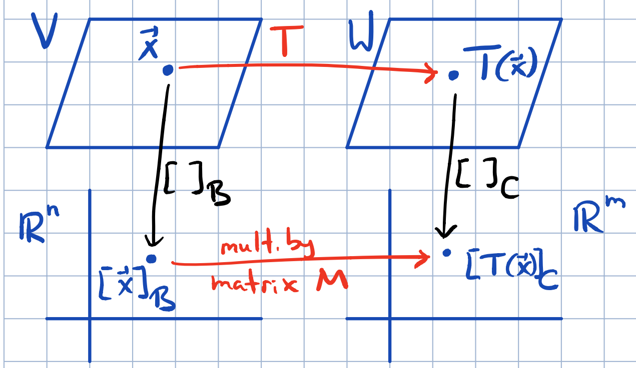
3/4/2020 | 5.4 Eigenvectors and linear transformations

Idea: if $A = PDP^{-1}$, the lin. transf. $\vec{x} \mapsto A\vec{x}$ is "essentially the same" as a simple lin. transf. $\vec{u} \mapsto D\vec{u}$.

- Matrix of a lin. transf. $T: \underset{\text{v.sp.}}{V} \to \underset{\text{v.sp}}{W}$

  $\quad\quad\quad\quad\quad\quad\quad\quad\quad \mathcal{B} \quad\quad \mathcal{C}$ - bases



how to connect $[\vec{x}]_{\mathcal{B}}$ and $[T(\vec{x})]_{\mathcal{C}}$?

$$\vec{x} = r_1\vec{b}_1 + \dots + r_n\vec{b}_n \Rightarrow [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}$$

$$T(\vec{x}) = r_1 T(\vec{b}_1) + \dots + r_n T(\vec{b}_n)$$

$$\Rightarrow [T(\vec{x})]_{\mathcal{C}} = r_1 [T(\vec{b}_1)]_{\mathcal{C}} + \dots + r_n [T(\vec{b}_n)]_{\mathcal{C}}$$

Thus $[T(\vec{x})]_{\mathcal{C}} = M[\vec{x}]_{\mathcal{B}}$ with $M = [\,T(\vec{b}_1)_{\mathcal{C}} \cdots T(\vec{b}_n)_{\mathcal{C}}\,]$

- matrix of T relative to bases $\mathcal{B}, \mathcal{C}$
= matrix representation of T

Ex: $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ basis for V

$\mathcal{C} = \{\vec{c}_1, \vec{c}_2, \vec{c}_3\}$ basis for W. Let $T: V \to W$ be a lin. transf. s.t.

$T(\vec{b}_1) = 3\vec{c}_1 - 2\vec{c}_2 + 5\vec{c}_3$, $T(\vec{b}_2) = 4\vec{c}_1 + 7\vec{c}_2 - \vec{c}_3$. **Q:** find the matrix of T rel. to $\mathcal{B}, \mathcal{C}$

Sol: $M = \begin{bmatrix} 3 & 4 \\ -2 & 7 \\ 5 & -1 \end{bmatrix}$

$[T(\vec{b}_1)]_{\mathcal{C}}$ $\quad$ $[T(\vec{b}_2)]_{\mathcal{C}}$

- If $V = W$ and $T(\vec{x}) = \vec{x}$ the identity matrix, then the matrix M is just $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ - change-of-coordinates matrix

- Lin. transformations $T: \underset{\mathcal{B}}{V} \to \underset{\mathcal{B}}{V}$ same space - <u>same</u> basis

In this case, $M =: [T]_{\mathcal{B}}$ - "matrix of T relative to $\mathcal{B}$" or "$\mathcal{B}$-matrix of T"

We have $[T(\vec{x})]_{\mathcal{B}} = [T]_{\mathcal{B}}\,[\vec{x}]_{\mathcal{B}}$ for all $\vec{x} \in V$

$$\begin{tikzcd}
\vec{x} \arrow[r, "T"] \arrow[d] & T(\vec{x}) \arrow[d] \\
{[\vec{x}]_{\mathcal{B}}} \arrow[r, "\text{mult. by}", "{[T]_{\mathcal{B}}}"'] & {[T(\vec{x})]_{\mathcal{B}}}
\end{tikzcd}$$

