2/17/2020 | Vector spaces and subspaces. Null spaces and column spaces.

4.1. Vector spaces and subspaces

def A vector space – a nonempty set V, objects – "vectors", with two operations:

* addition $\vec{u}+\vec{v} \in V$ for $\vec{u},\vec{v} \in V$
* multiplication by scalars $c\vec{u} \in V$ for $\vec{u} \in V$, $c \in \mathbb{R}$,

Such that:
- $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$,
- $\vec{u}+\vec{v} = \vec{v}+\vec{u}$,
- $c(\vec{u}+\vec{v}) = c\vec{u}+c\vec{v}$, $(c+d)\vec{u} = c\vec{u}+d\vec{u}$
- $c(d\vec{u}) = (cd)\vec{u}$
- $\exists$ zero-vector $\vec{0} \in V$ s.t. $\vec{u}+\vec{0}=\vec{u}$ $\forall \vec{u}$
- $\forall \vec{u}$ $\exists$ negative $-\vec{u}$ s.t. $\vec{u}+(-\vec{u})=\vec{0}$
- $1\cdot\vec{u}=\vec{u}$

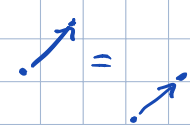
Corollaries: $\vec{0}$ is unique, $-\vec{u}$ is unique

$0\cdot\vec{u}=\vec{0}$, $c\vec{0}=\vec{0}$, $-\vec{u}=(-1)\vec{u}$.

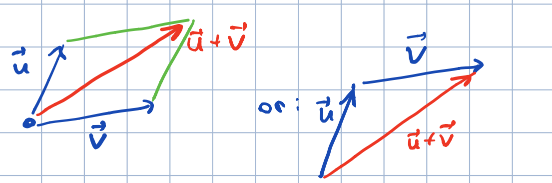
(Main example up to now)

Ex: spaces $\mathbb{R}^n$, $n \geq 1$

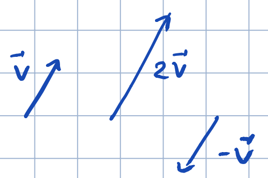
Ex: set of arrows (directed line segments) in $\mathbb{R}^2$. Two arrows are considered equal if they have same length and point in the same direction



addition: by parallelogram rule



Scalar multiplication:



Ex: $\mathbb{S}$ – space of all doubly-infinite sequences of numbers $y_k = \{\dots y_{-2}\, y_{-1}\, y_0\, y_1\, y_2 \dots\}$ – "discrete-time signals"

$\{y_k\}+\{z_k\}=\{y_k+z_k\}$, $c\{y_k\}=\{cy_k\}$

Ex: for $n \geq 0$, $\mathbb{P}_n$ – set of polynomials of degree $\leq n$, $\vec{p}(t) = a_0 + a_1 t + \dots + a_n t^n$ (*)

real coeffs

degree of $\vec{p}$ = highest power of t whose coeff in (*) is $\neq 0$.

for $\vec{p}(t) = a_0 \neq 0$, degree $= 0$. If all $a_j = 0$, $\vec{p} \equiv 0$ is called the zero polynomial (its degree is not defined)

Sum: for $\vec{q}(t) = b_0 + b_1 t + \dots + b_n t^n$,

$\vec{p}+\vec{q}$ is: $(\vec{p}+\vec{q})(t) = \vec{p}(t)+\vec{q}(t) = (a_0+b_0)+(a_1+b_1)t+\dots+(a_n+b_n)t^n$

$c\cdot\vec{p}$: $(c\vec{p})(t) = c\,\vec{p}(t) = ca_0 + (ca_1)t + \dots + (ca_n)t^n$

$\vec{0}$ = zero polynomial $\quad (-1)\vec{p} = -\vec{p}$ negative.

Ex: V = the set of all real-valued functions on a set $\mathbb{D}$

$(f+g)(t) = f(t) + g(t)$, $(cf)(t) = cf(t)$

e.g. $\mathbb{D} = \mathbb{R}$, $f = 1 + \sin 3t$, $g = 2 + 7t$

then: $(f+g)(t) = 3 + \sin 3t + 7t$, $(2g)(t) = 4 + 14t$

zero vector: $f(t) = 0 \quad \forall t$

- each function is a "point" of V

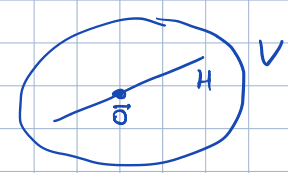
def A subspace of a vector space V is a subset $H \subset V$ satisfying:

(a) $\vec{0} \in H$

(b) $\vec{u} + \vec{v} \in H$ if $\vec{u}, \vec{v} \in H$ (H closed under addition)

(c) $c\vec{u} \in H$ if $\vec{u} \in H$, $c \in \mathbb{R}$ (H closed under scalar multiplication)

