4.2 | Null spaces, column spaces and linear transformations

Recall: for A $m\times n$ matrix,

$\text{Nul}\,A = \{\vec{x} \in \mathbb{R}^n \text{ s.t. } A\vec{x} = \vec{0}\}$ — subspace of $\mathbb{R}^n$

$\text{Col}\,A = \text{Span}\{\vec{a}_1, \dots, \vec{a}_n\} = \{\vec{b} \in \mathbb{R}^m \text{ s.t. } \vec{b} = A\vec{x} \text{ for some } \vec{x} \in \mathbb{R}^n\}$ — subspace of $\mathbb{R}^m$

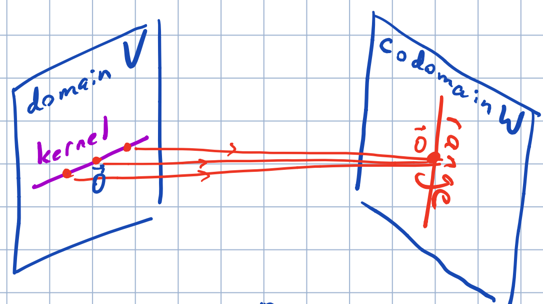
Can describe as $\text{Span}\{\vec{v}_1, \dots, \vec{v}_p\}$ with $\vec{v}_1, \dots, \vec{v}_p$ from the parametric vector solution of $A\vec{x} = \vec{0}$.

def A lin. transf. T from a v. sp. V into a v. sp. W is a rule assigning to each vector $\vec{x}$ in V a vector $T(\vec{x})$ in W s.t.

(i) $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ any $\vec{u}, \vec{v}$ (ii) $T(c\vec{u}) = c\,T(\vec{u})$

kernel (or null space) of T: set of all $\vec{u}$ in V s.t. $T(\vec{u}) = \vec{0}$. ← subspace of V.

range of T: all vectors of form $T(\vec{x})$ in W ← subspace of W.



Ex: $T: V = \mathbb{R}^n \longrightarrow W = \mathbb{R}^m$
$\vec{x} \longmapsto \underbrace{A}_{m\times n \text{ matrix}}\vec{x}$

$\ker T = \text{Nul}\,A$
$\text{range}\,T = \text{Col}\,A$

Ex: V = functions on $[a,b]$ which have continuous derivatives
W = continuous functions on $[a,b]$

$D: V \to W$ linear transf.
$f \mapsto f'$

$\ker D = \{\text{constant functions on } [a,b]\}$
$\text{range}\,D = W$.

4.3 Linearly independent sets, bases.

Vectors $\vec{v}_1, \ldots, \vec{v}_p$ in a vect. sp. V are linearly independent, iff the vect. eq. $\boxed{c_1\vec{v}_1 + \ldots + c_p\vec{v}_p = \vec{0}}$ $(*)$ has only the triv. sol. $c_1 = \ldots = c_p = 0$

otherwise, set $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is lin. dep. and $(*)$ (with not all weights zero) is a lin. dep. rel. among $\vec{v}_1, \ldots, \vec{v}_p$.

- as in $\mathbb{R}^n$: $\{\vec{v}\}$ is lin. indep. iff $\vec{v} \neq \vec{0}$

  $\{\vec{u}, \vec{v}\}$ lin. dep. iff $\vec{v} = c\vec{u}$ or $\vec{u} = d\vec{v}$.

  $\{\vec{0}, \vec{v}_1, \ldots, \vec{v}_p\}$ is lin. dep.

Thm Set $\{\vec{v}_1, \ldots, \vec{v}_p\}$ of $p \geq 2$ vectors in V with $\vec{v}_1 \neq \vec{0}$ is lin. dep. iff some $\vec{v}_j$ $(j > 1)$ is a lin. comb. of $\vec{v}_1, \ldots, \vec{v}_{j-1}$.

Note: for $V \neq \mathbb{R}^n$, $(*)$ cannot be cast as matrix eq. $A\vec{x} = \vec{0}$. We must rely on def. and Thm for lin. (in)dependence

Ex: $V = \mathbb{P}$ $\quad p_1(t) = 1 \quad p_2(t) = t \quad p_3(t) = 2 - 3t$

then $\{p_1, p_2, p_3\}$ is lin. dep. because $\boxed{p_3 = 2p_1 - 3p_2}$.

