1.3 Classification of differential equations

• Ordinary and partial diff. eq. (ODE & PDE)

ODE: unknown fun. depends on a single variable t

Ex: falling object ① $v(t)$; mice-owls ② $p(t)$; ③ $L\frac{d^2Q(t)}{dt^2}+R\frac{dQ(t)}{dt}+\frac{1}{C}Q(t)=E(t)$

eq. on $Q(t)$ - charge on a capacitor in an electric circuit

PDE: ④ $\alpha^2\frac{\partial^2 u(x,t)}{\partial x^2}=\frac{\partial u(x,t)}{\partial t}$ "heat equation"

⑤ $a^2\frac{\partial^2 u(x,t)}{\partial x^2}=\frac{\partial^2 u(x,t)}{\partial t^2}$ "wave equation"

• Systems of diff. eq. - if there are several unknown functions.

Lotka-Volterra equations (predator-prey) ⑥ $\begin{cases}\frac{dx}{dt}=ax-\alpha xy\\ \frac{dy}{dt}=-cy+\gamma xy\end{cases}$ $x(t)$ - prey population, $y(t)$ - predator population

• Order of an eq. - highest derivative that appears in the eq.

(*) $F(t,y,y',\dots,y^{(n)})=0$ ODE of order n. (we assume, (*) can be solved for $y^{(n)}$ as $y^{(n)}=f(t,y,y',\dots,y^{(n-1)})$)

Ex ⑦ $y'''+2e^t y''+yy'=t^4$ - 3rd order ODE on $y(t)$

• Linear vs nonlinear diff. eq.

an ODE $F(t,y,y',\dots,y^{(n)})=0$ is linear if F is a linear function in $y,y',\dots,y^{(n)}$. No linearity in t is assumed!

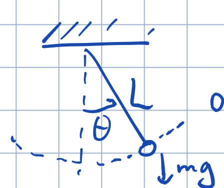
General linear n-th order ODE:

$$a_0(t)y^{(n)}+a_1(t)y^{(n-1)}+\dots+a_n(t)y=g(t)$$

Ex: falling object, mice-owls, ③ - lin. ODE. (4, 5) - lin. PDE

Ex (7) is non-linear because of yy' term.

eqs (6) are non-linear due to xy terms (each eq.)

Ex  oscillating pendulum (8) $\frac{d^2\theta}{dt^2}+\frac{g}{L}\sin\theta=0$

$\theta(t)$ - unknown function - non-linear ODE due to $\sin\theta$ term

linear eq. - easier, well-developed theory
non-linear - much harder, less satisfactory methods of solution
non-linear can (sometimes) be approximated by linear.
E.g. for θ small, $\sin\theta \approx \theta$ $\rightarrow$ (8) can be approximated by

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$$ - linear eq.
("linearization" of (8))

Solutions a solution of n^{th} order ODE $y^{(n)} = f(t, y, y', \dots, y^{(n-1)})$ $(*)$
on the interval $\alpha < t < \beta$ is a fun. φ s.t. $\varphi', \varphi'', \dots, \varphi^{(n)}$ exist and satisfy
$\varphi^{(n)}(t) = f(t, \varphi(t), \varphi'(t), \dots, \varphi^{(n-1)}(t))$ for every $\alpha < t < \beta$

Ex: $\dfrac{dp}{dt} = \dfrac{p}{2} - 450$ has the sol $p(t) = 900 + c\, e^{t/2}$ $(**)$, c - arbitrary constant

Given an eq., it is generally not easy to find a sol.
Given a fun., easy to verify whether it is a sol. (by substitution)

Ex: $y'' + y = 0$ Q: is $y_1(t) = \cos t$ a solution?
Sol: $y_1'(t) = -\sin t$, $y_1''(t) = -\cos t$ $\Rightarrow$ $y_1'' + y_1 = 0$ ✓

Questions

- Existence: $(*)$ does not always have sols (but for some classes of eqns, it does)
- Uniqueness usually, sols come in a family, like $(**)$, but one might ask about uniqueness for the init. value problem
- Determining actual solutions (explicitly) - not always possible. Sometimes, can only do numerically.

