2/24/2020

4.5 Dimension of a vector space

Recall: for V - v.sp., $\mathcal{B}$ - basis with n vectors, coord. mapping $V \to \mathbb{R}^n$ - isomorphism

dimension (intrinsic property of V)

Thm*: If a v.sp. V has a basis $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$, then any set of $p > n$ vectors in V must be lin. dep.

[Idea: $\{[\vec{v}_1]_{\mathcal{B}}, \dots, [\vec{v}_p]_{\mathcal{B}}\}$ - set of $p > n$ vectors in $\mathbb{R}^n$ $\Rightarrow$ lin. dep. in $\mathbb{R}^n$
$\Rightarrow \{\vec{v}_1, \dots, \vec{v}_p\}$ lin. dep. in V]

Thm: If a v.sp. V has a basis of n vectors, then any basis for V has exactly n vectors.

[Idea: let $\underset{n \text{ vectors}}{\mathcal{B}_1}, \underset{p \text{ vectors}}{\mathcal{B}_2}$ two bases, $p > n$. By thm*, $\mathcal{B}_2$ is lin. dep $\Rightarrow$ Contradiction!]

Recall: if V is spanned by a finite set S, then a subset of S is a basis for V.

def If V is spanned by a finite set, then V is finite-dimensional

dimension $\dim V$ = number of vectors in a basis for V.

- $\dim\{\vec{0}\} = 0$ (convention)
- if V is not spanned by a finite set, then V is infinite-dimensional.

Ex:
- stand. basis for $\mathbb{R}^n$ consists of n vectors, so $\dim \mathbb{R}^n = n$
- for $\mathbb{P}_2$, $\{1, t, t^2\}$ - stand. basis, thus $\dim \mathbb{P}_2 = 3$. Generally, $\dim \mathbb{P}_n = n+1$
- $\mathbb{P}$ is infinite-dimensional

Ex $H = \text{Span}\left\{ \underset{\vec{v}_1}{\begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}} \right\}$ - plane in $\mathbb{R}^3$ $\quad \{\vec{v}_1, \vec{v}_2\}$ - basis for H

$\Rightarrow \dim H = 2$

Ex: $H = \left\{ \begin{bmatrix} a + 3b + 6c \\ 2a + 5d \\ 4b + 8c - d \\ 9d \end{bmatrix} \,\middle|\, a, b, c, d \in \mathbb{R} \right\}$ Find $\dim H$.

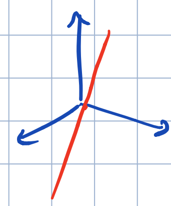
Sol: $H = \text{Span}\left\{ \underset{\vec{v}_1}{\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix} 3 \\ 0 \\ 4 \\ 0 \end{bmatrix}}, \underset{\vec{v}_3}{\begin{bmatrix} 6 \\ 0 \\ 8 \\ 0 \end{bmatrix}}, \underset{\vec{v}_4}{\begin{bmatrix} 0 \\ 5 \\ -1 \\ 9 \end{bmatrix}} \right\} \overset{\text{Spanning set thm}}{=} \text{Span} \underbrace{\{\vec{v}_1, \vec{v}_2, \vec{v}_4\}}_{\text{lin. ind. set}}$

note: $\vec{v}_3 = 2\vec{v}_2$

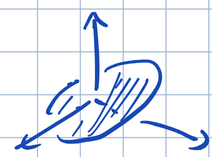
$\Rightarrow \{\vec{v}_1, \vec{v}_2, \vec{v}_4\}$ - basis for H $\Rightarrow$ $\dim H = 3$

<u>Ex:</u> subspaces of $\mathbb{R}^3$ classified by dimension

- 0 - dimensional $\{\vec{0}\}$
- 1 - dim. $H = \text{Span}\{\vec{v}\}$ ← nonzero - lines through $\vec{0}$



- 2 - dim. $H = \text{Span}\underbrace{\{\vec{u}, \vec{v}\}}_{\text{Lin. ind. set}}$ - planes through $\vec{0}$



- 3 - dim $\mathbb{R}^3$ itself

Subspaces of a fin. dim. space

<u>Thm**</u> Let H be a subspace of a fin. dim. v. sp. V.
Any lin. ind. set S can be expanded (if necessary) to a basis in H.
Also, H is fin. dim. and $\dim H \le \dim V$.

[<u>Idea:</u> $S = \{\vec{v}_1, \ldots, \vec{v}_p\}$ - lin. ind. set in H. If S spans H, we are done: S - basis.
($p \le \dim V$ by Thm*.)
If not, take $\vec{v}_{p+1}$ - some vector in $H \setminus \text{Span}\, S$
and adjoin to S. repeat until we span the entire H.]

"The basis thm": Let V be a p-dimensional v. sp., $p \ge 1$. Then:

1. any lin. indep. set of p vectors in V is a basis for V. (from thm **)
2. any spanning set of p vectors in V —— " —— (from spanning set thm)

<u>Recall:</u> $\dim \text{Col}\, A = \#$ pivot columns in A

$\dim \text{Nul}\, A = \#(\text{free var in } A\vec{x} = \vec{0}) = \#$ non-pivot columns in A.

Ex: $A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{REF}}$

$\text{Col}\, A \subset \mathbb{R}^3$
$\dim = 2$

$\text{Nul}\, A \subset \mathbb{R}^5$
$\dim = 3$

<u>Practice problem:</u> $H = \text{Span}\{\sin^2 t, \cos^2 t, 1\}$ in $C[0,1]$

(a) find a basis $\mathcal{B}$ for H $\dim H = ?$

(b) if $f = \cos 2t$ in H (c) if yes, find $[\cos 2t]_{\mathcal{B}}$

Sol: (a) $\vec{v}_1 = \sin^2 t$, $\vec{v}_2 = \cos^2 t$, $\vec{v}_3 = 1$ $\quad \vec{v}_1 + \vec{v}_2 = \vec{v}_3$

So, $H = \text{Span}\{\vec{v}_1, \vec{v}_2\} = \text{Span}\underbrace{\{\sin^2 t, \cos^2 t\}}_{\mathcal{B}}$

$\mathcal{B}$ - lin. indep. set

$\dim H = 2$

(b) $\cos 2t = \cos^2 t - \sin^2 t \in H$

(c) $[\cos 2t]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$