2/26/2020 | 4.6 Rank

Row space A $m\times n$ matrix. Rows have n entries; each row is a vector in $\mathbb{R}^n$.

Def. row space, $\text{Row } A = \text{Span}\{\text{rows of } A\}$ - subspace of $\mathbb{R}^n$.

$$\boxed{\text{Row } A = \text{Col } A^T}$$

Ex: $A = \begin{bmatrix} 1 & -2 & 3 & 1 \\ 2 & -4 & 7 & 7 \\ 3 & -6 & 8 & -2 \end{bmatrix}$ $\begin{matrix} \leftarrow \vec{r}_1 = (1,-2,3,1) \\ \leftarrow \vec{r}_2 = (2,-4,7,7) \\ \leftarrow \vec{r}_3 = (3,-6,8,-2) \end{matrix}$ $\text{Row } A = \text{Span}\{\vec{r}_1, \vec{r}_2, \vec{r}_3\} \subset \mathbb{R}^4$

Warning: row operations change lin. dependence relations of rows!
$\Rightarrow$ cannot figure out which rows to exclude from REF.

THM: If $A \sim B$ then $\text{Row } A = \text{Row } B$.
If B is in REF, then nonzero rows of B form a basis for $\text{Row } B = \text{Row } A$.

Ex: $A \sim \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B$ basis for $\text{Row } A = \{(1,-2,3,1), (0,0,1,5)\}$

basis for $\text{Col } A = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 8 \end{bmatrix} \right\}$
pivot columns of A

basis for $\text{Nul } A$: $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 14 \\ 0 \\ -5 \\ 1 \end{bmatrix} \right\}$

from RREF $\rightarrow$ param. vector solution of homog. eq.

Recall: $\text{rank } A = \dim \text{Col } A = \#\text{pivots}$ $= \dim \text{Row } A$
$= \dim \text{Col } A^T$

Note: $\text{rank } A = \text{rank } A^T$

Rank theorem: for A $m\times n$ matrix, $\boxed{\text{rank } A + \dim \text{Nul } A = n}$

Ex: Can a 3×7 matrix have a 2-dimensional null space?

Sol: $\underbrace{\text{rank } A}_{\leq 3} + \dim \text{Nul } A = 7$ $\Rightarrow$ NO!

Ex: A 40×42, $\dim \text{Nul } A = 2$. Q: Is it true that $A\vec{x} = \vec{b}$ has a sol. for any $\vec{b} \in \mathbb{R}^{40}$?

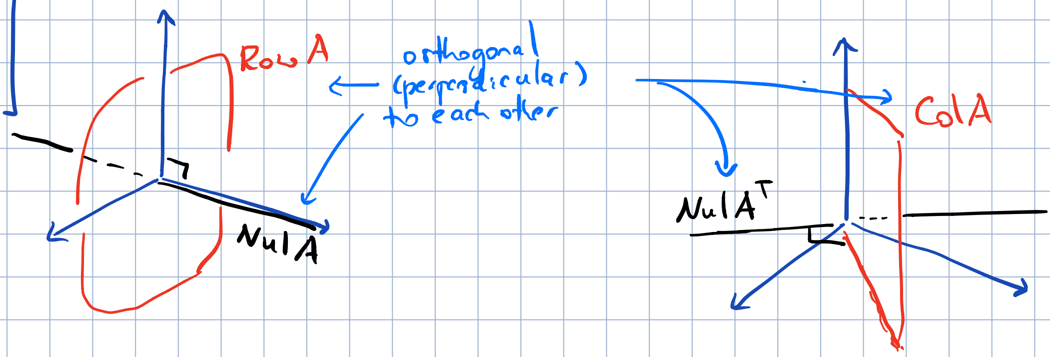
Sol: $\text{rank } A = 42 - \dim \text{Nul } A = 40$ $\Rightarrow$ $\text{Col } A$ is a 40-dim. subspace in $\mathbb{R}^{40}$ $\Rightarrow \text{Col } A = \text{entire } \mathbb{R}^{40} \Rightarrow$ YES!

<u>Ex:</u> A 5×7 $\dim \mathrm{Nul}\, A = 4$. Q: $\dim \mathrm{Nul}\, A^T = ?$

Sol: $\mathrm{rank} = 7 - 4 = 3$. rank thm for $A^T \Rightarrow 3 + \dim \mathrm{Nul}\, A^T = 5$

$\Rightarrow \dim \mathrm{Nul}\, A^T = 2$

Ex: $A = \begin{bmatrix} 3 & 0 & -1 \\ 3 & 0 & -1 \\ 4 & 0 & 5 \end{bmatrix}$



4.7 Change of bases

<u>Ex:</u> V - v.sp. with two bases $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$, $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$ s.t.

