CFT EXERCISES, 3/1/2019

1. Quantum harmonic oscillator

Assume that operators $\hat{a}, \hat{a}^+$ acting on a Hilbert space $\mathcal{H}$ are mutually Hermitian conjugate and satisfy the commutation relation

$$[\hat{a}, \hat{a}^+] = \mathrm{id} \tag{1}$$

Further, assume that there is a vacuum vector $|0\rangle \in \mathcal{H}$ of unit norm and satisfying $\hat{a}|0\rangle = 0$.

- Show that, for $n \geq 0$, the vector

$$(\hat{a}^+)^n|0\rangle \in \mathcal{H} \tag{2}$$

  has norm $\sqrt{n!}$. In other words, prove the equality

$$\langle 0|\hat{a}^n(\hat{a}^+)^n|0\rangle = n!$$

  (where in Dirac's notation multiplication with co-vector $\langle 0|$ one the left stands for evaluating the Hermitian pairing with $|0\rangle$).
- In Schrödinger representation, one identifies $\mathcal{H}$ with square-integrable complex-valued functions on the real line (*wavefunctions*), $\mathcal{H} = L^2(\mathbb{R}) = \{\psi(x)\}$. One identifies

$$\hat{a} = \frac{1}{\sqrt{2}}(x + \frac{d}{dx}), \qquad \hat{a}^+ = \frac{1}{\sqrt{2}}(x - \frac{d}{dx})$$

  Check that these operators are Hermitian conjugate to each other and satisfy (1). Find a wavefunction in $L^2(\mathbb{R})$ representing the vacuum vector. Is it uniquely characterized by its properties (being annihilated by $\hat{a}$ and being of unit norm)? Find the expression for the vector (2) in $L^2(\mathbb{R})$ in terms of Hermite polynomials $H_n(x) = (-1)^n e^{x^2} \frac{d}{dx^n} e^{-x^2}$.[1]
- The Hamiltonian of the harmonic oscillator is given by $\hat{H} = \hat{a}\hat{a}^+ + \hat{a}^+\hat{a}$. Check that it satisfies the commutation relations

$$[\hat{H}, \hat{a}^+] = \hat{a}^+, \qquad [\hat{H}, \hat{a}] = -\hat{a}$$

  Show that the vector (2) is an eigenvector of the $\hat{H}$ with the eigenvalue $n + \frac{1}{2}$.

2. Free boson propagator

The field inserted at a point $z \in \mathbb{C}\backslash\{0\}$ is represented (in the canonical quantization formalism) by the operator

$$\hat{\phi}(z, \bar{z}) = \hat{\phi}_0 - i\hat{\pi}_0 \log(z\bar{z}) + \sum_{n \neq 0} \frac{i}{n}(\hat{a}_n z^{-n} + \hat{\bar{a}}_n \bar{z}^{-n})$$

[1] Hint: notice that $\hat{a}^+$ can be written as a conjugation of the pure derivative: $\hat{a}^+ = -\frac{1}{\sqrt{2}} e^{\frac{x^2}{2}} \frac{d}{dx} e^{-\frac{x^2}{2}}$.

where operators $\{\hat{a}_n, \hat{\bar{a}}_n\}_{n\neq 0}, \hat{\phi}_0, \hat{\pi}_0$ satisfy the commutation relations

$$[\hat{a}_n, \hat{a}_m] = n\delta_{n,-m}, \quad [\hat{\bar{a}}_n, \hat{\bar{a}}_m] = n\delta_{n,-m}, \quad [\hat{a}_n, \hat{\bar{a}}_m] = 0, \quad [\hat{\pi}_0, \hat{\phi}_0] = -i$$

(and all other commutators vanish). These operators are represented on a Hilbert space (Fock space)

$$\mathcal{H} = \Big\{ \sum_{1\leq n_1\leq\cdots\leq n_r; 1\leq m_1\leq\cdots\leq m_s} \int d\underline{\pi}_0 \; \underbrace{\Psi_{n_1\cdots n_r; m_1\cdots m_s}(\underline{\pi}_0)}_{\text{wavefunction}} \; \underbrace{\hat{\bar{a}}_{-m_1}\cdots\hat{\bar{a}}_{-m_s}\hat{a}_{-n_r}\cdots\hat{a}_{-n_1}|\underline{\pi}_0\rangle}_{\text{basis vector } |n_1,\ldots,n_r;m_1,\ldots,m_s;\underline{\pi}_0\rangle} \Big\}$$

Here $|\underline{\pi}_0\rangle$, with $\underline{\pi}_0 \in \mathbb{R}$ is an eigenvector of $\hat{\pi}_0$ with eigenvalue $\underline{\pi}_0$, annihilated by all "annihilation operators" $\hat{a}_{>0}, \hat{\bar{a}}_{>0}$.

- We define the normal ordering $:\cdots:$ of a product of basic operators as a reordering that puts annihilation operators $\hat{a}_{>0}, \hat{\bar{a}}_{>0}$ to the right and creation operators $\hat{a}_{<0}, \hat{\bar{a}}_{<0}$ to the left (we also supplement this by the prescription that the normal ordering puts $\hat{\pi}_0$ to the right and $\hat{\phi}_0$ to the left). Prove that, for $z, w \in \mathbb{C}\backslash\{0\}$, $|z| > |w|$, one has
$$\hat{\phi}(z,\bar{z})\hat{\phi}(w,\bar{w}) - :\hat{\phi}(z,\bar{z})\hat{\phi}(w,\bar{w}): = -2\log|z-w|$$
- Show that, for $|z| > |w| > 0$, one has
$$\langle \mathrm{vac}|\partial\hat{\phi}(z,\bar{z})\partial\hat{\phi}(w,\bar{w})|\mathrm{vac}\rangle = -\frac{1}{(z-w)^2}$$
Here $|\mathrm{vac}\rangle$ is the state $|\underline{\pi}_0 = 0\rangle$ normalized to unit norm.
- Find the Hermitian inner product on $\mathcal{H}$ such that the the Hermitian conjugate of $\hat{a}_n$ is $\hat{a}_{-n}$, conjugate of $\hat{\bar{a}}_n$ is $\hat{\bar{a}}_{-n}$, and $\hat{\phi}_0$, $\hat{\pi}_0$ are self-adjoint.