frac{1}{\sqrt{n!}}}(\hat{a}^+)^n|0\rangle$ $\Rightarrow \hat{H}|n\rangle = (n+\frac{1}{2})\hbar\omega|n\rangle$

$\mathcal{H} = \mathrm{Span}_{\mathbb{C}}\{|0\rangle, |1\rangle, |2\rangle, \dots\}$

Rem: we can calculate $L^2$-norms of states ; eg normalize $|0\rangle$ s.t. $\|\cdot|0\rangle\|^2 = 1$
$=: \langle 0|0\rangle$

then ~~$\langle 1|1\rangle$~~ $=$ ~~$\langle 0|a a^+$~~ $\langle a^+|0\rangle, a^+|0\rangle\rangle_{L^2} = \langle|0\rangle, a a^+|0\rangle\rangle = \langle 0|\underbrace{a a^+}_{a^+a+1 \text{ by } [a,a^+]=1}|0\rangle \Leftrightarrow$

$\Leftrightarrow \langle 0|a^+\underbrace{a|0\rangle}_{0} + \underbrace{\langle 0|0\rangle}_{1} = 1$

In fact, ~~$\| \underbrace{|n\rangle}_{(a^+)^n|0\rangle} \|^2 = n!$~~ $\|(a^+)^n|0\rangle\|^2 = n!$ ← Exercise! $\Rightarrow$ basis $\{|n\rangle\}$ is orthonormal in $\mathcal{H}$

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Evolution operator $U(t) = e^{-\frac{i\hat{H}t}{\hbar}} : \psi = \sum_{n\geq 0} c_n|n\rangle \longmapsto \sum_{n\geq 0} c_n e^{-i(n+\frac{1}{2})\omega t}|n\rangle$

* normal ordering : word in $a, a^+ \longmapsto$ reshuffled word, where all $a$'s are on the right
all $a^+$'s are on the left

$W \longmapsto :W:$

$\mathrm{Heis}^{\otimes n} \longrightarrow \mathrm{Heis}^{\otimes n}$ (does not descend to $U\,\mathrm{Heis}$ )

In particular, $(:\hat{H}: = \hbar\omega a^+ a)$ $(n_0 + \frac{1}{2})$

so that $:\hat{H}:|n\rangle = n\hbar\omega|n\rangle$ , in particular $:\hat{H}:|0\rangle = 0$

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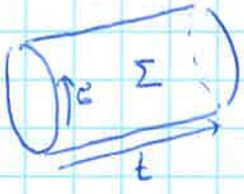
## Free boson on Minkowski cylinder



$\sigma \in \mathbb{R}/2\pi\mathbb{Z}$ "space coordinate" $t \in \mathbb{R}$ "time"

$$g_{\mu\nu} = \begin{array}{c|cc} & t & \sigma \\ \hline & 1 & \\ & & -1 \end{array}$$

$S[\varphi] = \frac{\overset{\swarrow \text{normalization constant, choose later}}{\varkappa}}{2}\int dt\, d\sigma\, (\dot{\varphi}^2 - (\partial_\sigma\varphi)^2)$ = class. mechanics with config. space $X = C^\infty(S^1)$ $\overset{\psi}{\varphi(\sigma)}$ and Lagrangian $\boxed{L = \frac{\varkappa}{2}\oint d\sigma(\dot{\varphi}^2 - (\partial_\sigma\varphi)^2)}$

- can expand $\varphi$ in Fourier nodes:

$$\boxed{\varphi(\sigma, t) = \sum_{n\in\mathbb{Z}} \varphi_n(t) e^{in\sigma}}$$

(with $\varphi_{-n}(t) = \overline{\varphi_n(t)}$ - reality condition )

Then: $L = \frac{\varkappa}{2} \cdot 2\pi \sum_{n\in\mathbb{Z}}(\dot{\varphi}_n\dot{\varphi}_{-n} - n^2\varphi_n\varphi_{-n})$

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Hamiltonian For

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## Hamiltonian formalism

phase space $\Phi = T^*X$ with coordinates $\varphi(\sigma), \underset{\uparrow\,\text{"momentum"}}{\pi(\sigma)}$ ; $\{\varphi(\sigma), \pi(\sigma')\} = -\delta_{per}(\sigma-\sigma')$

Legendre transform: $\pi(\sigma) = \frac{\delta L}{\delta\dot\varphi(\sigma)} = \varkappa\dot\varphi(\sigma)$

Hamiltonian: $H = \oint d\sigma \left(\frac{\pi(\sigma)^2}{2\varkappa} + \frac{\varkappa}{2}(\partial_\sigma\varphi)^2\right)$  Ham. eq.: $\dot\varphi = \frac{1}{\varkappa}\pi$, $\dot\pi = \varkappa\,\partial_\sigma^2\varphi$

Rem: in terms of $T_{\mu\nu}$: $T_{00} = T_{11} = \frac{\varkappa}{2}\left(\dot\varphi^2 + (\partial_\sigma\varphi)^2\right)$  $H = \oint d\sigma\, T_{00}$ - total energy

