2/28/2020 | 5.1 Eigenvectors and eigenvalues

def An eigenvector of an $n\times n$ matrix A is a nonzero vector $\vec{x}$ s.t. $\boxed{A\vec{x}=\lambda\vec{x}}$

some scalar

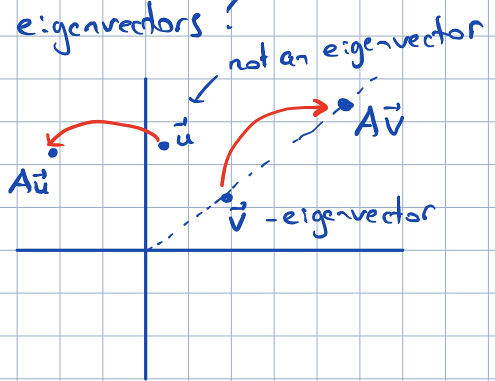
A scalar λ is called an eigenvalue of A if $A\vec{x}=\lambda\vec{x}$ has a nontriv. solution $\vec{x}$.

Such $\vec{x}$ is called an eigenvector corresponding to λ.

Ex*: $A=\begin{bmatrix}1&6\\5&2\end{bmatrix}$ $\vec{u}=\begin{bmatrix}6\\-5\end{bmatrix}$ $\vec{v}=\begin{bmatrix}3\\-2\end{bmatrix}$ Q: are $\vec{u},\vec{v}$ eigenvectors?

Sol: $A\vec{u}=\begin{bmatrix}-30\\20\end{bmatrix}=(-4)\vec{u} \Rightarrow$ $\vec{u}$ - eigenvector with $\lambda=-4$ eigenvalue

$A\vec{v}=\begin{bmatrix}-9\\11\end{bmatrix}\neq\lambda\vec{v} \Rightarrow$ $\vec{v}$ not an eigenvector!



Ex: Show that $\lambda=7$ is an eigenvalue for A; find corresponding eigenvectors.

Sol $\lambda=7$ is an e-value iff $A\vec{x}=7\vec{x}$ has a nontriv. sol. $\Leftrightarrow A\vec{x}-7\vec{x}=\vec{0}$

$\Leftrightarrow \boxed{(A-7I)\vec{x}=\vec{0}}$

$A-7I=\begin{bmatrix}1&6\\5&2\end{bmatrix}-\begin{bmatrix}7&0\\0&7\end{bmatrix}=\begin{bmatrix}-6&6\\5&-5\end{bmatrix}$ ← columns are lin.dep. $\Rightarrow$ there are nontriv. sol. to homog. eq.

$\Rightarrow \lambda=7$ is an eigenvalue!

Aug. Mat: $\begin{bmatrix}-6&6&0\\5&-5&0\end{bmatrix}\sim\begin{bmatrix}1&-1&0\\0&0&0\end{bmatrix}$ general sol: $\vec{x}=x_2\begin{bmatrix}1\\1\end{bmatrix}$ each such vector with $x_2\neq 0$ is an eigenvector for $\lambda=7$

WARNING: We used row reduction to find eigenvectors but it cannot be used to find eigenvalues. REF A does not display the eigenvalues of A!

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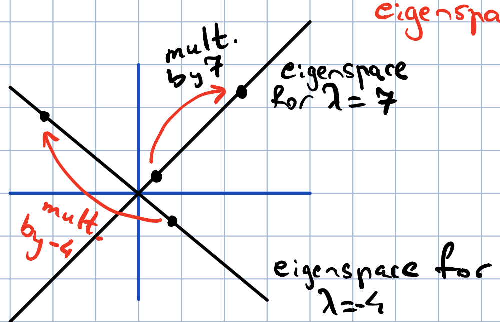
for A $n\times n$, λ is an e.v. iff $\boxed{(A-\lambda I)\vec{x}=\vec{0}}$ (**) has a nontriv. sol.

set of sol. of (**) $=$ $\underbrace{Nul(A-\lambda I)}_{\text{eigenspace of } A \text{ corresp. to } \lambda}\subset\mathbb{R}^n$ $=\vec{0}\cup\{\text{all eigenvectors for }\lambda\}$

In Ex*:



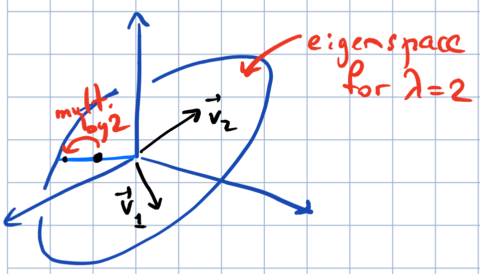
Ex: $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$, $\lambda = 2$ find a basis for the eigenspace H.

