CFT EXERCISES, 4/5/2019

1. Using Ward identity to compute correlators of descendants

Recall that the general Ward identity has the form

$$\delta_{\epsilon(z)\frac{\partial}{\partial z}}\langle\Phi_1(z_1,\bar z_1)\cdots\Phi_n(z_n,\bar z_n)\rangle := \tag{1}$$
$$=\sum_{k=1}^n\langle\Phi_1(z_1,\bar z_1)\cdots\Big(\delta_{\epsilon(z)\frac{\partial}{\partial z}}\Phi_k(z_k,\bar z_k)\Big)\cdots\Phi_n(z_n,\bar z_n)\rangle\quad=0$$

for $\epsilon(z)\frac{\partial}{\partial z}$ any meromorphic vector field with poles allowed at $z_1,\ldots,z_n$. Fields $\Phi_1,\ldots,\Phi_n$ in (1) are not necessarily primary. Here the action of the vector field on a field Φ_k is defined via

$$\delta_{\epsilon(z)\frac{\partial}{\partial z}}\Phi_k(z_k,\bar z_k) := -\frac{1}{2\pi i}\oint_{C_{z_k}} dz\ \epsilon(z)T(z)\Phi_k(z_k,\bar z_k)$$

where the integral is over a closed simple contour C_{z_k} going around z_k once in counterclockwise direction.

(a) Specializing (1) to the case when Φ_k is primary, of conformal dimension $(h_k,\bar h_k)$, for each $1\le k\le n$, and the meromorphic vector field is $\epsilon(z)\frac{\partial}{\partial z}=-(z-z_1)^{-p+1}\frac{\partial}{\partial z}$ (with $p\geq 1$), obtain the relation

$$\langle\Big(L_{-p}\Phi_1\Big)(z_1,\bar z_1)\Phi_2(z_2,\bar z_2)\cdots\Phi_n(z_n,\bar z_n)\rangle=\mathcal{D}\ \langle\Phi_1(z_1,\bar z_1)\cdots\Phi_n(z_n,\bar z_n)\rangle \tag{2}$$

with $\mathcal{D}$ certain differential operator on functions of $z_2,\ldots,z_n$. Find $\mathcal{D}$ explicitly.

(b) Specialize (2) further, to the case $\Phi_1=\mathbf{1}$ the identity field and $p=2$. Obtain a relation of the form

$$\langle T(z_1)\Phi_2(z_2,\bar z_2)\cdots\Phi_n(z_n,\bar z_n)\rangle=\mathcal{D}\ \langle\Phi_1(z_1,\bar z_1)\cdots\Phi_n(z_n,\bar z_n)\rangle$$

Find the differential operator $\mathcal{D}$ explicitly.

2. Mutual locality of formal distributions

Consider the following two formal powers series ("formal distributions")

$$\alpha=\frac{1}{z}+\frac{w}{z^2}+\frac{w^2}{z^3}+\cdots,\quad \beta=-\frac{1}{w}-\frac{z}{w^2}-\frac{z^2}{w^3}-\cdots\qquad\in\mathbb{C}[[z,z^{-1},w,w^{-1}]]$$

(a) Show that α and β are "mutually local":

$$(z-w)\alpha=(z-w)\beta$$

(b) Show that α arises as Taylor expansion of $\frac{1}{z-w}$ in variable w (with fixed nonzero z), whereas β arises as Taylor expansion of the same function $\frac{1}{z-w}$ but in variable z.

3. A matrix element in free boson theory

For the free boson, prove that one has the matrix element

$$\begin{aligned}(3)\quad &\Big\langle \hat{a}_{-1}|\mathrm{vac}\rangle \ ,\ \mathcal{R}\Big(i\partial\hat{\phi}(z)\ i\partial\hat{\phi}(w)\Big)\ \hat{a}_{-1}|\mathrm{vac}\rangle\Big\rangle = \\ &= \langle\mathrm{vac}|\hat{a}_1\ \mathcal{R}\Big(i\partial\hat{\phi}(z)\ i\partial\hat{\phi}(w)\Big)\ \hat{a}_{-1}|\mathrm{vac}\rangle \quad = \frac{1}{(z-w)^2}+\frac{1}{z^2}+\frac{1}{w^2}\end{aligned}$$

Hint: show that $\lim_{x\to\infty} x^2\langle\mathrm{vac}|i\partial\hat{\phi}(x) = \langle\mathrm{vac}|\hat{a}_1$ (using the expansion of $i\partial\phi(x)$ in terms of creation/annihilation operators). Use this together with $\lim_{y\to 0} i\partial\hat{\phi}(y)|\mathrm{vac}\rangle = \hat{a}_{-1}|\mathrm{vac}\rangle$ to reduce the computation (3) to the following:

$$\langle\mathrm{vac}|\hat{a}_1\ \mathcal{R}\Big(i\partial\hat{\phi}(z)\ i\partial\hat{\phi}(w)\Big)\ \hat{a}_{-1}|\mathrm{vac}\rangle = \lim_{x\to\infty,y\to 0} x^2\langle\mathrm{vac}|\mathcal{R}\ \ i\partial\hat{\phi}(x)\ i\partial\hat{\phi}(z)\ i\partial\hat{\phi}(w)\ i\partial\hat{\phi}(y)|\mathrm{vac}\rangle$$

Here on the r.h.s. we have the 4-point correlation function in the free boson theory that we already know.

Calculate the Laurent expansion ζ of the r.h.s. of (3) in z (at $z=0$) and the expansion θ of (3) in w (at $w=0$). Show that the two resulting formal distributions are mutually local, i.e., satisfy

$$(z-w)^2\zeta = (z-w)^2\theta$$

For which values of z, w does the formal series ζ actually converge? Also, when does one have absolute convergence and when it is just a conditional convergence? What about θ - when does it converge?

4. Local Virasoro action on fields in the free boson theory

Find

$$L_{-p}\ \partial\phi(w)$$

for $p\geq 1$ explicitly (as certain normally ordered differential polynomials in ϕ at w). *Hint*: compute the OPE $T(z)\partial\phi(w)$ explicitly using Wick's lemma (including the regular terms and expanding fields at z in terms of fields at w) – then one can read off $L_{-p}\partial\phi$ as the coefficient of $(z-w)^{p-2}$ in this OPE.[1]

[1]The answer is:

$$L_{-p}\partial\phi(w) = \frac{1}{p!}\partial^{p+1}\phi(w) - \frac{1}{2}\sum_{k+l=p-2;\ k,l\geq 0}\frac{1}{k!\ l!} :\partial^{k+1}\phi(w)\ \partial^{l+1}\phi(w)\ \partial\phi(w):$$