Ex: $T: \mathbb{P}_2 \to \mathbb{P}_2$ given by

$T(a_0 + a_1 t + a_2 t^2) = a_1 + 2a_2 t$ (differentiation in t)

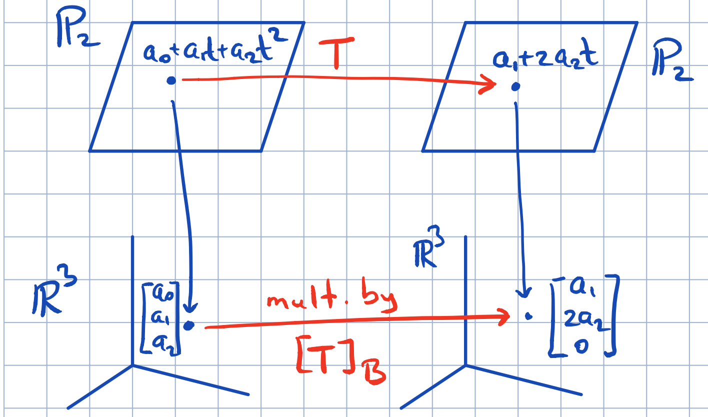
Q: find the $\mathcal{B}$-matrix for T, for $\mathcal{B} = \{1, t, t^2\}$

Sol: $T(1) = 0$ $\quad [T(1)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$T(t) = 1$ $\quad [T(t)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$T(t^2) = 2t$ $\quad [T(t^2)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$

$\Rightarrow [T]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$



Lin. transformations of $\mathbb{R}^n$

$T: \mathbb{R}^n \to \mathbb{R}^n$
$\vec{x} \mapsto A\vec{x}$

if A diagonalizable, $\mathcal{B}$-matrix of A is diagonal, for $\mathcal{B}$ - basis of eigenvectors

Thm (diagonal matrix representation)

Suppose $A = PDP^{-1}$ with D a diagonal $n \times n$ matrix. If $\mathcal{B}$ is the basis of $\mathbb{R}^n$ formed from the columns of P, then D is the $\mathcal{B}$ - matrix of the transf $\vec{x} \mapsto A\vec{x}$.

Ex: $T: \mathbb{R}^2 \to \mathbb{R}^2$, $\vec{x} \mapsto A\vec{x}$ $\quad A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$ find a basis $\mathcal{B}$ for $\mathbb{R}^2$ s.t. $[T]_{\mathcal{B}}$ is diagonal.

Sol: $A = PDP^{-1}$, $P = \begin{bmatrix} \underset{\vec{b}_1}{\boxed{\begin{matrix} 1 \\ -1 \end{matrix}}} & \underset{\vec{b}_2}{\boxed{\begin{matrix} 1 \\ -2 \end{matrix}}} \end{bmatrix}$, $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$ Thus, for $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$,

$[T]_{\mathcal{B}} = D$

I.e. mappings $\vec{x} \mapsto A\vec{x}$ and $\vec{u} \mapsto D\vec{u}$ describe the same lin. transf. rel. to different bases.

Similarity of matrix representations

Thm above does not in fact require D to be diagonal:

if $A \underset{\text{similar}}{\simeq} C$, i.e. $A = PCP^{-1}$ and $T: \mathbb{R}^n \to \mathbb{R}^n$, $\vec{x} \mapsto A\vec{x}$, then $[T]_{\underset{\uparrow}{\mathcal{B}}} = C$

basis of columns of P.

$$\begin{tikzcd}
\vec{x} \arrow[r, "\text{mult. by}", "A"'] \arrow[d, "\text{mult. by } P^{-1}"'] & A\vec{x} \\
{[\vec{x}]_{\mathcal{B}}} \arrow[r, "\text{mult. by}", "C"'] & {[A\vec{x}]_{\mathcal{B}}} \arrow[u, "\text{mult. by } P"']
\end{tikzcd}$$

Conversely, matrix of T rel. to <u>any</u> basis $\mathcal{B}$ is <u>similar to A</u>.