6.1 Inner product, length, orthogonality

- Want to generalize geometric notions of length, distance, perpendicularity from $\mathbb{R}^2, \mathbb{R}^3$ to $\mathbb{R}^n$

def for $\vec{u}, \vec{v} \in \mathbb{R}^n$, the "inner product" ("dot product") is $\underbrace{\vec{u}^T}_{1\times n \text{ mat.}} \underbrace{\vec{v}}_{n\times 1 \text{ mat.}} =: \vec{u}\cdot\vec{v}$ - a number

If $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$, $\quad \vec{u}\cdot\vec{v} = [u_1 \cdots u_n]\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = u_1v_1 + \dots + u_nv_n$

Ex: $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix}$

$\vec{u}\cdot\vec{v} = 1\cdot 3 + 2\cdot 5 + 3(-1) = 10$

$\vec{v}\cdot\vec{u} = 3\cdot 1 + 5\cdot 2 + (-1)\cdot 3 = 10$:) !

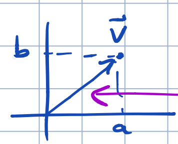
Thm
(a) $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$
(b) $(\vec{u}+\vec{v})\cdot\vec{w} = \vec{u}\cdot\vec{w} + \vec{v}\cdot\vec{w}$
(c) $(c\vec{u})\cdot\vec{v} = c(\vec{u}\cdot\vec{v})$

$\Big\}\Rightarrow (c_1\vec{u}_1 + \dots + c_p\vec{u}_p)\cdot\vec{v} = c_1(\vec{u}_1\cdot\vec{v}) + \dots + c_p(\vec{u}_p\cdot\vec{v})$

(d) $\vec{u}\cdot\vec{u} \geq 0$ and $\vec{u}\cdot\vec{u} = 0$ iff $\vec{u} = \vec{0}$

def Length ("norm") of $\vec{v}\in\mathbb{R}^n$ is $\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}} = \sqrt{v_1^2 + \dots + v_n^2} \geq 0$, $\quad \|\vec{v}\|^2 = \vec{v}\cdot\vec{v}$

Ex: $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$



$\|\vec{v}\| = \sqrt{a^2+b^2}$ = length of the line segment (Pythagorean theorem)

- $\|c\vec{v}\| = |c|\,\|\vec{v}\|$ for $c\in\mathbb{R}$
- a vector of length 1 - "unit vector". For $\vec{v} \neq \vec{0}$, $\vec{v} \longrightarrow \vec{u} = \frac{1}{\|\vec{v}\|}\vec{v}$ unit vector in the direction of $\vec{v}$

"normalizing $\vec{v}$"

Ex: $\vec{v} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$ Q: find $\vec{u}$ a unit vector in the direction of $\vec{v}$.

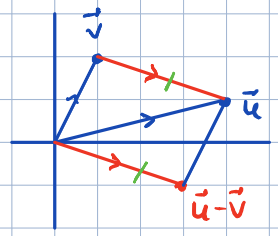
Sol: $\|\vec{v}\|^2 = \vec{v}\cdot\vec{v} = 1^2 + (-2)^2 + 2^2 = 9$, $\|\vec{v}\| = 3$, $\quad \vec{u} = \frac{1}{3}\vec{v} = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}$

Check: $\|\vec{u}\|^2 = \left(\frac{1}{3}\right)^2 + \left(\frac{-2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 = \frac{1+4+4}{9} = 1$

def for $\vec{u}, \vec{v}\in\mathbb{R}^n$, the distance between $\vec{u}$ and $\vec{v}$ is:

$\text{dist}(\vec{u},\vec{v}) := \|\vec{u}-\vec{v}\|$

Ex: $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ $\quad \text{dist}(\vec{u},\vec{v}) = \|\vec{u}-\vec{v}\| = \left\|\begin{bmatrix} -2 \\ -3 \end{bmatrix}\right\| = \sqrt{(-2)^2 + (-3)^2} = \sqrt{13}$



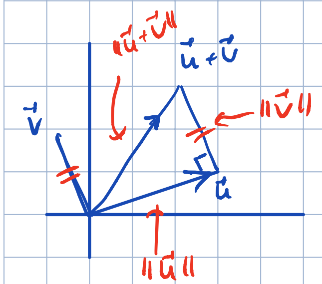
Orthogonal vectors

def vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ are orthogonal (to each other) if $\boxed{\vec{u}\cdot\vec{v} = \vec{0}}$
(perpendicular)

- $\vec{u}$ and $\vec{v}$ are orthogonal iff $\underbrace{\|\vec{u}+\vec{v}\|^2}_{(\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v})} = \|\vec{u}\|^2 + \|\vec{v}\|^2$ — Pythagorean thm.