- A subspace $H \subset V$ is automatically a vector space



Ex: $H = \{\vec{0}\} \subset V$ is a subspace - zero subspace of V

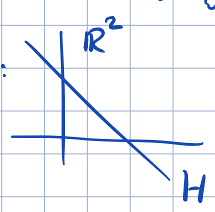
Ex: let $\mathbb{P}$ = all polynomials in t

- $\mathbb{P} \subset \{\text{functions on } \mathbb{R}\}$ - subspace
- $\mathbb{P}_n \subset \mathbb{P}$, $n \geq 0$ subspace

Ex: $\mathbb{R}^2$ is not a subspace of $\mathbb{R}^3$ (not even a subset!)

But: $H = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} \middle| \, s, t \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^3$ which "looks and acts" like $\mathbb{R}^2$

Ex:



a line not through $\vec{0}$ in $\mathbb{R}^2$ - not a subspace

- Subspace spanned by a set

THM: If $\vec{v}_1, \dots, \vec{v}_p$ are in V, then $H = \text{Span}\{\vec{v}_1, \dots, \vec{v}_p\}$ is a subspace of V

$$\text{Span}\{\vec{v}_1, \dots, \vec{v}_p\} = \{c_1\vec{v}_1 + \dots + c_p\vec{v}_p\}$$

H - subspace spanned/generated by $\{\vec{v}_1, \dots, \vec{v}_p\}$

$\{\vec{v}_1, \dots, \vec{v}_p\}$ - spanning/generating set for H.

Ex: $H = \{$ vectors of form $(a-3b, b-a, a, b) \mid a, b \in \mathbb{R}\}$ show that $H \underset{\text{subspace}}{\subset} \mathbb{R}^4$

Sol: $\begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} = a\underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}}_{\vec{v}_1} + b\underbrace{\begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v}_2}$. Thus $H = \text{Span}\{\vec{v}_1, \vec{v}_2\}$ - subspace of $\mathbb{R}^4$

4.2 | Null spaces, column spaces and linear transformations

<u>Recall</u>: for A $m\times n$ matrix, $\mathrm{Nul}\, A = \{\vec{x}\in\mathbb{R}^n \text{ s.t. } A\vec{x}=\vec{0}\}$ — subspace of $\mathbb{R}^n$

$\mathrm{Col}\, A = \mathrm{Span}\{\vec{a}_1,\dots,\vec{a}_n\} = \{\vec{b}\in\mathbb{R}^m \text{ s.t. } \vec{b}=A\vec{x} \text{ for some } \vec{x}\in\mathbb{R}^n\}$ — subspace of $\mathbb{R}^m$

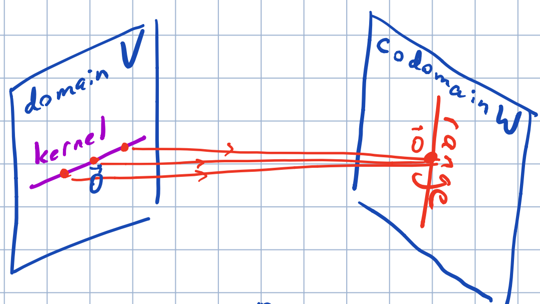
Can describe as $\mathrm{Span}\{\vec{v}_1,\dots,\vec{v}_p\}$ with $\vec{v}_1,\dots,\vec{v}_p$ from the parametric vector solution of $A\vec{x}=\vec{0}$.

<u>def</u> A lin. transf. T from a v. sp. V into a v.sp. W is a rule assigning to each vector $\vec{x}$ in V a vector $T(\vec{x})$ in W s.t.

(i) $T(\vec{u}+\vec{v}) = T(\vec{u})+T(\vec{v})$ any $\vec{u},\vec{v}$ (ii) $T(c\vec{u}) = c\,T(\vec{u})$

<u>kernel</u> (or null space) of T: set of all $\vec{u}$ in V s.t. $T(\vec{u})=\vec{0}$. ← subspace of V.

<u>range</u> of T: all vectors of form $T(\vec{x})$ in W ← subspace of W.



<u>Ex</u>: $T: V=\mathbb{R}^n \longrightarrow W=\mathbb{R}^m$ $\qquad \ker T = \mathrm{Nul}\, A$

$\vec{x} \longmapsto \underbrace{A}_{m\times n \text{ matrix}}\vec{x}$ $\qquad \text{range } T = \mathrm{Col}\, A$

<u>Ex</u>: V = functions on $[a,b]$ which have continuous derivatives
W = continuous functions on $[a,b]$

$D: V\to W$ linear transf. $\qquad \ker D = \{\text{constant functions on } [a,b]\}$

$f \mapsto f'$ $\qquad \text{range } D = W$.