Sol: $A - 2I = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}$ Aug. mat. for $(**)$: $\begin{bmatrix} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{1}{2} & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$x_1 = \frac{1}{2}x_2 - 3x_3$
x_2, x_3 free

$$\vec{x} = x_2 \underbrace{\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}}_{\vec{v}_1} + x_3 \underbrace{\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v}_2}$$

Thus, H - plane in $\mathbb{R}^3$,
$\{\vec{v}_1, \vec{v}_2\}$ - basis.



THM Eigenvalues of a triangular matrix are the diagonal entries

Idea: $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$ $A - \lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ 0 & a_{22}-\lambda & a_{23} \\ 0 & 0 & a_{33}-\lambda \end{bmatrix}$

λ e.v. $\Leftrightarrow$ $\det(\quad) = 0$ $\Leftrightarrow$ $\lambda \in \{a_{11}, a_{22}, a_{33}\}$

$\quad \| $
$(a_{11}-\lambda)(a_{22}-\lambda)(a_{33}-\lambda)$ ☐

Ex: $A = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 0 & 0 \\ -2 & -1 & 2 \end{bmatrix}$ lower-triang.
$\lambda = 3, 0, 2$

Note: $\lambda = 0$ is an eigenvalue $\Leftrightarrow$ $A\vec{x} = \vec{0}$ has a nontriv. sol. $\Leftrightarrow$ A non-invertible

Thm: If $\{\vec{v}_1, \dots, \vec{v}_r\}$ eigenvectors that correspond to distinct eigenvalues $\lambda_1, \dots, \lambda_r$ of a $n \times n$ mat. A, then $\{\vec{v}_1, \dots, \vec{v}_r\}$ is a lin. indep. set.

5.2 Characteristic equation

Ex: find the eigenvalues of $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$

Sol: λ e.v. $\Leftrightarrow$ $(A - \lambda I)\vec{x} = \vec{0}$ has a nontriv. sol. $\Leftrightarrow$ $A - \lambda I$ non-invertible

$\Leftrightarrow \boxed{\det(A - \lambda I) = 0}$

$A - \lambda I = \begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix}$ $\det = (2-\lambda)(-6-\lambda) - 3 \cdot 3 = \lambda^2 + 4\lambda - 21 = (\lambda+7)(\lambda-3)$

thus $\det = 0$ iff $\lambda \in \{3, -7\}$

So, $\lambda = 3$, $\lambda = -7$ - eigenvalues.

Inv. Mat. Thm (Cont'd): A $n \times n$ matrix is invertible iff

(s) 0 is not an eigenvalue of A (t) $\det A \neq 0$

- λ is an eigenvalue of A iff λ satisfies the characteristic equation $\boxed{\det(A-\lambda I)=0}$

Ex: $A=\begin{bmatrix}3&1&2\\0&1&5\\0&0&3\end{bmatrix}$ Q: find the char. eq. $\det(A-\lambda I)=\begin{vmatrix}3-\lambda&1&2\\0&1-\lambda&5\\0&0&3-\lambda\end{vmatrix}=-(\lambda-3)^2(\lambda-1)$

So, char. eq: $\underbrace{-(\lambda-3)^2(\lambda-1)}_{\text{char. polynomial of } A}=0$

Note: $\lambda=3$ - e.v. with (algebraic) multiplicity 2 (multiplicity as a root of char. eq.)

Ex: A 6×6 char. poly $=\lambda^6-4\lambda^5-12\lambda^4$. Q: find eigenvalues & their multiplicities

Sol: char. poly $=\lambda^4(\lambda-6)(\lambda+2)$. So, e.v. are: $\lambda=0$ (mult. 4)
$\lambda=6$ (mult. 1)
$\lambda=-2$ (mult. 1)

- for A $n\times n$, char. eq. has n roots (counting with multiplicities). Some of them can be complex.

Similarity def. A is similar to B if there is an invertible P s.t.

$$P^{-1}AP=B \quad (\text{or equivalently } A=PBP^{-1})$$

- $A\to P^{-1}AP$ "similarity transformation" Note: $\underset{\text{similar}}{A\approx B} \Rightarrow B\approx A$

THM If A and B are similar, they have the same char poly and hence same e.v. (with same multiplicities)

WARNING 1. $\begin{bmatrix}2&1\\0&2\end{bmatrix}\not\approx\begin{bmatrix}2&0\\0&2\end{bmatrix}$ same eigenvalues but not similar

2. similarity is not the same as row equivalence.
row operations change eigenvalues.