$T_{01} = T_{10} = \varkappa\dot\varphi\,\partial_\sigma\varphi$  $P = \oint d\sigma\, T_{01}$ - total momentum

Fourier modes: $\varphi(\sigma,t) = \sum_{n\in\mathbb{Z}} \varphi_n(t)e^{in\sigma}$  $\pi(\sigma,t) = \sum_{n\in\mathbb{Z}} \pi_n(t)e^{in\sigma}\frac{1}{2\pi}$ , $\boxed{\{\varphi_n, \pi_m\} = -\delta_{n,-m}}$

reality: $\varphi_{-n} = \bar\varphi_n$ , $\pi_{-n} = \bar\pi_n$

$$H = \sum_{n\in\mathbb{Z}} \frac{1}{2}\underbrace{\frac{1}{2\pi\varkappa}}_{1}\pi_n\pi_{-n} + \underbrace{\frac{1}{2}2\pi\varkappa}_{1/4} n^2\varphi_n\varphi_{-n} = (\pi_0)^2 + 2\sum_{n>0}^{\mathrm{Re}\,\&\,\mathrm{Im}}\left(|\pi_n|^2 + \frac{n^2}{4}|\varphi_n|^2\right) \quad **$$

choose $\varkappa = \frac{1}{4\pi}$

Ham. eq.: $\left.\begin{aligned}\dot\varphi_n &= \underbrace{\frac{1}{2\pi\varkappa}}_{2}\pi_n\\ \dot\pi_n &= -\underbrace{2\pi\varkappa}_{1/2} n^2\varphi_n\end{aligned}\right\}(*)$  $(\varphi_0, \pi_0)$ - free particle $(m = \frac{1}{2})$

$(\varphi_n, \pi_{-n})$ , $n \neq 0$ - (complex) harmonic oscillator with $\boxed{\omega_n = |n|}$

* Real oscillators: set $\varphi_n = \varphi_n^{(1)} + i\varphi_n^{(2)}$, $\pi_n = \frac{1}{2}(\pi_n^{(1)} + i\pi_n^{(2)})$ for $n > 0$

$\left(\varphi_n^{(1)} = \frac{\varphi_n + \varphi_{-n}}{2}, \quad \varphi_n^{(2)} = \frac{\varphi_n - \varphi_{-n}}{2i}\right)$

$\rightsquigarrow$ $\{\varphi_n^{(\alpha)}, \pi_m^{(\beta)}\} = -\delta_{n,-m}\delta_{\alpha\beta}$

$$H = \pi_0^2 + \sum_{n>0}\sum_{\alpha=1}^{2}\left(\frac{(\pi_n^{(\alpha)})^2}{2} + \frac{n^2}{2}(\varphi_n^{\alpha})^2\right) = H_{\substack{\text{free}\\ \text{particle}\\ m=\frac{1}{2}}} + \sum_{\alpha=1}^{2}\sum_{n>0} H_{\substack{\text{harmonic}\\ \text{oscillator},\\ \omega = n}}$$

Also: $P = \sum_{n\in\mathbb{Z}} in\,\pi_{-n}\varphi_n$

- Solution of $(*)$: $\varphi(\sigma,t) = \underbrace{\sum_{n\neq 0}\left(A_n e^{in(t+\sigma)} + B_n e^{in(-t+\sigma)}\right)}_{\varphi_n(t)e^{in\sigma}} + \underbrace{Ct + D}_{\varphi_0(t)}$ , $\pi(\sigma,t) = \cdots$

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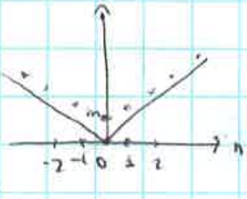
** massive scalar field:

$H = \sum_{n\in\mathbb{Z}}\left(\pi_n\pi_{-n} + \frac{1}{4}\underbrace{\omega_n^2}_{\substack{n^2+m^2\\ \uparrow\,\text{mass}}}\varphi_n\varphi_{-n}\right)$ ← collection of oscillators, one for each $n \in \mathbb{Z}$, $\omega_n = \sqrt{n^2+m^2}$



for $m \to 0$, $n=0$ oscillator becomes a free particle.

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## Canonical quantization

Promote $\varphi_n, \pi_n$ to operators $\hat{\varphi}_n, \hat{\pi}_n$ s.t. $[\hat{\pi}_n, \hat{\varphi}_m] = -i\delta_{n,-m}$ (we set $\hbar = 1$)

introduce creation/annihilation operators $a_n$, ~~$\bar{a}_n$~~ $\bar{a}_n$, $n \neq 0$:

$$(\hat{a}_n)^{+} = \hat{a}_{-n}$$
$$(\hat{\bar{a}}_n)^{+} = \hat{\bar{a}}_{-n}$$

