1/24/2020 | 1.5. Solution sets of linear systems

$A\vec{x} = \vec{0}$ - homogeneous system. $\vec{x} = \vec{0} \in \mathbb{R}^n$ always a solution - "trivial solution"

A: $m \times n$ matrix

Q: are there non-trivial solutions?

Answer: Homogeneous equation $A\vec{x}=\vec{0}$ has a non-trivial solution iff the eq. has at least one free variable

Ex 1

$$\begin{aligned} 2x_1 - x_2 + 3x_3 &= 0 \\ 2x_1 + 2x_2 + 3x_3 &= 0 \\ -6x_1 \quad - 9x_3 &= 0 \end{aligned}$$

Q: is there a nontrivial solution? describe the sol. set

Solution: Aug. Mat.

$$\underbrace{[A\ \vec{0}]}_{\text{coeff. mat.}} = \begin{bmatrix} 2 & -1 & 3 & 0 \\ 2 & 2 & 3 & 0 \\ -6 & 0 & -9 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 3 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -3 & 0 & 0 \end{bmatrix} \sim \begin{array}{c} \begin{bmatrix} 2 & -1 & 3 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \quad x_1 \quad x_2 \quad x_3 \qquad \end{array} \quad \text{REF}$$

x_3: free variable → there are nontriv. solutions

$$\sim \begin{bmatrix} 1 & 0 & \frac{3}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

RREF

$$\begin{aligned} x_1 \quad + \tfrac{3}{2}x_3 &= 0 \\ x_2 \quad &= 0 \\ 0 &= 0 \end{aligned} \Rightarrow \begin{aligned} x_1 &= -\tfrac{3}{2}x_3 \\ x_2 &= 0 \\ x_3 &\text{ free} \end{aligned}$$

solve for basic vars

Sol. in vector form

$$\Rightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2}x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \underbrace{\begin{bmatrix} -3/2 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v}} = x_3\vec{v}$$

$\Rightarrow$ Solution set is the line $\text{Span}\left\{\vec{v} = \begin{bmatrix} -3/2 \\ 0 \\ 1 \end{bmatrix}\right\}$ in $\mathbb{R}^3$

Note: a nontriv. solution can have some (not all!) zero entries

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Ex 2 $3x_1 + x_2 - 6x_3 = 0$ "system" of 1 eq. on 3 vars

Q: describe the solution set

Implicit description of the plane

Sol: x_1 basic x_2, x_3 free. solve for x_1: $x_1 = -\frac{1}{3}x_2 + 2x_3$, x_2 free, x_3 free

General solution: $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3}x_2 + 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} -1/3 \\ 1 \\ 0 \end{bmatrix}}_{\vec{u}} + x_3 \underbrace{\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v}}$

Explicit "parametric vector form" of the plane, $\vec{x} = s\vec{u} + t\vec{v}$, $s,t \in \mathbb{R}$ parameters

Thus, sol. set is $\text{Span}\{\vec{u}, \vec{v}\}$ - a plane in $\mathbb{R}^3$

- Sol. set to any homog. eq. $A\vec{x} = \vec{0}$ has the form $\text{Span}\{\vec{v}_1, \dots, \vec{v}_p\}$ for suitable vectors $\vec{v}_1, \dots, \vec{v}_p \in \mathbb{R}^n$.

  Special case: sol. set $= \text{Span}\{\vec{0}\} = \{\vec{0}\}$ - case when there are no non-triv. solutions

Solutions of non-homogeneous systems

Ex: Describe all solutions of $A\vec{x} = \vec{b}$ with

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 2 & 2 & 3 \\ -6 & 0 & -9 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}$$

matrix of coefficients from Ex 1.

Sol:
$$[A\,\vec{b}] = \left[\begin{array}{ccc|c} 2 & -1 & 3 & 3 \\ 2 & 2 & 3 & 0 \\ -6 & 0 & -9 & -6 \end{array}\right] \sim \begin{bmatrix} 1 & 0 & \frac{3}{2} & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x_1 + \tfrac{3}{2}x_3 &= 1 \\ x_2 &= -1 \\ 0 &= 0 \end{aligned} \Rightarrow \begin{cases} x_1 = 1 - \frac{3}{2}x_3 \\ x_2 = -1 \\ x_3 \text{ free} \end{cases}$$

$$\Rightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 - \frac{3}{2}x_3 \\ -1 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}_{\vec{p}} + x_3 \underbrace{\begin{bmatrix} -3/2 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v}} = \vec{p} + x_3\vec{v}$$

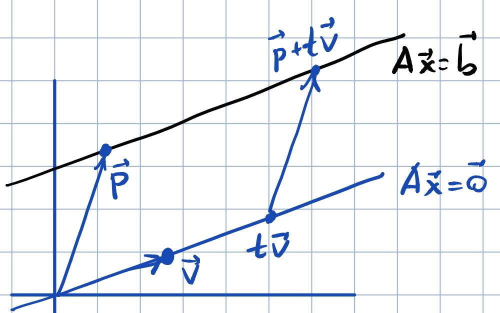
Sol. set in parametric form: $\vec{x} = \vec{p} + t\vec{v}$, $t \in \mathbb{R}$

$(= x_3)$

Recall: solution set of $A\vec{x} = \vec{0}$ was $\vec{x} = t\vec{v}$

add $\vec{p}$ to a generic sol. of $A\vec{x} = \vec{0}$.

same $\vec{v}$!



"addition = translation"

Solution set of homog. eq. $A\vec{x} = \vec{0}$ - a line through the origin

——— " ——— nonhomog. eq. $A\vec{x} = \vec{b}$ - a parallel line through $\vec{p}$

Theorem: Suppose the eq. $A\vec{x} = \vec{b}$ (*) is consistent for some given $\vec{b}$ and let $\vec{p}$ be a solution. Then the sol. set of $A\vec{x} = \vec{b}$ is the set of vectors of the form $\vec{x} = \vec{p} + \vec{v}_h$ where $\vec{v}_h$ is any sol of the homog. eq. $A\vec{x} = \vec{0}$.

Note: If (*) is inconsistent, the sol. set is empty. - Thm applies only to consistent eqs.

- Writing a sol. set in parametric form - the algorithm

1. Aug. Mat. $\sim$ RREF
2. solve for basic vars in terms of free vars [if any]
3. write a general sol. $\vec{x}$ as a vector w/ entries depending on free vars
4. decompose $\vec{x}$ into a lin. comb. of vectors (with numeric entries) with free vars as parameters.