CFT EXERCISES, 2/8/2019

1. Euler-Lagrange equations

Find

- the Euler-Lagrange equations,
- Noether 1-form (as a 1-form on TX),
- the Hamiltonian (as a function on TX)

in models of classical mechanics defined by the following action functionals:

(a) (Particle on $\mathbb{R}^n$ in a force potential.)

$$S[\mathbf{x}(t)] = \int_{t_0}^{t_1} dt \left(\frac{m|\dot{\mathbf{x}}|^2}{2} - U(\mathbf{x}) \right)$$

Here $\mathbf{x} \in \mathrm{Maps}([t_0, t_1], \mathbb{R}^n)$ is a path in $X = \mathbb{R}^n$ parameterized by the interval $[t_0, t_1]$; $U \in C^\infty(\mathbb{R}^n)$ is a smooth function (the "potential"), $m > 0$ a fixed "mass."

(b) (Particle on a Riemannian manifold.)

$$S[x(t)] = \int_{t_0}^{t_1} dt \frac{m(\dot{x}, \dot{x})_g}{2}$$

for $x : [t_0, t_1] \to X$ with (X, g) a Riemannian manifold. In particular, prove that the Euler-Lagrange equation is the equation of a geodesic on X with standard parametrization.

(c) (Relativistic particle.)

$$S[x(t)] = \int_{t_0}^{t_1} dt\, m\sqrt{(\dot{x}, \dot{x})_g}$$

with notations as in (b).

2. Integrals of motion

Construct (by Noether theorem) the integral of motion (Noether charge) associated to a symmetry. Check explicitly that the integral of motion is indeed independent of t if $\mathbf{x}(t)$ satisfies the Euler-Lagrange equation.

(a) (Spherically symmetric force potential.)

$$S[\mathbf{x}(t)] = \int_{t_0}^{t_1} dt \left(\frac{m|\dot{\mathbf{x}}|^2}{2} - U(\mathbf{x}) \right)$$

with $U(\mathbf{x}) = U(|\mathbf{x}|)$ an $O(n)$-invariant function on $\mathbf{R}^n$ (i.e. a function only of the distance to the origin). Find the integral of motion associated to the [target] symmetry

$$F_\lambda : \mathbf{x} \mapsto e^{\lambda M} \mathbf{x}$$

with $M \in \mathfrak{o}(n)$ a fixed skew-symmetric matrix (thus, the symmetry is a 1-parametric family of rotations).[1]

[1] For simplicity, first consider cases $n = 2, 3$ of this problem.

(b) (Mixed target/source scaling symmetry for a free particle.) Find the integral of motion for the free particle

$$S[\mathbf{x}(t)] = \int_{t_0}^{t_1} dt\, \frac{m|\dot{\mathbf{x}}|^2}{2}$$

associated to the mixed symmetry

$$\Big[\mathbf{x}(t)\Big]_{t_0}^{t_1} \mapsto \mathbf{x}'(t) = \Big[e^{\lambda/2}\cdot \mathbf{x}(e^{-\lambda}t)\Big]_{e^{\lambda}t_0}^{e^{\lambda}t_1}$$

3. Integral of motion in Hamiltonian mechanics

Consider the Hamiltonian system on the symplectic plane $\Phi = T^*\mathbb{R}$ with coordinates x, p (for the base/fiber) and standard symplectic structure $\omega = dx \wedge dp$, defined by the Hamiltonian function

$$H = \frac{p^2}{2m} + \mu x \qquad \in C^\infty(\Phi)$$

with $m > 0$ and μ a constant. Find the corresponding Hamiltonian vector field $\check{H}$ defined by $\iota_{\check{H}}\omega = dH$ and write explicitly Hamilton's equations of motion $\dot{x} = \cdots, \dot{p} = \cdots$.[2] Show that the family of symplectomorphisms

$$\Psi_\lambda : (x, p) \mapsto (x + \lambda, p)$$

is a symmetry (commutes with the flow of $\check{H}$) and is itself given by flow in time λ of some Hamiltonian vector field $\check{\psi}$ for certain $\psi \in C^\infty(\Phi)$. Find the corresponding integral of motion.[3]

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[2] Recall that Hamilton's equations of motion read $\dot{Y} = \{H, Y\}$ for Y any function (in particular, a coordinate function x or p) on Φ, with $\{f, g\} := L_{\check{f}}(g)$ the Poisson bracket.

[3] Observe that, if $\{H, \psi\} = C$ with C a constant, then $I(x, p; t) := \psi - Ct$ is an integral of motion, i.e. $I(x, p; t) = I(\mathrm{Flow}_{t'}(\check{H}) \circ (x, p); t + t')$.