CFT EXERCISES, 3/29/2019

1. State corresponding to the stress-energy tensor

In the free boson theory show that one has

$$\lim_{z\to 0} \hat{T}(z)|\mathrm{vac}\rangle = \frac{1}{2}\hat{a}_{-1}\hat{a}_{-1}|\mathrm{vac}\rangle \tag{1}$$

(Recall that $\hat{T}(z) = -\frac{1}{2} :\partial\hat{\phi}(z)\partial\hat{\phi}(z):$.)

Show that the state in the r.h.s. of (1) is $\hat{L}_{-2}|\mathrm{vac}\rangle$.

2. Vertex operators

Recall that the vertex operator in the free boson theory are defined as

$$\hat{V}_\alpha(z,\bar{z}) =: e^{i\alpha\hat{\phi}(z,\bar{z})} : \tag{2}$$

with $\alpha \in \mathbb{R}$ a parameter ("charge"). Show that:

(a) One has the OPE

$$\mathcal{R}\ i\partial\hat{\phi}(z)\ \hat{V}_\alpha(w,\bar{w}) \sim \frac{\alpha}{z-w}\hat{V}_\alpha(w,\bar{w}) + \mathrm{reg.}$$

(Hint: prove it by expanding the exponent in 2 in Taylor series and using the Wick's lemma. Recall that $\mathcal{R}\ \hat{\phi}(z,\bar{z})\hat{\phi}(w,\bar{w}) - :\hat{\phi}(z,\bar{z})\hat{\phi}(w,\bar{w}): = -2\log|z-w|$ and thus, by taking derivative in z, $\mathcal{R}\ \partial\hat{\phi}(z)\hat{\phi}(w,\bar{w}) - :\partial\hat{\phi}(z)\hat{\phi}(w,\bar{w}): = -\frac{1}{z-w}$.)

(b) One has the OPE

$$\mathcal{R}\ \hat{T}(z)\ \hat{V}_\alpha(w,\bar{w}) \sim \frac{\alpha^2/2}{(z-w)^2}\hat{V}_\alpha(w,\bar{w}) + \frac{\partial\hat{V}_\alpha(w,\bar{w})}{z-w} + \mathrm{reg.}$$

In particular, this together with the similar OPE with T replaced with $\bar{T}$ implies that V_α is, by definition, a primary field of conformal dimension $(h = \frac{\alpha^2}{2}, \bar{h} = \frac{\alpha^2}{2})$.

3. Correlators of primary fields

Assume that we have a function $F(z_1,\bar{z}_1,\ldots,z_n,\bar{z}_n) =$ "$\langle\Phi_1(z_1,\bar{z}_1)\cdots\Phi_n(z_n,\bar{z}_n)\rangle$" (the correlator) on the configuration space of n distinct points $z_1,\ldots,z_n$ on $\bar{\mathbb{C}} = \mathbb{C}\cup\{\infty\} = \mathbb{C}P^1$. Assume that for any Möbius transformation $f : \mathbb{C}P^1 \to \mathbb{C}P^1$ it satisfies the relation

(3)

$$F(f(z_1),\bar{f}(\bar{z}_1),\ldots,f(z_n),\bar{f}(\bar{z}_n)) \cdot \prod_{k=1}^{n} (\partial f(z_k))^{h_k} (\bar{\partial}\bar{f}(\bar{z}_k))^{\bar{h}_k} = F(z_1,\bar{z}_1,\ldots,z_n,\bar{z}_n)$$

for $h_1,\bar{h}_1,\ldots,h_n,\bar{h}_n$ some real numbers (conformal dimensions of the primary fields $\Phi_1,\ldots,\Phi_n$) with $h_k - \bar{h}_k \in \mathbb{Z}$.[1]

[1] Note that the bar in $\bar{h}_k$ does not stand for complex conjugation, it is just a notation for the anti-holomorphic conformal dimension. One often imposes the condition $h_k, \bar{h}_k > 0$ which

(a) Show that (3) implies the following differential equation ("Ward identity") for the correlator

(4)
$$\sum_{k=1}^{n}\left(\epsilon(z_k)\frac{\partial}{\partial z_k}+\bar{\epsilon}(z_k)\frac{\partial}{\partial \bar{z}_k}+h_k\cdot(\partial\epsilon)(z_k)\cdot+\bar{h}_k\cdot(\bar{\partial}\bar{\epsilon})(\bar{z}_k)\cdot\right)F(z_1,\bar{z}_1,\ldots,z_n,\bar{z}_n)=0$$

for $\epsilon(z)\frac{\partial}{\partial z}$ any holomorphic vector field on $\mathbb{C}P^1$.

(b) Consider the case $n=2$ of (3). Show that invariance (3) of the correlator under translations, rotations and scalings implies that

$$F(z_1,\bar{z}_1,z_2,\bar{z}_2)=\frac{C}{(z_1-z_2)^{h_1+h_2}(\bar{z}_1-\bar{z}_2)^{\bar{h}_1+\bar{h}_2}} \tag{5}$$

with C a constant.

(c) Show that invariance of the correlator for $n=2$ under the full Möbius group (i.e. adjoining special conformal transformations) imposes a further constraint on (5), that C must vanish unless $h_1=h_2$ and $\bar{h}_1=\bar{h}_2$.

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implies a physically meaningful statement $\lim_{z_k\to\infty}F(z_1,\bar{z}_1\ldots,z_n,\bar{z}_n)=0$ (no long-distance correlatation).