Linear transformations

- A transformation (or function, or mapping) T from $\underbrace{\mathbb{R}^n}_{}$ to $\mathbb{R}^m$ is a rule assigning to each vector $\vec{v} \in \mathbb{R}^n$ some vector $\underbrace{T(\vec{v})}_{\text{image of } \vec{v}} \in \mathbb{R}^m$

domain codomain

- A transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is <u>linear</u> if

(a) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ for any $\vec{u}, \vec{v} \in \mathbb{R}^n$

(b) $T(\underset{\substack{\uparrow \\ \text{scalar}}}{c}\cdot\vec{v}) = c\,T(\vec{v})$

<u>Ex:</u> for A an $m\times n$ matrix, $\boxed{T(\vec{v}) := A\vec{v}}$ (*) is a <u>linear</u> transf. (matrix transf. determined by A).

check: $\big(T(\vec{u}+\vec{v}) = A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v} = T(\vec{u}) + T(\vec{v})$,
$T(c\vec{v}) = A(c\vec{v}) = cA\vec{v} = cT(\vec{v})\big)$

<u>Theorem</u> Any linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transf. (*), with

$$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \end{bmatrix}$$

where $\vec{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i^{th}\text{ place} \in \mathbb{R}^n$

$= i^{th}$ column of I_n.

"standard matrix" of T

Notation: $[T] := A$
- the stand. matrix of A.

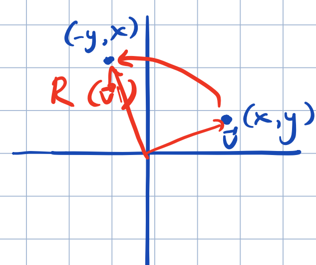
<u>Ex:</u> $id: \mathbb{R}^n \to \mathbb{R}^n$, $\vec{v} \mapsto \vec{v}$ - identity map, $[id] = I_n$.

<u>Ex:</u> $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(\vec{v}) = 3\vec{v}$ $\Rightarrow$

stand. matrix $= \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

$\begin{pmatrix} 3 \\ 0 \end{pmatrix} = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = T(\vec{e}_1)$, $\begin{pmatrix} 0 \\ 3 \end{pmatrix} = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = T(\vec{e}_2)$

Ex: $R: \mathbb{R}^2 \to \mathbb{R}^2$ rotation by $90°$ counterclockwise
$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} -y \\ x \end{bmatrix}$



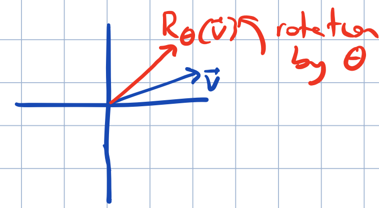
stand. matrix: $R\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $R\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$\Rightarrow [R] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Rem: Rotation by an angle θ counterclockwise

$R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$

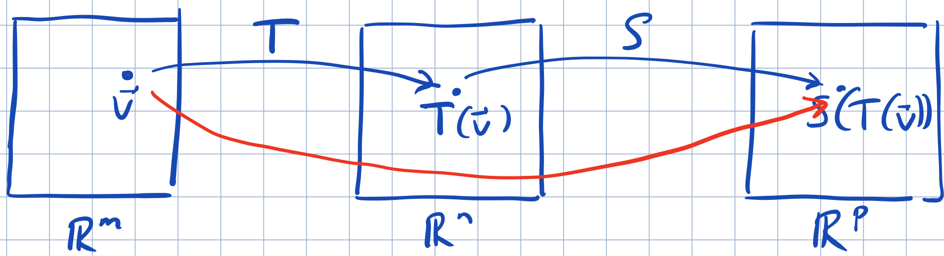
has stand. matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$



- Composition of linear transformations

Let $T: \mathbb{R}^m \to \mathbb{R}^n$, $S: \mathbb{R}^n \to \mathbb{R}^p$ be two lin. transformations. Then one has the lin. transf. $S \circ T : \mathbb{R}^m \to \mathbb{R}^p$, mapping $\vec{v} \in \mathbb{R}^m$ to $S(T(\vec{v}))$.

 - composition of S and T



WARNING: $S \circ T$ first applies T to a vector and then S. So we "read" $S \circ T$ from right to left.

Thm The stand. matrix of the composition $S \circ T$ is the product of stand. matrices of S and T:

$$\underset{p \times m}{[S \circ T]} = \underset{p \times n}{[S]}\ \underset{n \times m}{[T]}$$

[Argument:

$$\vec{v} \xmapsto{T} T(\vec{v}) = [T]\vec{v} \xrightarrow{S} S([T]\vec{v}) = [S]([T]\vec{v})$$

$$= ([S][T])\vec{v}$$

$\Rightarrow [S \circ T] = [S][T]$]

Ex: $R: \mathbb{R}^2 \to \mathbb{R}^2$ rotation by $90°$

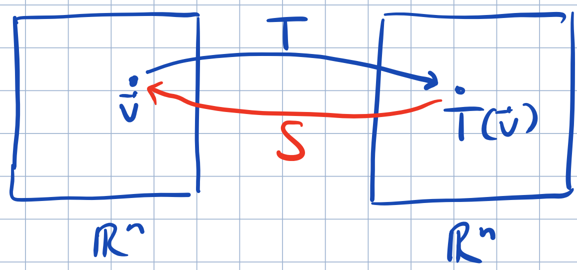
$S: \mathbb{R}^2 \to \mathbb{R}^2$ scaling by 3

$(S \circ R)\begin{bmatrix} x \\ y \end{bmatrix} = S\left(\begin{bmatrix} -y \\ x \end{bmatrix}\right) = \begin{bmatrix} -3y \\ 3x \end{bmatrix}$

stand. mat. $[S \circ R] = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix}$

def A lin. transf. $T : \mathbb{R}^n \to \mathbb{R}^n$ is "invertible" if there exists a lin. transf. $S : \mathbb{R}^n \to \mathbb{R}^n$ s.t. $S \circ T = \underbrace{id}_{\text{denoted } I_n \text{ in Poole}} = T \circ S$. Then $S =: T^{-1}$ is called the "inverse" of T.