$\|\vec{u}\|^2 + \|\vec{v}\|^2 = \vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{v}$

$(\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) = \vec{u}\cdot\vec{u} + \boxed{2\vec{u}\cdot\vec{v}} + \vec{v}\cdot\vec{v}$

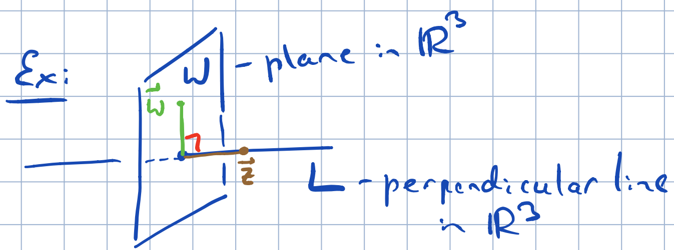


Note: $\vec{0} \perp \vec{u}$ for any $\vec{u}$.

Orthogonal complements.

- If $\vec{z} \in \mathbb{R}^n$ is orthogonal to every vector in $W \subset \mathbb{R}^n$ (a subspace), then $\vec{z}$ is orthogonal to W.
- Set of all vectors in $\mathbb{R}^n$ orthogonal to W - "orthogonal complement of W", $W^\perp$ – notation

Ex:



$W^\perp = L$, $L^\perp = W$.

- $\vec{x}$ is in $W^\perp$ (for any W) if it is orthog. to any vector in a set which spans W.
- $W^\perp$ is a subspace of $\mathbb{R}^n$

Ex: for A $m\times n$ matrix, $\operatorname{Nul} A$ and $\operatorname{Row} A \subset \mathbb{R}^n$ - orthog. complements of each other

$\operatorname{Nul} A^T$ and $\operatorname{Col} A \subset \mathbb{R}^m$ - orthog. complements of each other

- for $W \subset \mathbb{R}^n$, $\boxed{\dim W + \dim W^\perp = n}$

6.2 Orthogonal sets

A set of vectors $\{\vec{u}_1, \dots, \vec{u}_p\}$ in $\mathbb{R}^n$ is an orthogonal set if

$\vec{u}_i \cdot \vec{u}_j = 0$ for each pair $i \neq j$.

Ex*: $\vec{u}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$ $\vec{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ $\vec{u}_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}$ Q: check that $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthog. set.

Sol: $\vec{u}_1 \cdot \vec{u}_2 = 3(-1) + 1 \cdot 2 + 1 \cdot 1 = 0$ ✓ $\vec{u}_1 \cdot \vec{u}_3 = 3(-\frac{1}{2}) + 1(-2) + 1(\frac{7}{2}) = 0$ ✓

$\vec{u}_2 \cdot \vec{u}_3 = (-1)(-\frac{1}{2}) + 2(-2) + 1(\frac{7}{2}) = 0$ ✓

THM If $S = \{\vec{u}_1, \dots, \vec{u}_p\}$ is an orthog. set of nonzero vectors in $\mathbb{R}^n$, then S is lin. indep. Hence, S is a basis for Span S.

- an orthogonal basis for a subspace $W \subset \mathbb{R}^n$ is a basis which is also an orthogonal set.

THM Let $\{\vec{u}_1, \dots, \vec{u}_p\}$ be an orthog. basis for $W \subset \mathbb{R}^n$. For each $\vec{y} \in W$, weights in $y = c_1 \vec{u}_1 + \dots + c_p \vec{u}_p$ are: $\boxed{c_j = \dfrac{\vec{y} \cdot \vec{u}_j}{\vec{u}_j \cdot \vec{u}_j}}$, $j = 1, \dots, p$

< Idea: $y \cdot \vec{u}_1 = c_1 \vec{u}_1 \cdot \vec{u}_1 + \cancel{c_2 \vec{u}_2 \cdot \vec{u}_1} + \dots + \cancel{c_p \vec{u}_p \cdot \vec{u}_1}$ $\Rightarrow c_1 = \dfrac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1}$; similarly for c_j >

Ex: orthog set from Ex* is a basis in $\mathbb{R}^3$. $\underbrace{}_{S = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}}$ $\vec{y} = \begin{bmatrix} 6 \\ 1 \\ -8 \end{bmatrix}$ Q: express $\vec{y}$ as a lin. comb. of vectors in S

Sol: $\vec{y} = \underbrace{\dfrac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1}}_{\frac{11}{11}} \vec{u}_1 + \underbrace{\dfrac{\vec{y} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2}}_{\frac{-12}{6}} \vec{u}_2 + \underbrace{\dfrac{\vec{y} \cdot \vec{u}_3}{\vec{u}_3 \cdot \vec{u}_3}}_{\frac{-33}{33/2}} \vec{u}_3 = \boxed{\vec{u}_1 - 2\vec{u}_2 - 2\vec{u}_3}$

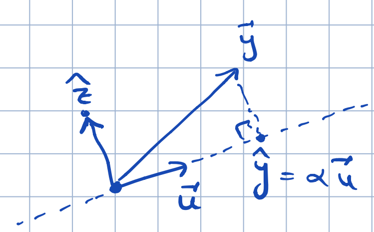
← did not need to solve the lin. sys. to compute the weights

Orthogonal projection onto a vector / a line

Given $\vec{u} \neq \vec{0}$ in $\mathbb{R}^n$, want to write $\vec{y} \in \mathbb{R}^n$ as

