2/10/2020 **3.1 Determinants**

Recall: $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is invertible iff $\underbrace{a_{11}a_{22} - a_{12}a_{21}}_{\det A} \neq 0$

$\det A$ - "determinant"

For a 3×3 matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\boxed{\begin{aligned} \det A := {} & a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ & - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} \end{aligned}}$$

terms: $\begin{bmatrix} \odot & \cdot & \cdot \\ \cdot & \odot & \cdot \\ \cdot & \cdot & \odot \end{bmatrix} + \begin{bmatrix} \cdot & \odot & \cdot \\ \cdot & \cdot & \odot \\ \odot & \cdot & \cdot \end{bmatrix} + \begin{bmatrix} \cdot & \cdot & \odot \\ \odot & \cdot & \cdot \\ \cdot & \odot & \cdot \end{bmatrix} - \begin{bmatrix} \odot & \cdot & \cdot \\ \cdot & \cdot & \odot \\ \cdot & \odot & \cdot \end{bmatrix} - \begin{bmatrix} \cdot & \odot & \cdot \\ \odot & \cdot & \cdot \\ \cdot & \cdot & \odot \end{bmatrix} - \begin{bmatrix} \cdot & \cdot & \odot \\ \cdot & \odot & \cdot \\ \odot & \cdot & \cdot \end{bmatrix}$

- A invertible iff $\det A \neq 0$

- $\det A = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$

$$= a_{11} \det \underbrace{\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}}_{A_{11}} - a_{12} \det \underbrace{\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}}_{A_{12}} + a_{13} \det \underbrace{\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}}_{A_{13}}$$

A_{11} - A with row 1 and column 1 deleted

$$= a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13}$$

- for A $n \times n$ square matrix, A_{ij} - submatrix formed by deleting row i and column j from A

Ex: $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$ $\quad A_{23} = \begin{bmatrix} 1 & 2 & 4 \\ 9 & 10 & 12 \\ 13 & 14 & 16 \end{bmatrix}$

- for a 1×1 matrix, $\det [a_{11}] = a_{11}$

- recursive definition of the determinant:

def: for $A = [a_{ij}]$ an $n \times n$ matrix, $n \geq 2$, the determinant is:

$$(*) \quad \det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \dots + (-1)^{1+n} a_{1n} \det A_{1n} = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}$$

<u>Ex:</u> $A = \begin{bmatrix} 2 & 3 & 0 \\ 5 & 1 & -7 \\ 0 & 0 & -2 \end{bmatrix}$ Q: compute $\det A$

<u>Sol:</u> $\det A = 2 \underbrace{\begin{vmatrix} 1 & -7 \\ 0 & -2 \end{vmatrix}}_{\det(\ldots)\text{ notation}} - 3 \begin{vmatrix} 5 & -7 \\ 0 & -2 \end{vmatrix} + 0 \cdot \begin{vmatrix} 5 & 1 \\ 0 & 0 \end{vmatrix} = 2\cdot(-2) - 3(-10) + 0\cdot 0 = 26$

· for $A = [a_{ij}]$, the (i,j)-cofactor of A is: $\boxed{C_{ij} = (-1)^{i+j} \det A_{ij}}$

By (*), we have: $\det A = a_{11}C_{11} + a_{12}C_{12} + \ldots + a_{1n}C_{1n}$ - "cofactor expansion across the first row of A"

<u>Thm</u> $\det A$ can be computed by a cofactor expansion across any row:

$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}$ (row i)

or down any column: $\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}$ (column j)

<u>Ex:</u> $A = \begin{bmatrix} 2 & 5 & 0 \\ 3 & 1 & 0 \\ 0 & -7 & -2 \end{bmatrix}$ Q: compute $\det A$ (by cofactor expansion)

<u>Sol:</u> cofactor expansion down 3^{rd} column:

$\det A = a_{31}\underbrace{C_{31}}_{\det A_{31}} + a_{32}\underbrace{C_{32}}_{-\det A_{32}} + a_{33}\underbrace{C_{33}}_{\det A_{33}} = 0 \cdot \begin{vmatrix} 3 & 1 \\ 0 & -7 \end{vmatrix} - 0 \cdot \begin{vmatrix} 2 & 5 \\ 0 & -7 \end{vmatrix} + (-2)\begin{vmatrix} 2 & 5 \\ 3 & 1 \end{vmatrix}$

$= (-2)(-13) = 26$

· It is useful to do a cofactor expansion across a row/col. containing <u>many zeros</u>!

<u>Ex:</u> $A = \begin{bmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}$

$\det A \underset{1^{st}\text{ col}}{=} 3 \begin{vmatrix} 2 & -5 & 7 & 3 \\ 0 & 1 & 5 & 0 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & -2 & 0 \end{vmatrix} + 0\cdot(\ldots) + 0\cdot(\ldots) + 0\cdot(\ldots) + 0\cdot(\ldots)$

$\underset{1^{st}\text{ col}}{=} 3\cdot 2 \begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix} \underset{3^{rd}\text{ row}}{=} 3\cdot 2\cdot \underbrace{(-1)}_{(-1)^{3+2}}\cdot(-2)\underbrace{\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix}}_{-1} = -12$

<u>Thm</u> if A is a triangular matrix, then $\det A$ is the product of diagonal entries of A

<u>Ex:</u> $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{vmatrix} = 1 \cdot \begin{vmatrix} 4 & 5 \\ 0 & 6 \end{vmatrix} = 1\cdot 4 \cdot |6| = 1\cdot 4\cdot 6$

Practice problem:

compute $\begin{vmatrix} 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 0 & 5 & 0 & -6 \end{vmatrix}$

3.2 Properties of determinants

How does det change under row operations?

