3/2/2020

LAST TIME: for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\underset{\nearrow \text{ inverse matrix}}{A^{-1}} = \dfrac{1}{\boxed{ad - bc}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$\nwarrow$ det A - "determinant of A"

Thm (a) An $n \times n$ matrix A is invertible iff $A \sim I_n$ ← argument for (a): A invertible iff $A\vec{x} = \vec{b}$ has a sol. for each $\vec{b}$ ⇒ pivot in every row; sol. is unique ⇒ pivot in each column

(b) In this case, any sequence of row operations that reduces A to I_n also transforms I_n to A^{-1}.

↖ argument - via elem. matrices (p.3)

Algorithm for finding A^{-1}

Row reduce the "augmented matrix" $[A \,\vdots\, I]$ - $n \times 2n$ matrix

if A is invertible, RREF is: $[I \,\vdots\, A^{-1}]$.

Ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$[A \,\vdots\, I] = \left[\begin{array}{cc:cc} \underbrace{\begin{matrix} 1 & 2 \\ 3 & 4 \end{matrix}}_{A} & \underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{I} \end{array}\right] \sim \left[\begin{array}{cc:cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right] \sim \left[\begin{array}{cc:cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right]$$

$$\sim \left[\begin{array}{cc:cc} \underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{I} & \underbrace{\begin{matrix} -2 & 1 \\ 3/2 & -1/2 \end{matrix}}_{A^{-1}} \end{array}\right]$$

2.3 Characterizations of invertible matrices

Thm ("the invertible matrix theorem")

Let A be a square, $n \times n$, matrix. The following are equivalent:

(a) A is invertible

(b) $A \sim I_n$

(c)* A has n pivots ← most useful

(d) $A\vec{x} = \vec{0}$ has only the trivial solution

(e) columns of A are lin. indep.

(f) lin. transf. $\vec{x} \longmapsto A\vec{x}$ is 1-1.

(g) $A\vec{x} = \vec{b}$ has a sol. for each $\vec{b} \in \mathbb{R}^n$ $\rightsquigarrow$ (g') $A\vec{x} = \vec{b}$ has a unique sol. $\forall \vec{b} \in \mathbb{R}^n$

(h) columns of A span $\mathbb{R}^n$

(i) lin. transf. $\vec{x} \longmapsto A\vec{x}$ is onto $\mathbb{R}^n$

(j) there is C s.t. $CA = I$

(k) there is D s.t. $AD = I$

(l) A^T invertible.

If A, B $n \times n$ matrices and $AB = I$, then
A, B are both invertible, with $B = A^{-1}$, $A = B^{-1}$

Ex: $A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix}$ invertible?

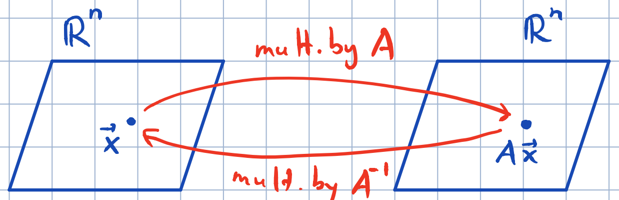
Sol: $A \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} \enclose{circle}{1} & 0 & -2 \\ 0 & \enclose{circle}{1} & 4 \\ 0 & 0 & \enclose{circle}{3} \end{bmatrix}$

$\underset{\text{(c) of THM}}{\Longrightarrow}$ A invertible

Invertible linear transformations

for A invertible, we have

$A^{-1} A \vec{x} = \vec{x}$



- transforms $A\vec{x}$ back into $\vec{x}$

- a lin. transf. $T: \mathbb{R}^n \to \mathbb{R}^n$ is <u>invertible</u> iff there exists $S: \mathbb{R}^n \to \mathbb{R}^n$ s.t.
  $S(T(\vec{x})) = \vec{x}$ for all $\vec{x} \in \mathbb{R}^n$
  $T(S(\vec{x})) = \vec{x}$ ——— " ———

- if such S exists, it is <u>unique</u> and is "the inverse of T", $S = T^{-1}$
- if T has stand. mat. A, then T is invertible iff A invertible.
  In this case, S is given by $S(\vec{x}) = A^{-1}\vec{x}$

<u>Ex:</u> let $T: \mathbb{R}^n \to \mathbb{R}^n$ one-to-one. By Thm (f), A is invertible $\Rightarrow$ T invertible

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<u>Practice questions:</u>

Ⓘ $A = \begin{bmatrix} 2 & 3 & 0 \\ 3 & 5 & 0 \\ -4 & -6 & 1 \end{bmatrix}$ is it invertible? If yes, find the inverse

Ⅱ Is $\begin{bmatrix} 1 & 2 & 7 \\ 1 & 2 & 7 \\ 1 & 2 & 7 \end{bmatrix}$ invertible?

Is $\begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}$ invertible? (if yes, find the inverse)

Ⅲ Assume A, B $n \times n$, invertible. Is $(AB)^T$ invertible? What is the inverse? (in terms of A^{-1}, B^{-1})

Sol: I:

$$\left[\begin{array}{ccc|ccc} 2 & 3 & 0 & 1 & 0 & 0 \\ 3 & 5 & 0 & 0 & 1 & 0 \\ -4 & -6 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 2 & 3 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{3}{2} & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & \frac{3}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & -3 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & -3 & 0 \\ 0 & 1 & 0 & -3 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{array}\right] \implies A^{-1} = \begin{bmatrix} 5 & -3 & 0 \\ -3 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

I $\qquad$ A^{-1}

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- **Elementary matrices**

an elem. matrix is the result of a single elem. row operation on the identity matrix.

Ex: $E_1 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$

$r_2 \mapsto r_2 - 3r_1$ $\qquad$ $r_1 \leftrightarrows r_2$ $\qquad$ $r_2 \mapsto 5r_2$

for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $E_1 A = \begin{bmatrix} a & b \\ c-3a & d-3b \end{bmatrix}$, $E_2 A = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$, $E_3 A = \begin{bmatrix} a & b \\ 5c & 5d \end{bmatrix}$

- If an elem row op is performed on $m \times n$ mat. A, the resulting matrix is EA, where E is the $m \times m$ elem. mat. created by doing the <u>same</u> row op. on I_m.

$A \underset{\rho}{\sim} EA$ $\qquad$ where $I \underset{\rho}{\sim} E$.

- Each E is invertible. The inverse is an elem. mat. of the same type.

Argument for (b) of Thm on p. 1:

$$A \sim E_1 A \sim E_2(E_1 A) \sim \dots \sim \underbrace{E_p \cdots E_1}_{A^{-1}} A = I$$

$$I \sim E_1 I = E_1 \sim E_2 E_1 \sim \quad \dots \sim E_p \cdots E_1$$