$\vec{b}_1 = 4\vec{c}_1 + \vec{c}_2$, $\vec{b}_2 = -6\vec{c}_1 + \vec{c}_2$. Suppose $\vec{x} = 3\vec{b}_1 + \vec{b}_2$ $(*)$, i.e. $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

Q: find $[\vec{x}]_{\mathcal{C}}$

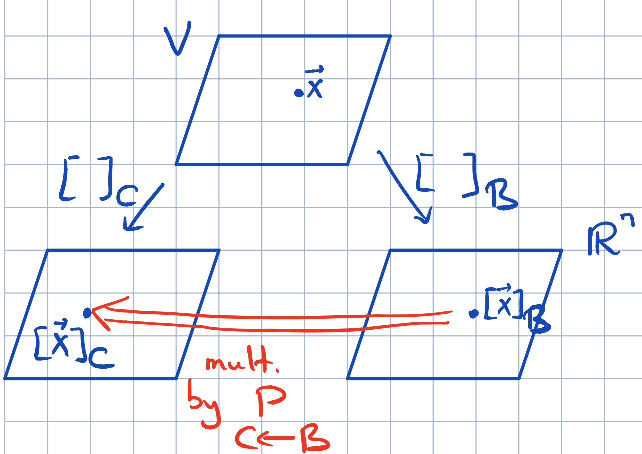
<u>Sol:</u> Apply coord. mapping defined by $\mathcal{C}$ to $(*)$:

$[\vec{x}]_{\mathcal{C}} = 3[\vec{b}_1]_{\mathcal{C}} + [\vec{b}_2]_{\mathcal{C}}$ i.e. $[\vec{x}]_{\mathcal{C}} = \boxed{\left[[\vec{b}_1]_{\mathcal{C}} \; [\vec{b}_2]_{\mathcal{C}} \right]} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix}} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$

<u>Thm</u> Let $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$, $\mathcal{C} = \{\vec{c}_1, \dots, \vec{c}_n\}$ be bases in V.

Then there is a unique $n\times n$ mat. $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ s.t. $[\vec{x}]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} [\vec{x}]_{\mathcal{B}}$ $(**)$

Explicitly: $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \left[[\vec{b}_1]_{\mathcal{C}} \cdots [\vec{b}_n]_{\mathcal{C}} \right]$ - change-of-coordinate matrix from $\mathcal{B}$ to $\mathcal{C}$



Also, $(**)$ implies

$\left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right)^{-1} [\vec{x}]_{\mathcal{C}} = [\vec{x}]_{\mathcal{B}}$

hence: $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right)^{-1}$

Change of basis in $\mathbb{R}^n$

Recall: if $\mathcal{B}=\{\vec{b}_1,\dots,\vec{b}_n\}$, $\mathcal{E}=\{\vec{e}_1,\dots,\vec{e}_n\}$ stand. basis in $\mathbb{R}^n$, then $[\vec{b}_i]_{\mathcal{E}}=\vec{b}_i$

and $\underset{\mathcal{E}\leftarrow\mathcal{B}}{P} = P_{\mathcal{B}} = [\vec{b}_1 \cdots \vec{b}_n]$

Change between two nonstandard bases in $\mathbb{R}^n$:

Ex: $\underbrace{\vec{b}_1=\begin{bmatrix}-9\\1\end{bmatrix} \quad \vec{b}_2=\begin{bmatrix}-5\\-1\end{bmatrix}}_{\mathcal{B}}$; $\underbrace{\vec{c}_1=\begin{bmatrix}1\\-4\end{bmatrix} \quad \vec{c}_2=\begin{bmatrix}3\\-5\end{bmatrix}}_{\mathcal{C}}$ two bases in $\mathbb{R}^2$

Q: find $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$

Sol: we need $[\vec{b}_1]_{\mathcal{C}}=\begin{bmatrix}x_1\\x_2\end{bmatrix}$, $[\vec{b}_2]_{\mathcal{C}}=\begin{bmatrix}y_1\\y_2\end{bmatrix}$.

By def., $[\vec{c}_1\, \vec{c}_2]\begin{bmatrix}x_1\\x_2\end{bmatrix}=\vec{b}_1$, $[\vec{c}_1\, \vec{c}_2]\begin{bmatrix}y_1\\y_2\end{bmatrix}=\vec{b}_2$.

To solve two systems simultaneously, augment the coeff. mat. with $\vec{b}_1$ and $\vec{b}_2$:

$$[\vec{c}_1\ \vec{c}_2 \mid \vec{b}_1\ \vec{b}_2]=\left[\begin{array}{cc|cc}1&3&-9&-5\\-4&-5&1&-1\end{array}\right]\sim\left[\begin{array}{cc|cc}1&0&6&4\\0&1&-5&-3\end{array}\right]$$

thus: $[\vec{b}_1]_{\mathcal{C}}=\begin{bmatrix}6\\-5\end{bmatrix}$, $[\vec{b}_2]_{\mathcal{C}}=\begin{bmatrix}4\\-3\end{bmatrix}$ and $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}=\begin{bmatrix}6&4\\-5&-3\end{bmatrix}$

Observe $[\vec{c}_1\ \vec{c}_2 \mid \vec{b}_1\ \vec{b}_2] \sim [\,I \mid \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\,]$ ← works analogously for any two bases in $\mathbb{R}^n$

Another description of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$:

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \underset{\mathcal{C}\leftarrow\mathcal{E}}{P}\cdot\underset{\mathcal{E}\leftarrow\mathcal{B}}{P} = (P_{\mathcal{C}})^{-1}P_{\mathcal{B}}$$

or: $\begin{cases}\vec{x}=P_{\mathcal{B}}[\vec{x}]_{\mathcal{B}}\\ \vec{x}=P_{\mathcal{C}}[\vec{x}]_{\mathcal{C}} \Rightarrow [\vec{x}]_{\mathcal{C}}=P_{\mathcal{C}}^{-1}\vec{x}\end{cases}$

$\Rightarrow [\vec{x}]_{\mathcal{C}}=\underbrace{(P_{\mathcal{C}})^{-1}P_{\mathcal{B}}}_{\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}}[\vec{x}]_{\mathcal{B}}$