2.1 Integrating Factors

divide by $P(t)$ if $P(t) \neq 0$

General 1^{st} order linear ODE: $\boxed{\dfrac{dy}{dt} + p(t)\, y = g(t)}$ or $\boxed{P(t)\dfrac{dy}{dt} + Q(t)\, y = G(t)}$ $(*)$

Ex: $\underbrace{(4+t^2)\dfrac{dy}{dt} + 2ty}_{= \frac{d}{dt}\left((4+t^2)y\right)} = 4t$ (1) $\iff$ $\dfrac{d}{dt}\left((4+t^2)y\right) = 4t$ $\xrightarrow[\text{w.r.t. } t]{\text{integrate}}$ $(4+t^2)y = 2t^2 + \underbrace{C}_{\text{arbitrary const. of integration}}$

derivative of a product

$\xrightarrow{\text{solve for } y}$ $\boxed{y = \dfrac{2t^2}{4+t^2} + \dfrac{C}{4+t^2}}$ - general sol. of (1)

Generally, l.h.s. of (*) is not a derivative of a product.

Idea (Leibniz): find a function $\mu(t)$ s.t. once we multiply (*) by $\mu(t)$, l.h.s. becomes a der. of a product.

Ex: $$\frac{dy}{dt} + \frac{1}{2}y = \frac{1}{2}e^{t/3} \qquad (2)$$ find the gen. sol.

Sol: $\underbrace{\mu(t)}_{\text{(yet) undetermined function}} \cdot (2)$: $$\underbrace{\mu(t)\frac{dy}{dt} + \boxed{\frac{1}{2}\mu(t)}\,y}_{} = \frac{1}{2}\mu(t)\,e^{t/3} \qquad (4)$$

$$\overset{\text{WANT}}{=} \frac{d}{dt}\left(\mu(t)\,y\right) = \mu(t)\frac{dy}{dt} + \boxed{\frac{d\mu(t)}{dt}}\,y$$

– works iff $\frac{d\mu(t)}{dt} = \frac{1}{2}\mu(t)$ (3)

Solve (3): $$\frac{1}{\mu(t)}\frac{d\mu(t)}{dt} = \frac{1}{2}$$

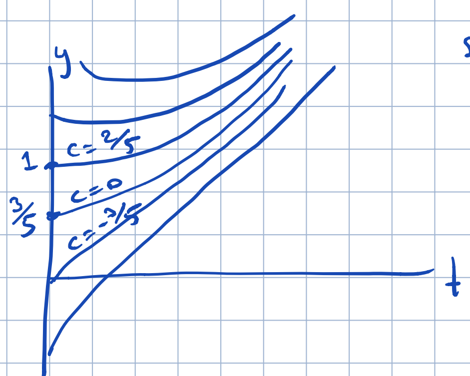
$$\rightarrow \frac{d}{dt}\ln|\mu(t)| = \frac{1}{2} \longrightarrow \ln|\mu(t)| = \frac{t}{2} + C \rightarrow \mu(t) = c\,e^{t/2}$$

• we don't need the general $\mu(t)$, need just one $\mu \neq 0$.

→ choose $c = 1$ → $\mu(t) = e^{t/2}$

(4): $$\underbrace{e^{t/2}\frac{dy}{dt} + \frac{1}{2}e^{t/2}y}_{\frac{d}{dt}\left(e^{t/2}y\right)} = \frac{1}{2}\underbrace{e^{t/2}e^{t/3}}_{e^{5t/6}} \xrightarrow[\text{integrate}]{} e^{t/2}y = \frac{3}{5}e^{5t/6} + C$$

$$\xrightarrow{\text{solve for } y} \boxed{y = \frac{3}{5}e^{t/3} + c\,e^{-t/2}} \qquad (5)$$ – general solution



Ex: (2) + init. cond. $y(0) = 1$

$$y(0) = \frac{3}{5} + C = 1 \Rightarrow C = \frac{2}{5}$$

$$\Rightarrow y(t) = \frac{3}{5}e^{t/3} + \frac{2}{5}e^{-t/2}$$