THM Let A be a square matrix

(a) if $A \underset{\text{row replacement}}{\sim} B$, then $\det B = \det A$

(b) if $A \underset{\text{row interchange}}{\sim} B$, then $\det B = -\det A$

(c) if $A \underset{r_i \mapsto k \cdot r_i}{\sim} B$, then $\det B = k \cdot \det A$

Ex compute $\det A$, $A = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}$

Sol: $\det A = \begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix} \underset{\text{row replacements}}{=} \begin{vmatrix} 1 & 4 & -2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{vmatrix} \underset{r_2 \rightleftarrows r_3}{=} - \underbrace{\begin{vmatrix} 1 & 4 & -2 \\ 0 & 3 & 2 \\ 0 & 0 & -5 \end{vmatrix}}_{\text{triangular}} = -1 \cdot 3 \cdot (-5) = 15$

- Suppose A was reduced to REF U using only row replacements & interchanges (always possible!)

Then: $\det A = \underset{\#\text{interchanges}}{(-1)^r} \det U \underset{\text{since } U \text{ triangular}}{=} (-1)^r u_{11} u_{22} \cdots u_{nn}$

$U = \begin{bmatrix} \bullet & * & * & * \\ 0 & \bullet & * & * \\ 0 & 0 & \bullet & * \\ 0 & 0 & 0 & \bullet \end{bmatrix}$ A invertible $\Rightarrow$ $\det A = (-1)^r \cdot (\text{product of pivots})$

$U = \begin{bmatrix} \bullet & * & * & * \\ 0 & \bullet & * & * \\ 0 & 0 & 0 & \bullet \\ 0 & 0 & 0 & 0 \end{bmatrix}$ A non-invertible $\Rightarrow$ $\det A = 0$

Thm A square matrix A is invertible iff $\det A \neq 0$

- $\det A = 0 \iff$ columns of A are lin. dep. $\iff$ rows of A are lin. dep.

Ex: $\begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 2 & 3 & 4 \end{vmatrix} = 0$ since $r_3 = r_1 + r_2$

Ex: (Combining row reduction and cofactor expansion)

$$\begin{vmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{vmatrix} \underset{r_4 \mapsto r_4 + r_2}{=} \begin{vmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ 0 & 0 & -3 & 1 \end{vmatrix} \underset{\substack{\text{cofactor}\\\text{expansion}\\\text{down col. 1}}}{=} -2 \begin{vmatrix} 1 & 2 & -1 \\ 3 & 6 & 2 \\ 0 & -3 & 1 \end{vmatrix} \underset{r_2 \mapsto r_2 - 3r_1}{=} -2 \begin{vmatrix} 1 & 2 & -1 \\ 0 & 0 & 5 \\ 0 & -3 & 1 \end{vmatrix}$$

$$\underset{r_2 \leftrightarrow r_3}{=} 2 \underbrace{\begin{vmatrix} 1 & 2 & -1 \\ 0 & -3 & 1 \\ 0 & 0 & 5 \end{vmatrix}}_{\text{triangular}} = 2 \cdot 1 \cdot (-3) \cdot 5 = -30$$

Thm: $\det A^T = \det A$

Thm: $\det(AB) = (\det A)(\det B)$

Ex: $A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$ $\quad B = \begin{bmatrix} 2 & 5 \\ 0 & 1 \end{bmatrix}$ $\quad AB = \begin{bmatrix} 2 & 5 \\ 4 & 13 \end{bmatrix}$

$\det A = 3$ $\quad \det B = 2$ $\quad \det AB = 2 \cdot 13 - 5 \cdot 4 = 6 = 3 \cdot 2$ ✓

WARNING: $\det(A+B) \neq \det A + \det B$ generally

(E.g. in ex. above, $A + B = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$, $\det(A+B) = 2 \neq 3 + 2$)

- one can perform column operations on A, similar to row operations

  $\det A$ changes in the same way as for row op.

Linearity property of the determinant function

Suppose, j-th column of A can vary: $A = [\vec{a}_1 \cdots \vec{a}_{j-1}\ \vec{x}\ \vec{a}_{j+1} \cdots \vec{a}_n]$

Set $T: \mathbb{R}^n \to \mathbb{R}$

$\vec{x} \mapsto T(\vec{x}) = \det[\vec{a}_1 \cdots \vec{a}_{j-1}\ \vec{x}\ \vec{a}_{j+1} \cdots \vec{a}_n]$

Then: $T(c\vec{x}) = c\,T(\vec{x})$

$T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$

I.e., T is a lin. mapping.

Practice problem:

Compute $\begin{vmatrix} 1 & -3 & 1 & -2 \\ 2 & -5 & -1 & -2 \\ 0 & -4 & 5 & 1 \\ -3 & 10 & -6 & 8 \end{vmatrix}$

Sol: $\underset{\substack{r_2 \to r_2 - 2r_1 \\ r_4 \to r_4 + 3r_1}}{=} \begin{vmatrix} 1 & -3 & 1 & -2 \\ 0 & 1 & -3 & 2 \\ 0 & -4 & 5 & 1 \\ 0 & 1 & -3 & 2 \end{vmatrix} = 0$