$$\left.\begin{aligned} \hat{\varphi}_n &= \frac{i}{n}(-\hat{a}_{-n} + \hat{\bar{a}}_n) \\ \hat{\pi}_n &= \frac{\hat{a}_{-n} + \hat{\bar{a}}_n}{2} \end{aligned}\right\} \text{ with } \cancel{[\hat{a}_n, \hat{a}_m] = 0 \quad [\hat{\bar{a}}_n, \hat{\bar{a}}_m]} \quad \left.\begin{aligned} [\hat{a}_n, \hat{a}_m] &= n\delta_{n,-m} \\ [\hat{\bar{a}}_n, \hat{\bar{a}}_m] &= n\delta_{n,-m} \\ [\hat{a}_n, \hat{\bar{a}}_m] &= 0 \end{aligned}\right\} (*) , \quad \hat{H} = \sum_{n \neq 0} \frac{\hat{a}_{-n}\hat{a}_n + \hat{\bar{a}}_{-n}\hat{\bar{a}}_n}{2} + (\hat{\pi}_0)^2$$

may define $\hat{a}_0 = \hat{\bar{a}}_0 := \hat{\pi}_0$

then $\hat{H} = \frac{1}{2}\sum_{n \in \mathbb{Z}}(\hat{a}_{-n}\hat{a}_n + \hat{\bar{a}}_{-n}\hat{\bar{a}}_n)$, total momentum operator $\hat{P} = \frac{1}{2}\sum_{n \in \mathbb{Z}}(\hat{a}_{-n}\hat{a}_n - \hat{\bar{a}}_{-n}\hat{\bar{a}}_n)$

- Lie algebra $\underbrace{\mathrm{Span}_{\mathbb{C}}(\{\hat{a}_n\}_{n \in \mathbb{Z}}, \mathbb{K})}_{\text{"Heisenberg Lie algebra"}}$ with comm. rel. $[\hat{a}_n, \hat{a}_m] = n\delta_{n,-m}$, $[\hat{a}_n, \mathbb{K}] = 0$

$\cong$ central extension of the abelian Lie alg. of formal Laurent series $\{f(z) = \sum_{n \in \mathbb{Z}} \underbrace{f_n z^{-n}}_{\text{"}a_n\text{"}}\}$

with $[f, g]' = \mathbb{K} \cdot \underbrace{\mathrm{res}_{z=0}}_{\text{coeff. of } z^{-1}dz}(f\,dg)$

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we have

$$[\hat{H}, \hat{a}_n] = -n\hat{a}_n \qquad [\hat{P}, \hat{a}_n] = -n\hat{a}_n$$
$$[\hat{H}, \hat{\bar{a}}_n] = -n\hat{\bar{a}}_n \qquad [\hat{P}, \hat{\bar{a}}_n] = +n\hat{\bar{a}}_n$$

| for $n > 0$: | annihilation operator | creation operator |
|---|---|---|
| right mover | $\hat{a}_n$ | $\hat{a}_{-n}$ |
| left mover | $\hat{\bar{a}}_n$ | $\hat{\bar{a}}_{-n}$ |

**Space of states:** $\mathcal{H} = \mathcal{H}_{\text{free particle}} \otimes \bigotimes_{n \neq 0} \mathcal{H}_{\text{Harm. osc.}, \omega_n = |n|}$ $= \cancel{\mathrm{Span}_{\mathbb{C}}\{\prod_{n, \bar{n} > 0}(\hat{a}_{-n})^{k_n}(\hat{\bar{a}}_{-\bar{n}})^{k_{\bar{n}}}}$

$$= \mathrm{Span}\{\underbrace{\prod_{i=1}^{r}\hat{a}_{-n_i}\prod_{j=1}^{s}\hat{\bar{a}}_{-\bar{n}_j}|\pi_0\rangle}_{|\pi_0; \{n_i\}, \{\bar{n}_j\}\rangle} \mid \begin{aligned} 1 \le n_1 \le n_2 \le \dots \le n_r \\ 1 \le \bar{n}_1 \le \bar{n}_2 \le \dots \le \bar{n}_s \\ \pi_0 \in \mathbb{R} \end{aligned}\}$$

$\longleftarrow$ "$(r+s)$-particle state"

occupation numbers

$$= \mathrm{Span}\{\prod_{n \geq 1}(\hat{a}_{-n})^{k_n} \cdot \prod_{\bar{n} \geq 1}(\hat{\bar{a}}_{-\bar{n}})^{k_{\bar{n}}} |\pi_0\rangle\}$$

- normally-ordered operators $:\hat{H}:$, $:\hat{P}:$ — put annihilation operators $a_{>0}, \bar{a}_{>0}$ to the right, creation operators $a_{<0}, \bar{a}_{<0}$ to the left.

$:\cdots: \circlearrowleft$ Free AssocAlg$(\{a_n, \bar{a}_n\}_{n \in \mathbb{Z}})$

$\mathcal{O} \longmapsto :\mathcal{O}:$

We have:

$$:\hat{H}: |\pi_0; \{n_i\}, \{\bar{n}_j\}\rangle = (\pi_0^2 + \sum_i n_i + \sum_j \bar{n}_j)\,|\pi_0; \{n_i\}, \{n_j\}\rangle$$
$$:\hat{P}: |\pi_0; \{n_i\}, \{\bar{n}_j\}\rangle = (\sum_i n_i - \sum_j \bar{n}_j)\,|\pi_0; \{n_i\}, \{n_j\}\rangle$$