Ex: • Set $\{\sin t, \cos t\}$ is lin. indep. in $C[0,1]$ (space of continuous functions on $0 \leq t \leq 1$)

since there is no scalar c s.t. $\cos t = c \cdot \sin t$ for all $t \in [0,1]$

• Set $\{\sin t \cdot \cos t, \sin 2t\}$ is lin. dep. in $C[0,1]$ since $\boxed{\sin 2t = 2 \sin t \cos t}$ $\forall t$.

def Let H be a subspace of v.sp. V. A set of vectors $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_p\}$ in V is a basis for H if

(i) $\mathcal{B}$ is a lin. indep. set

(ii) $H = \text{Span}\{\vec{b}_1, \ldots, \vec{b}_p\}$

saw this before

Ex: let A - invertible $n \times n$ matrix, $A = [\vec{a}_1 \cdots \vec{a}_n]$. Then, columns of A form a basis for $\mathbb{R}^n$ - they are lin. indep. & span $\mathbb{R}^n$ by Inv. Mat. Thm.

Ex: Let $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \vec{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$ be columns of I_n

$\{\vec{e}_1, \ldots, \vec{e}_n\}$ - standard basis for $\mathbb{R}^n$

Ex: $\vec{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}$ $\vec{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix}$ Q: Is $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ a basis for $\mathbb{R}^3$?

Sol: $A = [\vec{v}_1 \vec{v}_2 \vec{v}_3] \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ So, A invertible $\Rightarrow$ YES

Ex: let $S = \{1, t, t^2, \dots, t^n\}$ S - basis for $\mathbb{P}_n$ - the standard basis.

Indeed: S spans $\mathbb{P}_n$. Linear independence: assume $c_0 \cdot 1 + c_1 \cdot t + \dots + c_n t^n = \underbrace{\vec{0}(t)}_{\text{zero polynomial}}$

then $c_0 = c_1 = \dots = c_n = 0$.

The spanning set theorem.

Ex: $\vec{v}_1 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$ $\vec{v}_3 = \begin{bmatrix} 6 \\ 16 \\ -5 \end{bmatrix}$, $H = \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

Note: $\vec{v}_3 = 5\vec{v}_1 + 3\vec{v}_2$ Q: (a) show that $H = \text{Span}\{\vec{v}_1, \vec{v}_2\}$

(b) find a basis for H.

Sol: (a) a vector in $\text{Span}\{\vec{v}_1, \vec{v}_2\}$ is $c_1\vec{v}_1 + c_2\vec{v}_2$ $(+0\,\vec{v}_3)$ - in H

a vector in H is $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\underbrace{\vec{v}_3}_{5\vec{v}_1 + 3\vec{v}_2} = (c_1 + 5c_3)\vec{v}_1 + (c_2 + 3c_3)\vec{v}_2$ - in $\text{Span}\{\vec{v}_1, \vec{v}_2\}$

(b) $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ - lin. indep. (not multiples of one another)

- span H, due to (a)

THM (spanning set theorem)

Let $S = \{\vec{v}_1, \dots, \vec{v}_p\}$ be a spanning set for H.

(a) if a vector $\vec{v}_k$ from S is a lin. comb. of other vectors in S, then the set formed from S by deleting $\vec{v}_k$ still spans H.

(b) if $H \neq \{\vec{0}\}$, some subset of S is a basis for H.

A basis for H is: - the smallest spanning set for H

can delete lin.dep. vectors from a spanning set for H. When we arrive to a lin. indep. subset, if we delete one more, the result will no longer span H!

- the largest lin. indep. set in H.

further shrinking

shrink to a basis

Ex: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \right\} \xrightarrow[\text{to a basis}]{\text{enlarge}} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\} \xrightarrow[\text{enlargement}]{\text{further}} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \right\}$

lin. indep. set, does not span $\mathbb{R}^3$ | basis in $\mathbb{R}^3$ | spans $\mathbb{R}^3$, lin. dep.