$\vec{y} = \underbrace{\hat{y}}_{\alpha \vec{u} \text{ for some } \alpha} + \underbrace{\hat{z}}_{\text{orthog. to } \vec{u}}$ $\Rightarrow (\vec{y} - \alpha \vec{u}) \cdot \vec{u} = 0 \Rightarrow \alpha = \dfrac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}$

$\vec{y} \cdot \vec{u} - \alpha \vec{u} \cdot \vec{u}$



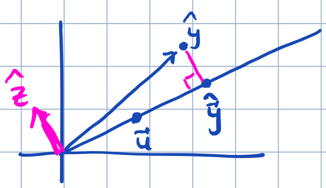
$\hat{y} = \dfrac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u} =: \text{proj}_L \vec{y}$ - orthogonal projection of $\vec{y}$ onto the line $L = \text{Span}\{\vec{u}\}$

does not change if $\vec{u} \mapsto c\vec{u}$, $c \neq 0$

Ex: $\vec{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ $\quad \vec{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ $\quad$ Q: write $\vec{y}$ as $\underset{\text{in Span}\{\vec u\}}{\hat{\vec y}} + \underset{\text{orthog to } \vec u}{\hat{\vec z}}$

Sol: $\vec y \cdot \vec u = 40$, $\vec u \cdot \vec u = 20$

$\Rightarrow \hat{\vec y} = \frac{40}{20}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$, $\quad \vec z = \vec y - \hat{\vec y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix} - \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. So: $\begin{bmatrix} 7 \\ 6 \end{bmatrix} = \underbrace{\begin{bmatrix} 8 \\ 4 \end{bmatrix}}_{\hat{\vec y}} + \underbrace{\begin{bmatrix} -1 \\ 2 \end{bmatrix}}_{\vec z}$



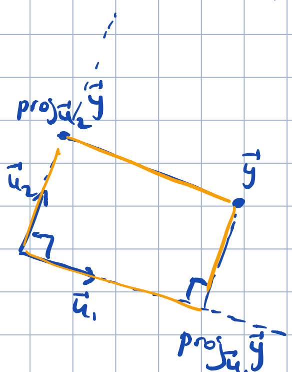
Q: find $\text{dist}(\vec y, L)$

Sol: $\text{dist}(\vec y, L) = \text{dist}(\vec y, \underset{\substack{\text{closest point on } L \\ \text{to } \vec y}}{\hat{\vec y}}) = \|\vec y - \hat{\vec y}\| = \|\vec z\| = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$

Ex: (THM, geometric picture)

$W = \mathbb{R}^2 = \text{Span}\{\vec u_1, \vec u_2\}$ (orthogonal) $\quad$ $\vec y \in \mathbb{R}^2$ can be written as

$$\vec y = \underbrace{\frac{\vec y \cdot \vec u_1}{\vec u_1 \cdot \vec u_1}\vec u_1}_{\text{proj}_{\vec u_1}\vec y} + \underbrace{\frac{\vec y \cdot \vec u_2}{\vec u_2 \cdot \vec u_2}\vec u_2}_{\text{proj}_{\vec u_2}\vec y}$$



So: THM decomposes $\vec y$ into a sum of orthog. projections onto 1-dim subspaces (which are mutually orthogonal)

Orthonormal sets $\quad S = \{\vec u_1, \dots, \vec u_p\}$ is orthonormal if it is an orthog. set of unit vectors. $\quad$ If $W = \text{Span}\, S$, then S - o/n basis for W.

Ex: $\{\vec e_1, \dots, \vec e_n\}$ - o/n basis for $\mathbb{R}^n$

Ex: $\vec v_1 = \begin{bmatrix} 3/\sqrt{11} \\ 1/\sqrt{11} \\ 1/\sqrt{11} \end{bmatrix}$ $\quad \vec v_2 = \begin{bmatrix} -1/\sqrt6 \\ 2/\sqrt6 \\ 1/\sqrt6 \end{bmatrix}$ $\quad \vec v_3 = \begin{bmatrix} -1/\sqrt{66} \\ -4/\sqrt{66} \\ 7/\sqrt{66} \end{bmatrix}$ $\quad$ $\{\vec v_1, \vec v_2, \vec v_3\}$ - o/n basis for $\mathbb{R}^3$ (obtained from Ex* by normalizing vectors to unit length, $\vec v_i = \frac{1}{\|\vec u_i\|}\vec u_i$)

THM An $m \times n$ matrix U has o/n columns iff $\boxed{U^T U = I}$

THM Let U be a $m \times n$ matrix with o/n columns and let $\vec x, \vec y \in \mathbb{R}^n$. Then:

(a) $\|U\vec x\| = \|\vec x\|$ $\quad$ (b) $(U\vec x)\cdot(U\vec y) = \vec x \cdot \vec y$ $\quad$ (c) $U\vec x \cdot U\vec y = 0$ iff $\vec x \cdot \vec y = 0$

I.e. mapping $\vec x \mapsto U\vec x$ preserves length and orthogonality.

Case $m=n$: square U with o/n columns is an orthogonal matrix. U orthogonal iff $\boxed{U^{-1} = U^T}$