Thm T is invertible iff $[T]$ is invertible matrix, and $[T^{-1}] = [T]^{-1}$.

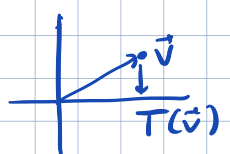
Ex: R - rot. by $90°$ $\qquad$ R^{-1} - rotation by $-90°$

$$[R] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \underset{[R^{-1}]}{\overset{\|}{[R]^{-1}}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ - inverse matrix}$$

Ex: $T : \mathbb{R}^2 \to \mathbb{R}^2$ projection to x-axis

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x \\ 0 \end{bmatrix}$$

$$[T] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ - non-invertible matrix!}$$



$\Rightarrow$ T not invertible.

More on matrix algebra

matrix powers

- for A $n \times n$ matrix, can form $A^2 = AA$, $A^3 = AAA$, $A^k = \underbrace{A \cdots A}_{k \text{ factors}}$

  if A is invertible, then A^{-1} = inverse of A,
  $A^{-2} = A^{-1}A^{-1}$, $A^{-k} = \underbrace{A^{-1} \cdots A^{-1}}_{k}$

Ex:
$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$A^3 = A^2 A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

Properties of matrix operations

$A(B+C) = AB + AC$

$(AB)^{-1} = B^{-1}A^{-1}$, A, B $n\times n$ invertible

$(AB)^T = B^T A^T$

$(A^T)^{-1} = (A^{-1})^T$

$(A^T)^T = A$

Ex: solve for unknown matrix X: $\boxed{AX^{-1}B = (AB)^2}$

assuming A, B, X $n\times n$, invertible

Sol: $AX^{-1}B = ABAB \Rightarrow X^{-1} = BA \Rightarrow X = (BA)^{-1} = A^{-1}B^{-1}$.

multiply by A^{-1} on the left, by B^{-1} on the right

Invertible matrix thm (v. 1)

Let A be $n\times n$ matrix. The following statements are equivalent:

a) A is invertible

b) $A\vec{x} = \vec{b}$ has a unique solution for every $\vec{b} \in \mathbb{R}^n$

c) $A\vec{x} = \vec{0}$ has only the trivial solution

d) RREF of A is I_n

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a) $\Rightarrow$ b) : for A invertible, the unique sol. of $A\vec{x} = \vec{b}$ is $\boxed{\vec{x} = A^{-1}\vec{b}}$.

Ex: $$\underbrace{\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ -1 \end{bmatrix}}_{\vec{b}} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A^{-1}\vec{b} = \frac{1}{1}\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}.$$

Block multiplication - Poole pp. 148-149.

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 3 \\ 0 & -3 \end{bmatrix}$$

Ex: $$\left[\begin{array}{cc|c} \underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{A_{11}} & & \underbrace{\begin{matrix} 1 \\ -1 \end{matrix}}_{A_{12}} \end{array}\right] \left[\begin{array}{c|cc} 0 & 1 & 0 \\ 1 & 0 & 1 \\ \hline 2 & 0 & 3 \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \end{bmatrix}$$

$B_{11} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $B_{12} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B_{21} = \begin{bmatrix} 2 \end{bmatrix}$, $B_{22} = \begin{bmatrix} 0 & 3 \end{bmatrix}$

$$= \left[\begin{array}{c|cc} 2 & 1 & 3 \\ -1 & 0 & -2 \end{array}\right]$$

Homogeneous linear systems

def A lin. sys. is homogeneous if the constant term in each equation is zero.

Aug. Mat. of a homog. sys. has the form $[A \mid \vec{0}]$ ← column of zeros

(equivalently: it is the matrix eq. $A\vec{x} = \vec{0}$)

Ex:
$$\begin{aligned} x + 2y &= 0 \\ 3x + 4y &= 0 \end{aligned}$$
– homog. sys.

Aug. Mat.: $\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 3 & 4 & 0 \end{array}\right] \Leftrightarrow \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\downarrow$$

$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \rightsquigarrow \begin{aligned} x_1 &= 0 \\ x_2 &= 0 \end{aligned}$

- $\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ is always a sol. for any homog. sys. $\Rightarrow$ any homog. sys. is consistent.
  – "trivial solution"

Ex:
$$\begin{aligned} x + 2y &= 0 \\ 2x + 4y &= 0 \end{aligned}$$

$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 4 & 0 \end{array}\right] \longrightarrow \begin{array}{c} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cc} x & y = s \\ \text{leading var.} & \text{free var.} \end{array} \end{array}$

$x = -2s$
$y = s$

Sol: $\begin{bmatrix} x \\ y \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

– infinitely many solutions ($s \neq 0 \Rightarrow$ nontrivial sol.)
since there is a free variable

- a homogeneous eq. $A\vec{x} = \vec{0}$ has ∞-many solutions if $n > m$.
  ↑ $m \times n$ matrix
  (because # free vars $= n - \underbrace{\text{rank}(A)}_{\leq m} > 0$)